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Flexible Plate and Foundation Modelling

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Abstract. In the most common mathematical model for a moving load on a continuously-supported flexible plate, the plate is assumed thin and elastic. An exception is the inclusion of viscoelasticity in the theory for the response of a floating ice plate, where the deflexion at the critical load speed corresponding to the minimum phase speed of hybrid flexural-gravity waves consequently approaches a steady state. This is in contrast to the elastic theory, where the response is predicted to grow continuously at this critical load speed. In the theory for a floating ice plate, the dominant pressure due to the underlying water is inertial, introduced via a velocity potential and the Bernoulli equation (assuming non-cavitation at the plate-water interface). On the other hand, the classical Winkler representation used in early railway engineering analysis corresponds to retaining a term which is generally negligible in the ice plate context. Critical load speeds are consequently predicted to be much higher, at wavelengths correspondingly much lower, for commonly accepted railway engineering parameters. Other models might be considered.

Keywords: Flexible Plate, Foundations, Transport Systems

1. Introduction

Much of the theory published to date on the response of a continuously-supported flexible plate to a moving load assumes the plate is elastic. In several areas of application, a thin elastic plate model has produced significant results, including elucidation of the wave forms which may be generated and the character of critical load speeds at which the response is most pronounced. My introduction to the field occurred in New Zealand, where a paper was written jointly with a Masters student and Alfred Sneyd at the University of Waikato, inter alia describing for the first time how two-dimensional wave patterns vary with the load speed (Davys *et al.* [2]). It is a privilege to contribute this present paper to the symposium in celebration of the 70th birthday of Alex McNabb, another of my eminent New Zealand mathematical friends.

An exception to the usual elastic theory is a two-parameter Boltzmann delay integral approach to incorporate viscoelasticity in the differential equation for the thin plate (Hosking *et al.* [5], Squire *et al.* [13]). The validity of this preferred viscoelastic model, in describing the flexural response of ice plates subjected to moving loads, is discussed in Squire *et al.* [13] – cf. section 2.5 on relevant mechanical properties of ice, and section 5.8 on the question of plate thickness, in particular. Viscoelastic dissipation produces an asymmetric steady state response at the critical load speed, namely the minimum phase speed c_{min} of flexural-gravity waves generated in the floating plate. In addition, by including viscoelasticity the response is rendered finite at that critical load speed, the shorter leading waves are generally more severely damped than the longer trailing waves, the maximum plate depression lags behind the load, and the wave pattern is "swept back" to some extent – all phenomena identified in extensive observations (Squire *et al.* [12], [13]; Takizawa [14], [15]). A summary of this Boltzmann integral approach is the first topic of this paper.

It is also notable that all of the published time-dependent theory assumes the floating ice plate is elastic. Schulkes & Sneyd [10] reviewed the pioneering analysis of Kheysin [8], and showed that for the one-dimensional response due to an impulsively-started steadily moving concentrated line load on a floating plate there are two load speeds at which the deflexion continuously grows with time. Recently, Nugroho *et al.* [9] developed the analogous time-dependent theory for the *two-dimensional* response to either a concentrated point or a distributed (uniform circular) load, which predicts that there is continuous growth only at the critical load speed c_{min}, whether the load is concentrated or not. Physically, in two dimensions energy can radiate away in directions other than the line of motion of the load. Viscoelastic time-dependent theory for the response of a thin floating flexible plate is the subject of a new investigation (Wang [19]), and a few aspects are mentioned here.

The thorough experimental verification of the theory for a load moving over floating ice means one can be confident about its application to transport systems in cold regions, from conventional vehicles to landing aircraft and hovercraft used as ice-breakers (Squire *et al.* [13]). The underlying foundation in many land-based transport systems is of course not water, but nevertheless there may be similar phenomena such as a pronounced response at some particular load speed. Indeed, a resonant response at the critical speed corresponding to the minimum phase speed of flexural waves is predicted with the simple classical Winkler [20] representation for a continuous elastic foundation, an early model adopted in railway engineering (Timoshenko & Langer [17]). The longitudinal ladder sleeper, presently being trialled for use in future railways by engineers in Japan, is briefly discussed towards the end of this paper.

