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Research Article

A General Input Distance Function Based on Opportunity Costs

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There are several distance function definitions in a general production framework, including Data Envelopment Analysis, which can be used to describe the production technology and to define corresponding measures of technical efficiency (notably the Shephard and the directional distance functions). This paper introduces a generalisation of the distance function concept based on the idea of minimizing firm's opportunity cost. We further state a general dual correspondence between the cost function and this new general distance function, which encompasses all previously published duality results. All our results also hold under the assumption that we work in a Data Envelopment Analysis context.

1. Introduction

The theory of duality has acquired great popularity in microeconomics [1–4]. Duality theory has allowed to state the most common alternative ways of representing preferences and technologies, such as indirect utility and expenditure functions, cost and distance functions, and so forth. Having different ways to describe a technology seems very suitable since some types of mathematical arguments are easier to demonstrate by using, for example, a cost function instead of a distance function, that is, a direct representation of the technology [1, page 81]. (A firm produces outputs from a set of inputs. In order to analyze firm choices, it is necessary a convenient way to summarize the production possibilities of the firm, that is, which combinations (vectors) of inputs and outputs are feasible. A technology is the set of all these feasible combinations [1].) Both the cost function and the distance function are, by definition, optimization problems. The theory of duality studies under which conditions these two optimization problems are related.

Distance functions are natural representations of multiple-output and multiple-input technologies. Shephard [5] was the first to define a distance function in a production context. In particular, the Shephard input distance function measures the largest radial contraction of an input vector consistent with remaining technically feasible. Additionally, the Shephard input distance function is of common use in production theory and it also has a dual relation to the cost function. Nevertheless, the Shephard input distance function is not the unique known distance function in the literature.

During the last two decades, Luenberger [6], Chambers et al. [7], Briec and Lesourd [8] and Briec and Gardères [9] have proposed some new achievements with respect to duality theory and distance functions. To be precise, Luenberger [6] introduced the concept of benefit function as a representation of the amount that a consumer is willing to trade, in terms of a specific reference commodity bundle g, for moving from a utility threshold to a consumption bundle. Later, Chambers et al. [7] redefined the benefit function as an inefficiency measure, introducing the notion of directional input distance function allowing the move from consumer theory to production theory. Chambers et al. showed how the directional input distance function encompasses, among others, the Shephard input distance function. They derived also a dual correspondence between the directional input distance function and the cost function that generalizes all previous dual relationships. Subsequently, Briec and Lesourd [8] introduced the notion of Hölder metric distance function, and additionally an input oriented version, relating the concept of inefficiency and the concept of distance in topology. Following the work by Chambers et al. [7], Briec and Lesourd [8, pages 30–32] derived a duality result based upon the cost function and the input Hölder metric distance function. Another interesting related recent paper is Briec and Gardères [9], who tried to generalize Luenberger's benefit function in the context of consumer theory. Their generalized benefit function is intimately related to topological norms, which somehow represents a drawback. In fact, they cannot encompass the case of the benefit function when the reference vector g has some zero components. (Given two vectors $z,\ g \in R^n_+$, we define $\|z\|_g := z^T g$. It is straightforward to prove that if each component of g is strictly positive, then $\|z\|_g$ is a norm. However, if vector g has some zero components, then it is easy to find a numerical example where "$\|z\|_g = 0$ if and only if z is the null vector", a basic property of any norm (see [10]), does not hold.)

Since there are different families of distance functions in the literature, this paper is devoted to encompass all these alternatives in a unique general family. Additionally, we are interested in enlarging the measurement possibilities provided by the distance functions which currently exist. To achieve these aims, we introduce in this paper a general input distance function and study its mathematical properties. In this respect, we will focus the analysis on production theory and considering a model of cost-minimizing behaviour as in Chambers et al. [7].

