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Classroom Note

NUMERICAL SOLUTIONS OF NAGUMO'S EQUATION

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Abstract. Nagumo's equation is a third order non-linear ordinary differential equation $\frac{d^3u}{dx^3} - c\frac{d^2u}{dx^2} + f'(u)\frac{du}{dx} - (b/c)\,u = 0$ where $f(u) = u(1-u)(u-a)$, $0 < a < 1$. In this paper we have developed a technique to determine those values of the parameters a, b and c which permit non-constant bounded solutions.

1. INTRODUCTION

Hodgkin and Huxley [11] in their fundamental work on pulses in a squid axon were the first to give a mathematical description of this process. Their model was based on a concept derived from 'Kelvin's Cable Theory' that the nerve membrane is effectively an inductance-free line with a constant capacitance and a non-linear current flow element.

Later a simplified model for the process was proposed by Nagumo, A. Rimoto and Yoshizawa [16] to obtain the non-constant bounded solutions for the third order non-linear ordinary differential equation

$$\frac{d^3u}{dx^3} - c\frac{d^2u}{dx^2} + f'(u)\frac{du}{dx} - (b/c)\,u = 0 \tag{1}$$

where $f(u) = u(1-u)(u-a)$, $0 < a < 1$, and f is a cubic function of u and b is a positive constant, c is the speed of the travelling wave $u = u(x + ct)$. H. Cohen [13], J. Cooley and F. Dodge [4, 5], Green [9], Hagstrom [9] and R. Knight [14] have compiled extensive numerical results for a speed diagram. Many other authors Rinzel [17], Fitzhugh [6] and McKean [15] reviewed the subject for $0 < a < 1$, $b \leq 0$ and $c \geq 0$.

A natural tool for the mathematical simulation of such processes and phenomena is the theory of impulsive differential equations. At first this theory developed slowly. In the last decade, however, a considerable increase in the number of publications has been observed in various branches of the theory of impulsive differential equations such as Ciment [2].

2. THE ORIGIN, DEVELOPMENT AND SIGNIFICANCE OF NAGU EQUATION

The nervous system consists of nodal points (cell, soma and dendrites), lines (axons) and termini (receptors). The best known aspect of the nervous system is the conduction of the impulse along a single axon. The source of conduction is a pulse which is either rapidly damped out or is shaped into a characteristic form which then propagates down the axon without distortion like a traveling wave.

Hodgkin and Huxley [11, 12, 13], in their fundamental work on pulses in a squid axon were the first to give a successful mathematical description of the process. The Nagumo model is essentially an initial value problem for the following non-linear partial differential equation in the quarter plane $\{u(x,t)|x \geq 0,\ \ t \geq 0\}$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f - b\int u\,dt \qquad \text{where } f = u(1-u)(u-a), \qquad 0 < a < 1, \quad (2.1)$$

in which $u(x,t)$ is effectively the axon potential and a and b are physiological parameters.

Further to electronically simulate an animal nerve axon, Nagumo and others [16] made an active pulse transmission line using tunnel diodes. This line shapes the signal wave form during transmission, smaller signals are amplified, larger ones are attenuated, narrower ones are widened and those which are wider are shrunk, all approaching the above mentioned wave form.

Differentiating (2.1) with respect to t, we obtain the partial differential equation

$$\frac{\partial 3u}{\partial t\partial x^2} - c\frac{\partial^2 u}{\partial t^2} + f'(u)\frac{\partial u}{\partial t} - (b/c)u = 0. \quad (2.2)$$

Now if we look for traveling wave solutions $u = u(x,t)$, then on substitution in (2.2) we get the third order O.D.E.

$$\frac{d^3u}{dx^3} - c\frac{d^2u}{dx^2} + f'(u)\frac{du}{dx} - (b/c)u = 0. \quad (2.3)$$

The parameters a, b and c are to be determined which permit non-constant bounded solutions of (2.3).