2. The Differential Equation for a Thin Viscoelastic Plate

The small vertical deflexion $\eta(x,y,t)$ of a thin viscoelastic plate of thickness h and density ρ_0, due to a forcing function $f(x,y,t)$ representing a moving load, is

$$D\nabla^4\left(\eta(x,y,t)-\int_0^\infty \Psi(\tau)\eta(x,y,t-\tau)d\tau\right)+\rho_0 h\eta_{tt}=p-f(x,y,t) \qquad (1)$$

where $\Psi(t)$ is the viscoelastic memory function, p is the underlying pressure at $z=0$ due to the reaction of the foundation, and constant D is the effective flexural rigidity of the plate (Hosking *et al.* [5]; Squire *et al.* [13]).

If Ψ satisfies the fading memory hypothesis of Coleman and Noll [1], its general form is a finite sum of exponentials (Graffi [4])

$$\Psi(t) = \sum_{j=0}^{n} A_j e^{-\alpha_j t},$$

where $\alpha_j > 0$ such that Ψ tends to 0 monotonically as $t \to \infty$ and $A_j > 0$ to ensure positive energy dissipation (Hosking *et al.* [5]). The simplest possible memory function corresponds to $n = 0$, when there are two viscoelastic parameters A_0 and α_0 – which is a suitable descriptor of cyclic viscoelastic behaviour in an ice plate under dynamic loading (Squire *et al.* [13]), as mentioned in the Introduction. This model can be viewed as a spring of modulus E in series with a Voigt unit, consisting of a second spring of modulus $E(\alpha_0/A_0 - 1)$ in parallel with a dashpot of viscosity E/A_0 (Flugge [3]). The relation $A_0 \leq \alpha_0$ is required for the modulus of the spring in the Voigt unit to be positive (Squire *et al.* [13]).

3. A Floating Viscoelastic Plate

In the case where the underlying medium is water of finite depth H, so its motion is described by a velocity potential, on the assumption that there is no cavitation at the plate-water interface the time-dependent Fourier integral for the deflexion obtained from (1) is (Wang [19])

$$\eta(x,y,t) = -\frac{1}{(2\pi)^{\frac{3}{2}}\rho} \iiint \frac{g(k,\omega)\hat{f}(k_1,k_2,\omega)e^{-i(k_1x+k_2y-\omega t)}}{W(k,\omega)} d\omega dk_1 dk_2, \qquad (2)$$

where

$$g(k,\omega) = (\alpha_0 + i\omega)k\tanh(kH) \text{ and } W(k,\omega) = \omega^3 + ip\omega^2 + q\omega + ir, \text{ with}$$
$$p = -\alpha_0, \quad q = -(Dk^4 + \rho g)k\tanh(kH)/\rho,$$
$$\text{and} \quad r = [(Dk^4 + \rho g)\alpha_0 - Dk^4 A_0]k\tanh(kH)/\rho.$$

Here the contribution from the plate acceleration term $\rho_0 h \eta_{tt}$ in (1) has been omitted, on the assumption that the horizontal wavelength of the deflexion is much larger than the plate thickness ($|kh| << 1$).

Both q and r are real functions even in k (and p is constant), so the three roots ω of the equation $W(k,\omega) = 0$ are even functions in k. For any given wave number $k \neq 0$, two of these are complex roots symmetric about the imaginary axis, and the third root is pure imaginary.

When the viscoelastic parameter A_0 is zero, cancellation of the factor $\alpha_0 + i\omega$ leaves

$$\frac{g(k,\omega)}{W(k,\omega)} = \frac{ik\tanh(kH)}{\omega^2 + q} \qquad (3)$$

and the Fourier integral in (2) reduces to the form in the elastic limit, where the quadratic equation $\omega^2 + q = 0$ is the familiar dispersion relation for free flexural-gravity waves (Squire *et al.* [13]). Thus in the elastic limit when $A_0 = 0$, the two roots $\omega_{1,2} = \pm|k|c(k)$ are real, and the method of stationary phase may be invoked to evaluate the deflexion asymptotically as time $t \to \infty$ (see Schulkes & Sneyd [10]; Nugroho *et al.* [9]). With viscoelasticity included (i.e. $A_0 \neq 0$) however, as noted above both of these roots $\omega_{1,2}$ are complex (provided $k \neq 0$), and consequently the asymptotic analysis for $t \to \infty$ is different (Wang [19]).