One precursor of this paper is the work by Debreu [11]. Debreu introduced a well-known radial efficiency measure termed "coefficient of resource utilization." Nevertheless, he derived this coefficient from a much less well-known "dead loss function," that characterizes the monetary value sacrificed due to inefficiency; that is, it measures an opportunity cost. (The opportunity cost of an action is what you give up to obtain as a consequence of this action. When making any decision, decision makers should be aware of the opportunity costs that accompany each possible action (see [12]).) The minimization problem originally proposed by Debreu to measure such opportunity cost was $\mathrm{Min}_{z,p_z}\{p_z(z_0-z)\}$, where z_0 is a vector representing the actual allocation of resources, z is a vector belonging to the set of optimal allocations (the isoquant), and p_z is a vector of the corresponding set of shadow price vectors for z (see [11, page 284]). Debreu pointed out that "p_z is affected by an arbitrary positive scalar." The influence of this multiplicative scalar means that the minimum value can be driven to zero by appropriately scaling all components of p_z. In order to eliminate this problem, Debreu proposed to divide the objective function by a price index, reformulating the original problem as $\mathrm{Min}_{z,p_z}\{p_z(z_0 - z)/p_z z_0\}$ or, equivalently, as $\mathrm{Min}_{z,p_z}\{p_z(z_0 - z) : p_z z_0 = 1\}$. Then, Debreu proved that $z^* = \rho \cdot z_0$ is always an optimal solution to the above minimization problem, where the scalar ρ $(0 < \rho \leq 1)$ is known as Debreu's coefficient of resource utilization.

Inspired by the Debreu's work, in this paper, we minimize the dead loss function to evaluate the technical inefficiency of any producer, but considering a wide set of normalization conditions on the shadow prices instead of using a price index. Following this line, we will define the main concept of this work: the general input distance function. This new notion measures the opportunity cost associated to perform inefficiently, since it is based, by definition, on the Debreu's dead loss function. Moreover, there are numerous distance functions in the literature and we will show that the general distance function encompasses all these alternatives. In this sense, we make some order in this complexity.

The paper unfolds as follows. In Section 2, we list the usual requirements that the production set must satisfy and we define a general input distance function. This measure is formulated in a generic way since the normalization restrictions regarding the set of feasible shadow prices are not specifically formulated, allowing us to encompass a large class of normalization constraints. In Section 3, we study the basic properties of the general distance function. Section 4 is devoted to stating that the new distance function can be recovered from the cost function. Additionally, we will show that the general distance function encompasses the Shephard input distance function, the directional input distance function, and the input Hölder metric distance function, among others. In Section 5, we show that the sign of the new distance function is really significant to characterize the feasible allocations. Section 6 shows that Shephard's duality theorem and other similar results are special cases of a more general duality result. Finally, Section 7 concludes.

2. The Input Correspondence and a General Input Distance Function

Let R^n_+ and R^n_{++} denote the nonnegative orthant and the positive orthant in the n-dimensional space, respectively. Let 0_n denote a vector of n zeros. A vector of m inputs is denoted as $x = (x_1, \ldots, x_m) \in R^m_+$, and a vector of s outputs is denoted as $y = (y_1, \ldots, y_s) \in R^s_+$. Furthermore, $p^T x$ denotes the inner product of the vectors p and x. $\|\cdot\|$ denotes an arbitrary norm in R^n_+. Let also $2^{R^m_+}$ denote the class of all subsets of R^m_+.

For each output vector y, we can define $L(y) = \{x \in R^m_+ : x \text{ yields at least y}\}$, the set of feasible inputs. This characterizes the input correspondence $L : R^s_+ \to 2^{R^m_+}$, which maps each y in R^s_+ to an input set, $L(y) \subset R^m_+$, known as the input requirement set.

It is assumed here that the input correspondence L satisfies the following subset of axioms as suggested by Färe et al. [13, pages 23–27]:

(P1) $y \in R^s_+, y \neq 0_s \;\Rightarrow\; 0_m \notin L(y)$, and $L(0_s) = R^m_+$;

(P2) Let $\{y^n\}$ be a sequence such that $\lim_{n\to\infty}\|y^n\| = +\infty$. Then, $\bigcap_n L(y^n) = \emptyset$;

(P3) $x \in L(y), u \geq x \;\Rightarrow\; u \in L(y)$;

(P4) $L(y)$ is closed for all $y \in R^s_+$;

(P5) $L(y)$ is a convex set for all $y \in R^s_+$.

Postulate P1 states that positive output vectors cannot be obtained from a null input vector and any nonnegative input vector yields at least the null output vector. P2 states that finite inputs cannot produce infinite outputs. P3 implies strong disposability of inputs.

P4 is assumed in order to be able to define the boundary of $L(y)$ as a subset of the input requirement set. Postulate P5 states that, given any two input vectors in $L(y)$, the line segment joining them is also in $L(y)$. Although these are the usual general postulates in order to define an input correspondence in the literature, we want to point out that we do not really need P2 to prove the results that appear in this paper.

We assume the usual axioms because we want to generalize the most famous distance functions, which have been developed by assuming all these postulates.