3. METHOD OF SOLUTION.

We are dealing with two cases:

(a) When $b = 0$ then (2.3) reduces to the second order differential equation

$$\frac{d^2u}{dx^2} - c\frac{du}{dx} + f = 0. \quad (3.4)$$

(b) When $b > 0$, then the differential equation (2.3) is reduced to a system of simultaneous equations of the first order by introducing the new variables

$$u' = \frac{du}{dx} = v \qquad \text{and} \qquad u'' = \frac{d^2u}{dx^2} = w.$$

In vector notations, the system is written as

$$'\vec{u} = \vec{g}(\vec{u}) \qquad \text{where } '\vec{u} = \vec{g}(\vec{u}$$

$$'\vec{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad \text{and}$$

$$\vec{g}(\vec{u}) = \begin{bmatrix} v \\ w \\ (b/c)u + (3u^2 - 2(1+a)u + a) + cw \end{bmatrix}$$

In solving (3.2) we employed Hamming's [10] predictor corrector method, with Runge–Kutta quartic method to obtain three starting values (points) on the solution curve in addition to the initial point.

4. INITIAL VALUES.

a) For $b = 0$. The basic requirement now is to obtain initial conditions to get the numerical procedure started. Near $u = 0$, the linearized equation is

$$\frac{d^2u}{dx^2} - c\frac{du}{dx} - au = 0. \tag{4.1}$$

The auxiliary equation of (4.1) is

$$m^2 - cm - a = 0 \tag{4.2}$$

and has roots

$$m_1 = \frac{c + \sqrt{c^2 + 4a}}{2a}, \quad m_2 = \frac{c - \sqrt{c^2 + 4a}}{2a}$$

since parameters a and c are positive, so there is only one positive root $m_1 = \left(\frac{c+\sqrt{b^2+4a}}{2}\right)$. Therefore, the solution $u(x) \simeq Ae^{m_1 x}$ and $u'(x) \simeq Am_1 e^{m_1 x} = m_1 u$ for large negative values of x, $u \simeq 0$ and A is constant. Thus the initial conditions used are $u(0) = h$, $v(0) = u'(0) = m_1 h$ where h is a small step size used in the numerical solution and m_1 is the only positive root of (4.2). We used the step size h as $h = 0.001$ in our calculations.

b) Initial Conditions for $b > 0$. Near $u = 0$ the linearized equation is

$$u''' - cu'' - au' - (b/c) = 0. \tag{4.3}$$

The auxiliary equation is

$$m^3 - cm^2 - am - (b/c) = 0. \tag{4.4}$$

Now to obtain the desired behavior of the solution $u(x)$ as $x \to -\infty$, we require that the roots of the cubic (4.4) should be real. Further we also require that only one of the three roots should be positive. These considerations imply that $b < a^2/4$ and c exceeds the largest positive root of the equation

$$(a^2 - 4b)c^4 + 2a(2a^2 - 9b)c^2 - 27b^2 = 0 \qquad \text{Burnside [1].}$$

Thus if b and c are chosen satisfying these conditions and m_1 is the only positive root of the auxiliary equation (4.4), then for large negative values of x we must have

$$\begin{aligned} u &\simeq Ae^{m_1 x} \quad \text{for some constant } A \\ u' &\simeq Am_1 e^{m_1 x} = m_1 u \\ u'' &\simeq Am_1^2 e^{m_1 x} = m_1^2 u \end{aligned}$$

The initial conditions are

$$\left.\begin{aligned} u &= h \\ v &= m_1 h \\ w &= m_1^2 h \end{aligned}\right\} \tag{4.5}$$

where h is the small step size used in the subsequent computations. The value of m_1 was found for the different values of the parameters used by GRAEFFE's ROOT SQUARING procedure (see Froberg [7]).

CONCLUSION.

Our method worked very well and our numerical results supported the conjecture proposed by H.P. McKean [15], that for the value of the parameter 'a' > 0.5 and 'b' > 0, no non-constant bounded solution of (4.3) exists.

The underlying idea is that parameter 'a' plays the role of a doping parameter and the disappearance of the non-constant bounded solution corresponds to the physical fact, that if too much of novocaine is injected, the whole nerve goes dead.

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