Given $y \in R_+^s$ and $p \in R_+^m$, a vector of input prices, in $L(y)$ the input vector x is chosen so as to minimize costs. Therefore, we assume cost minimizing behaviour throughout the paper. In other words, the firm takes prices and outputs as fixed and selects a feasible input vector belonging to $L(y)$ which minimizes their cost. The resulting (optimum) cost is a function of y and p which we denote here by $C(y,p)$.

Definition 2.1. Given $y \in R_+^s$ and $p \in R_+^m$, the cost function is defined as

$$C(y,p) = \min_x \left\{ p^T x : x \in L(y) \right\}. \tag{2.1}$$

(We assume that (2.1) attains its minimum in the set $L(y)$. There exist several sufficient conditions in the literature which assure such result. For example, Shephard [14, page 223] assumed that the subset of Pareto-efficient points of $L(y)$ is bounded. A special case is when the technology is a polyhedral, as in a Data Envelopment Analysis (DEA) context. In this case, we do not need any assumption since (2.1) always attains its minimum in the set $L(y)$ (see [15, page 130]).)

The price vector p need not be strictly positive in the above definition. When $p_i = 0$ for some $i = 1, \ldots, m$, the corresponding factor of production is a free item of goods or service, situation of minor interest in economics but considered in the last definition. The same can be said regarding the case $L(y) = \emptyset$. Indeed, we focus our attention on a specific set of output vectors: $\operatorname{Dom} L = \{y \in R_+^s : L(y) \neq \emptyset\}$, that is, the set of producible outputs. On the other hand, if $L(y) = \emptyset$, then we define $C(y,p) = +\infty$, as usual. In addition, let us observe that $C(y,p) \geq 0$ since $p \in R_+^m$ and $L(y) \subset R_+^m$.

Obviously, not all input vectors belonging to an input requirement set are technologically efficient. Firms usually want to use the smallest levels of inputs to produce a given output vector. In fact, doing otherwise would be wasteful. In this respect, the measurement of inefficiency is necessary to compare the actual performance with respect to a certain reference set of the input requirement set. We are really referring to the boundary or isoquant of $L(y)$, as defined in what follows.

Definition 2.2. The isoquant of $L(y)$ is defined by

$$\operatorname{Isoq} L(y) = \{x : x \in L(y), \lambda < 1 \implies \lambda x \notin L(y)\}, \quad y \in R_+^s. \tag{2.2}$$

Using Definition 2.2, we say that (x, y) is input-isoquant efficient if x belongs to the isoquant of $L(y)$. Due to the assumed postulates (specially P3), if $y \in \operatorname{Dom} L$ and $p \in R_+^m$, then the optimal solution(s) of problem (2.1) are achieved necessarily at some input-isoquant efficient point of $L(y)$.

Now, we are ready to introduce the general input distance function. This new notion is defined with respect to a given normalization set denoted as NS. Nevertheless, before introducing the aforementioned notion, we need to introduce some notation.

Definition 2.3. Let $\overline{x} \in \text{Isoq}L(y)$. Then, $\overline{p} \in R_+^m \setminus \{0_m\}$ is called a shadow price vector for $\overline{x}$ if and only if $\overline{p}^T\overline{x} \leq \overline{p}^T u$ for all $u \in L(y)$.

By postulates P4 and P5 and applying the separating hyperplane theorem, we know that for each $\overline{x} \in \text{Isoq}L(y)$ there exists at least a shadow price vector $\overline{p} \in R_+^m \setminus \{0_m\}$.

Definition 2.4. Let $(x, y) \in R_+^m \times R_+^s$ be an input-output vector. Let $L(y)$ be an input requirement set satisfying P1–P5. And let also NS be a subset of R_+^m. Then, the function $G_I : R_+^m \times R_+^s \times 2^{R_+^m} \rightarrow [-\infty, +\infty]$ defined as

$$G_I(x, y; \text{NS}) = \inf_{\overline{x}, \overline{p}} \left\{ \overline{p}^T (x - \overline{x}) : \ \overline{x} \in \text{Isoq}L(y), \ \overline{p} \in Q_{\overline{x}} \cap \text{NS} \right\}, \tag{2.3}$$

where $Q_{\overline{x}} = \{p \in R_+^m : p \text{ is a shadow price vector of } \overline{x}\}$, is called the general input distance function.

As a referee points out, the formulation of $G_I(x, y; \text{NS})$ could be easier if we remove the constraint $\overline{x} \in \text{Isoq}L(y)$ in (2.3), deriving an equivalent expression. It is due to the fact that it follows already by the supposition that $\overline{p}$ is a shadow price of $\overline{x}$. However, in order to follow a formulation similar to that used by Debreu [11] to define the dead loss function, we will use (2.3) hereafter.

This general input distance function can be interpreted as a measure of the distance from $x \in R_+^m$ to the boundary of $L(y)$. To get a distance with an economic meaning, we evaluate $(x - \overline{x})$ times the shadow price vector $\overline{p}$, associated with $\overline{x} \in \text{Isoq}L(y)$. In economics terms, the general input distance function represents the monetary value sacrificed due to technical inefficiency. In this sense, the general input distance function could be understood as an opportunity cost for firms.

Obviously, G_I will take different values depending on the structure of the normalization set NS and the vector (x, y). In particular, if $(\bigcup_{\overline{x} \in \text{Isoq}L(y)} Q_{\overline{x}}) \cap \text{NS} = \emptyset$, then we set $L_I(x, y; \text{NS}) = +\infty$, as usual. Nevertheless, from now on, we will assume in our results that $(\bigcup_{\overline{x} \in \text{Isoq}L(y)} Q_{\overline{x}}) \cap \text{NS} \neq \emptyset$, which guarantees $G_I(x, y; \text{NS}) < +\infty$.

The above general input distance function has the same arbitrary multiplicative scalar problem pointed out by Debreu [11]. Debreu showed that "p_z is affected by an arbitrary positive scalar" in his formulation of the dead loss function. The influence of this multiplicative scalar in our formulation means that the minimum value can be driven to zero by appropriately scaling all components of $\overline{p}$. Therefore, the set NS in the definition of G_I should satisfy some property which avoids this problem in practice. To this end, the next condition must hold.

(C1) NS is a closed set and $0_m \notin \text{NS}$; that is, NS is bounded away from zero.

Obviously, any set $\text{NS} \subset R_+^m$ that satisfies C1 avoids the arbitrary multiplicative scalar problem in (2.3). In fact, it is a sufficient, but not necessary, condition. Since the vector 0_m does not belong to the closed set NS, the distance from 0_m to NS is strictly positive and we cannot achieve 0_m scaling any $p \in \text{NS}$.

Another regularity condition on the set NS, which will be used below in some results, states that NS must contain at least a "representative" of each ray that belongs to the cone R_+^m.

(C2) for all $p \in R_+^m \setminus \{0_m\}$ there exists $k > 0$ such that $kp \in \mathrm{NS}$.

3. Properties of the General Input Distance Function

This section shows the main properties of the general input distance function. It satisfies the weak monotonicity condition with respect to the inputs and is a continuous concave function as well as one-sided directional differentiable.

Proposition 3.1. *Let $(x, y) \in R_+^m \times R_+^s$ and $u \in R_+^m$. Let $L(y)$ be an input requirement set that satisfies P1–P5. Let NS be a subset of R_+^m that satisfies C1. Then, the function G_I satisfies the following properties:*

(a) $u \geq x \Rightarrow G_I(u, y; \mathrm{NS}) \geq \mathrm{G_I(x, y; NS)}$;

(b) $G_I(x, y; \mathrm{NS})$ *is concave in* x;

(c) $G_I(x, y; \mathrm{NS})$ *is continuous with respect to x on each open convex subset of R_+^m in which the general input distance function is finite;*

(d) *let $x \in R_+^m$ be a vector such that $G_I(x, y; \mathrm{NS})$ is finite. For each $u \in R_+^m$, there exists the one-sided directional derivative of the general input distance function at x with respect to the vector u.*

Proof. (a)$u \geq x \Rightarrow u - \overline{x} \geq x - \overline{x}$, for all $\overline{x} \in R_+^m$. Then, we have that $\overline{p}^T(u - \overline{x}) \geq \overline{p}^T(x - \overline{x})$, for all $\overline{x} \in \mathrm{Isoq}L(y)$, for all $\overline{p} \in Q_{\overline{x}} \cap \mathrm{NS}$. Therefore, $G_I(u, y; \mathrm{NS}) \geq G_I(x, y; \mathrm{NS})$.

(b) Let $x^1, x^2 \in R_+^m$ and $\lambda \in [0, 1]$. It holds that

$$\overline{p}^T\left(\lambda x^1 + (1-\lambda)x^2 - \overline{x}\right) = \lambda\left[\overline{p}^T\left(x^1 - \overline{x}\right)\right] + (1-\lambda)\left[\overline{p}^T\left(x^2 - \overline{x}\right)\right], \forall \overline{x} \in \mathrm{Isoq}L(y), \quad \forall \overline{p} \in Q_{\overline{x}} \cap \mathrm{NS} \tag{3.1}$$

since $\overline{x} = \lambda\overline{x} + (1-\lambda)\overline{x}$. Then, since $\overline{p}^T(x^1 - \overline{x}) \geq G_I(x^1, y; \mathrm{NS})$ and $\overline{p}^T(x^2 - \overline{x}) \geq G_I(x^2, y; \mathrm{NS})$, for all $\overline{x} \in \mathrm{Isoq}L(y)$ and for all $\overline{p} \in Q_{\overline{x}} \cap \mathrm{NS}$, we have that

$$\begin{aligned} G_I\left(\lambda x^1 + (1-\lambda)x^2, y; \mathrm{NS}\right) &= \inf_{\overline{x}, \overline{p}}\left\{\overline{p}^T\left(\left(\lambda x^1 + (1-\lambda)x^2\right) - \overline{x}\right) : \overline{x} \in \mathrm{Isoq}L(y), \overline{p} \in Q_{\overline{x}} \cap \mathrm{NS}\right\} \\ &\geq \lambda G_I\left(x^1, y; \mathrm{NS}\right) + (1-\lambda)G_I\left(x^2, y; \mathrm{NS}\right). \end{aligned} \tag{3.2}$$

(c) and (d) The concavity of the general input distance function implies directly these properties (see [15, page 62 for (c)] and [16, page 214 for (d)]). This concludes the proof. $\square$

4. How to Recover the General Input Distance Function from the Cost Function

Shephard [5, 14] developed a great deal of duality theory. In particular, he established a duality relationship between the cost function and the Shephard input distance function. In this section, we prove that the general input distance function can be recovered from the cost function, establishing a new duality result. Later, we will also show that the general input distance function encompasses a wide family of well-known distance functions.

Theorem 4.1. *Let* $(x, y) \in R_+^m \times \operatorname{Dom} L$. *Let* $L(y)$ *be an input requirement set that satisfies P1–P5. Let* NS *be a subset of* R_+^m *that satisfies C1. Then,*

$$G_I(x, y; \mathrm{NS}) = \inf_{p \in R_+^m} \left\{ p^T x - C(y, p) : p \in \mathrm{NS} \right\}. \tag{4.1}$$

Proof. For any $\overline{x} \in \operatorname{Isoq} L(y)$ and $\overline{p} \in Q_{\overline{x}} \cap \mathrm{NS}$, we have that $\overline{p}^T(x - \overline{x}) = \overline{p}^T x - \overline{p}^T \overline{x} = \overline{p}^T x - C(y, \overline{p})$, where the last equality holds because $\overline{p}$ is a shadow price vector of $\overline{x}$. In addition, $(\bigcup_{\overline{x} \in \operatorname{Isoq} L(y)} Q_{\overline{x}}) \cap \mathrm{NS} \subset \mathrm{NS}$. Therefore, it is apparent that $\inf_{p \in R_+^m}\{p^T x - C(y, p) : p \in \mathrm{NS}\} \leq G_I(x, y; \mathrm{NS})$.

To prove the other inequality, since $\overline{p} \in R_+^m$, we have that there exists $\overline{x} \in \operatorname{Isoq} L(y)$ such that $C(y, \overline{p}) = \overline{p}^T \overline{x}$. In other words, $\overline{p} \in Q_{\overline{x}} \cap \mathrm{NS}$. Additionally, we have that $\overline{p}^T x - C(y, \overline{p}) = \overline{p}^T(x - \overline{x})$. Then, $\overline{p}^T(x - \overline{x}) \geq G_I(x, y; \mathrm{NS})$. Finally, by the definition of infimum, we have that $\inf_p\{p^T x - C(y, p) : p \in \mathrm{NS}\} \geq G_I(x, y; \mathrm{NS})$. □

Theorem 4.1 states a first duality result: the general input distance function can be recovered from the cost function. The main implication of this result is that several interesting specific normalization sets satisfy their hypothesis. Hence, it will allow us to derive important relations between the general input distance function and some well-known distance functions.

Briec and Lesourd [8, page 31] defined the input Hölder metric distance function (it is denoted here as $D_T^{t,i}(x, y)$, where ℓ_q is the dual space of ℓ_t with $1/t + 1/q = 1$) and related it to the cost function. This distance function is a special case of the general input distance function, introduced in this paper, as we show next. If we consider as normalization set $\mathrm{NS} = \{p \in R_+^m : \|p\|_q \geq 1\}$, then we get, by Theorem 4.1, $G_I(x, y; \mathrm{NS}) = \inf_p\{p^T x - C(y, p) : \|p\|_q \geq 1\} = D_T^{t,i}(x, y)$, where the last equality holds thanks to Proposition 4.1 in Briec and Lesourd [8]. The conclusion is that G_I encompasses the input Hölder metric distance function.

Other special case of the general input distance function which is the interest is the directional input distance function [7]. In particular, when we consider $\mathrm{NS} = \{p \in R_+^m : p^T g = 1\}$, with $g \in R_+^m$ a nonzero vector, we get $G_I(x, y; \mathrm{NS}) = \inf_p\{p^T x - C(y, p) : p^T g = 1\}$ thanks to Theorem 4.1. Chambers et al. [7, page 413] proved that the directional input distance function (it is denoted here as $\vec{D}_i(x, y; g)$) can be recovered from the cost function by means of $\inf_p\{p^T x - C(y, p) : p^T g = 1\}$. As a direct consequence, we get that $G_I(x, y; \mathrm{NS}) = \vec{D}_i(x, y; g)$.

On the other hand, it is well known that the directional input distance function generalizes the Shephard input distance function (see [7, page 411]) when $g = x$. In this sense and thanks to the above paragraph, the general input distance function encompasses the Shephard input distance function as well as other families of distance functions.

5. Characterization of the Input Requirement Set

One of the main uses of the distance functions is to characterize whether an input vector $x \in R_+^m$ belongs or not to $L(y)$, assuming that y is producible. Next we characterize the input requirement set by means of the value of the general input distance function. The proof is based on the next two lemmas, which need different hypothesis.

Lemma 5.1. *Let $(x, y) \in R_+^m \times \operatorname{Dom} L(y)$. Let $L(y)$ be an input requirement set that satisfies P1–P5. Let* NS *be a subset of R_+^m. Then, if $x \in L(y)$, then one has that $G_I(x, y; \mathrm{NS}) \geq 0$.*

Proof. for all $\overline{x} \in \operatorname{Isoq}L(y)$ and for all $\overline{p} \in Q_{\overline{x}} \cap \mathrm{NS}$, we have, by definition of shadow price vector, $\overline{p}^T\overline{x} = C(y, \overline{p})$. In addition, we know that $\overline{p}^T x \geq C(y, \overline{p})$ since $x \in L(y)$. Therefore, $\overline{p}^T(x - \overline{x}) = \overline{p}^T x - \overline{p}^T\overline{x} \geq 0$ for all $\overline{x} \in \operatorname{Isoq}L(y)$ and for all $\overline{p} \in Q_{\overline{x}} \cap \mathrm{NS}$. Finally, by (2.3), $G_I(x, y; \mathrm{NS}) \geq 0$. □

Lemma 5.2. *Let $(x, y) \in R_+^m \times \operatorname{Dom} L$. Let $L(y)$ be an input requirement set that satisfies P1–P5. Let* NS *be a subset of R_+^m that satisfies C1 and C2. Then, if $x \notin L(y)$, then one has that $G_I(x, y; \mathrm{NS}) < 0$.*

Proof. Applying the separation of a convex set and a point theorem, we get that there exists $p \in R_+^m \setminus 0_m$ such that $p^T x < C(y, p)$. By C2 there exists $k > 0$ such that $\overline{p} = kp \in \mathrm{NS}$. Moreover, there exists $\overline{x} \in L(y)$ (in fact, $\overline{x} \in \operatorname{Isoq}L(y)$) which satisfies $p^T\overline{x} = C(y, p)$, that is, $\overline{p} \in Q_{\overline{x}}$. Hence, $\overline{p}^T(x - \overline{x}) = k[p^T x - p^T\overline{x}] < 0$. Finally, $G_I(x, y; \mathrm{NS}) < 0$ since we are seeking an infimum in (2.3). □

The following result is derived from the last two lemmas. As can be seen, the sign of the general input distance function allows us to characterize the input requirement set.

Proposition 5.3. *Let $(x, y) \in R_+^m \times \operatorname{Dom} L$. Let $L(y)$ be an input correspondence that satisfies P1–P5. Let* NS *be a subset of R_+^m that satisfies C1 and C2. Then, $x \in L(y)$ if and only if $G_I(x, y; \mathrm{NS}) \geq 0$.*

Finally, the following proposition states the intuitive result that $G_I(x, y; \mathrm{NS}) = 0$ is a necessary condition for input-isoquant efficiency.

Proposition 5.4. *Let $(x, y) \in R_+^m \times \operatorname{Dom} L$. Under the same hypothesis of Proposition 5.3, if (x, y) is input-isoquant efficient, then $G_I(x, y; \mathrm{NS}) = 0$.*

Proof. Let $(x, y) \in R_+^m \times \operatorname{Dom} L$ be an input-isoquant efficient point of the input requirement set $L(y)$. Then, it must exist $p \in R_+^m \setminus \{0_m\}$ such that $p \in Q_x$. Then, by C2, there exists $k > 0$ such that $kp \in \mathrm{NS}$ and, consequently, $kp \in Q_x \cap \mathrm{NS}$. This implies that $(kp)^T(x - x) = 0 \geq G_I(x, y; \mathrm{NS})$, by the definition of the general input distance function. Finally, $G_I(x, y; \mathrm{NS}) \geq 0$ since Lemma 5.1 holds and, necessarily, $G_I(x, y; \mathrm{NS}) = 0$. □

6. How to Recover the Cost Function from the General Input Distance Function

The dual "general" relationship (4.1) says that the general input distance function can be recovered from the cost function. This section is devoted to study the converse result, that is, how to derive the cost function from the general input distance function. Additionally, we will show that this result generalizes all previous dual connections between distance functions and the cost function.

Lemma 6.1. *Let* $(x, y) \in R_+^m \times \operatorname{Dom} L$. *Let* $L(y)$ *be an input requirement set that satisfies P1–P5. Let* NS *be a subset of* R_+^m *that satisfies C1. Let also* $p \in \mathrm{NS}$. *Then,*

$$C(y,p) = \inf_{x \in R_+^m} \left\{ p^T x - \; G_I(x, y; \mathrm{NS}) \right\}. \tag{6.1}$$

Proof. By Theorem 4.1, we have that $G_I(x, y; \mathrm{NS}) \leq p^T x - C(y,p)$, for all $x \in R_+^m$. It implies that $C(y,p) \leq p^T x - G_I(x, y; \mathrm{NS})$, for all $x \in R_+^m$. Hence,

$$C(y,p) \leq \inf_{x \in R_+^m} \left\{ p^T x - G_I(x, y; \mathrm{NS}) \right\}. \tag{6.2}$$

To prove the converse inequality, given any $\varepsilon > 0$ there exists $x \in L(y)$ such that $p^T x \leq C(y,p) + \varepsilon$ since $C(y,p) > -\infty$ (in fact, $C(y,p) \geq 0$). Since we know that $G_I(x, y; \mathrm{NS}) \geq 0$ by Lemma 5.1, $p^T x - G_I(x, y; \mathrm{NS}) \leq C(y,p) + \varepsilon$. Now, through the definition of ε, $\inf_{x \in L(y)} \{p^T x - G_I(x, y; \mathrm{NS})\} \leq C(y,p)$. Finally, since $L(y) \subset R_+^m$, we have that $\inf_{x \in R_+^m} \{p^T x - G_I(x, y; \mathrm{NS})\} \leq C(y,p)$. □

When a price vector p does not belong to the normalization set NS, the above result is true if there exists at least a representative vector in the ray associated with p which belongs to NS. This idea is formalized by the following theorem.

Theorem 6.2. *Let* $(x, y) \in R_+^m \times \operatorname{Dom} L$. *Let* $L(y)$ *be an input requirement set that satisfies P1–P5. Let* NS *be a subset of* R_+^m *that satisfies C1. Let also* $p \in R_+^m$ *be such that there exists* $k > 0$ *with* $kp \in \mathrm{NS}$. *Then,*

$$C(y,p) = \inf_{x \in R_+^m} \left\{ p^T x - k^{-1} G_I(x, y; \mathrm{NS}) \right\}. \tag{6.3}$$

Proof. Since $kp \in \mathrm{NS}$, by Lemma 6.1, $C(y, kp) = \inf_{x \in R_+^m} \{(kp)^T x - G_I(x, y; \mathrm{NS})\}$. By homogeneity of degree +1 of the cost function, $C(y, kp) = kC(y,p)$. It implies that

$$\begin{aligned} C(y,p) &= k^{-1} C(y, kp) \\ &= k^{-1} \inf_{x \in R_+^m} \left\{ (kp)^T x - G_I(x, y; \mathrm{NS}) \right\} \\ &= \inf_{x \in R_+^m} \left\{ p^T x - k^{-1} G_I(x, y; \mathrm{NS}) \right\}. \end{aligned} \tag{6.4}$$

□

Next, we apply Theorem 6.2 in order to get the well-known dual connections that allow to recover the cost function from a specific distance function, as stated by Shephard [5, 14], Chambers et al. [7], and Briec and Lesourd [8].

First, we study the case of the input Hölder metric distance function of Briec and Lesourd [8]. Given any $p \in R_+^m \backslash \{0_m\}$, $k := \|p\|_q^{-1} > 0$ yields that $kp \in \mathrm{NS} = \{p \in R_+^m : \|p\|_q \geq 1\}$. As a consequence, we can apply Theorem 6.2 and that $G_I(x, y; \mathrm{NS}) = D_T^{t,i}(x, y)$ (see Section 4) to get $C(y,p) = \inf_{x \in R_+^m} \{p^T x - \|p\|_q D_T^{t,i}(x, y)\}$. This result is similar, but not identical, to

Proposition 4.1 (a) due to Briec and Lesourd [8, page 31]. ($C(y,p) = \inf_x\{p^T x + D_T^{t,i}(x,y) : x \in L(y)\}$ in the paper of Briec and Lesourd [8]) Briec and Lesourd need to know $D_T^{t,i}(x,y)$ and $L(y)$ to derive the cost function of the firm. Unlike them, we only need to know $D_T^{t,i}(x,y)$ to recover the same cost function. In this sense, our result seems closer to the spirit of the duality in microeconomics.

In the case of the directional input distance function of Chambers et al. [7], taking into account that given any $p \in R_{++}^m$ and $g \in R_+^m \setminus \{0_m\}$, $k := (p^T g)^{-1}$ yields that $kp \in \mathrm{NS} = \{p \in R_+^m : p^T g = 1\}$, we have that $C(y,p) = \inf_{x\in R_+^m}\{p^T x - p^T g\vec{D}_i(x,y;g)\}$ by applying directly Theorem 6.2 and realizing that $G_I(x,y;\mathrm{NS}) = \vec{D}_i(x,y;g)$ (see Section 4). This derived result was first stated in Chambers et al. [7, page 413]. Hence, the dual correspondence developed by Chambers et al. between the cost function and the directional input distance function is a particular case of the general duality result (6.3).

Finally, taking $g = x$, it is possible to deduce Shephard's duality result ($D_i(x,y)$ denotes here the Shephard input distance function) $C(y,p) = \inf_{x\in R_+^m}\{p^T x/D_i(x,y)\}$ [14, Chapter 8] from $C(y,p) = \inf_{x\in R_+^m}\{p^T x - p^T g\vec{D}_i(x,y;g)\}$, as Chambers et al. [7] showed in their paper. Therefore, Theorem 6.2 encompasses Shephard's dual correspondence as well.

7. Conclusions

This paper has introduced a general input distance function and has shown that it encompasses other existing distance functions: the Shephard input distance function, the directional input distance function, and the input Hölder metric distance function. After developing its properties, we have outlined a series of general duality results that represent a generalization of the well-known relations between the cost function and different distance functions.

Thanks to the general distance function and since all the revised input distance functions have the same structure, it would be possible to study globally the achievement of certain properties, for example, units invariance. Additionally, we could derive generic economic relationships from the general input distance function (e.g., a general Fenchel-Mahler inequality for measuring profit inefficiency; see for more details [17, 18]).

As a byproduct, all our results also hold in a Data Envelopment Analysis (DEA) context, since this type of polyhedral technologies satisfy the postulates that we assume. (A polyhedral technology is a technology such that if it is represented in the space of m inputs and s outputs, then it is a polyhedron.) Hence, the input-oriented BCC and the input-oriented CCR DEA models collapse in to the general input distance function (see for more details [19]).

Flexibility is one of the features of the proposed general approach. In this respect, simply by varying the normalization set in the general framework we are able to derive new input distance functions in terms of opportunity costs. Nevertheless, the selection of a specific normalization condition deserves further research.

An additional potential extension of this paper is to focus on consumer theory instead of production theory and try to generalize the Luenberger benefit function [6]. So, we believe that this line may be a good avenue for further follow-up research.

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