In brief, this time-dependent viscoelastic theory describes both one-dimensional and two-dimensional responses, for concentrated loads impulsively reaching uniform speed V from rest. Thus in the one-dimensional case for example, a (y-independent) line load moving in the positive x-direction is represented by the loading function $f(x,t) = F_0\delta(x - Vt)H(t)$, where δ denotes the Dirac delta function and $H(t)$ is the Heaviside unit step function. After a contour integration in ω, the consequent deflexion obtained from (2) is

$$\eta(X,t) = \frac{F_0}{2\pi\rho}(I_0 + I_1 + I_2 + I_3), \tag{4}$$

where the coordinate $X = x - Vt$ is relative to the moving load, the integrals are ($j = 0, 1, 2, 3$)

$$I_j(X,t) = -i\int_{-\infty}^{\infty} \frac{g(k,\omega_j)e^{-i[kX-(\omega_j-kV)t]}}{W_L'(\omega_j)}dk, \tag{5}$$

$W_L(\omega) = (\omega - kV)W(k,\omega)$, and the prime denotes differentiation with respect to ω – so the derivatives in the denominator of (5) are evaluated at $\omega_0 = kV$ and the three roots of the equation $W(k,\omega) = 0$, respectively. The time-independent contribution obtained from I_0 is the steady-state viscoelastic result (11) of Hosking *et al.* [5]), and the time-dependent behaviour of the deflexion resides in the contribution from the sum of the remaining integrals I_1, I_2 and I_3.

In the elastic limit $A_0 \to 0$, the integral $I_3 \to 0$ and the sum

$$I_1 + I_2 \to \int_0^{\infty} \frac{\tanh(kH)}{c(k)}\left\{\frac{\cos(kX - \psi_1 t)}{\psi_1(k)} + \frac{\cos(kX + \psi_2 t)}{\psi_2(k)}\right\}dk$$

where $\psi_1(t) = k(c - V)$ and $\psi_2(t) = k(c + V)$ are the phase functions, when the dominant contributions as $t \to \infty$ arise from the neighbourhood of points of stationary phase (Schulkes & Sneyd [10]).

As previously mentioned however, with viscoelasticity the asymptotic analysis is different. In particular, it turns out that the time-dependent contributions to the deflexion in (4) are all transient at the critical load speed $V = c_{min}$. Thus the response at this critical speed also approaches a steady state (in the limit $t \to \infty$), given by the time-independent Fourier form (11) in Hosking *et al.* [5], when there is nevertheless a pronounced peak. The time dependence is quantitatively different in the two-dimensional analysis, but there is a similar outcome – and indeed, a steady state response is predicted at *all* load speeds. (As in the elastic theory, slow growth $O(t^{1/3})$ as $t \to \infty$ remains for a load moving at the gravity wave speed $V = \sqrt{gH}$ in the one-dimensional but not the two-dimensional case, since the viscoelasticity is ineffective at very large wavelength.)

4. Elastic Plate or Beam on a Winkler Foundation

The underlying pressure due to a Winkler foundation is $p = -\gamma\eta$ where γ is a constant, so it is quite suitable for the response of a plate floating on water to a static load, but *not* its dynamic response due to a moving load – i.e. there is only an hydrostatic term and no fluid inertia contribution, using the hydrodynamic terminology appropriate in the previous Section. For a Winkler foundation, a notable consequence is that the contribution from the plate acceleration term $\rho_0 h\eta_{tt}$ in (1) has traditionally been retained. Thus instead of (2), the Fourier integral for the deflexion of a plate on a Winkler foundation is

$$\eta(x,y,t) = -\frac{1}{(2\pi)^{3/2}}\int\int\int \frac{\hat{f}(k_1,k_2,\omega)e^{-i(k_1x+k_2y-\omega t)}}{\rho_0 h\omega^2 - (Dk^4+\gamma)}d\omega dk_1 dk_2, \tag{6}$$

if the viscoelastic delay term in (1) is also ignored. The corresponding dispersion relation for free flexural waves in an elastic plate on a Winkler foundation is

$$\omega^2 = \frac{Dk^4+\gamma}{\rho_0 h}. \tag{7}$$

Consequently, the phase speed $c(k) = \omega/k$ is asymptotically infinite at large wavelengths (as $k \to 0$) dominated by the reaction of the foundation and tends to increase linearly (with k) at small wavelengths (as $k \to \infty$), with a minimum $c_{min} = (4\gamma D/(\rho_0 h)^2)^{1/4}$ occurring at the wave number $k_{min} = (\gamma/D)^{1/4}$.

It is remarkable that the simple Winkler model for the foundation has played such a major role in railway engineering. A very early paper discussed the theoretical response of an elastic beam to a moving load in the context of a rail track with longitudinal sleepers (Schwedler [11]); and following some influential investigations (Timoshenko [16]; Timoshenko & Langer [17]), it became accepted that railways with the now much more common cross-tie transverse sleeper configuration could be modelled by a beam on a Winkler foundation. The differential equation for an elastic beam

$$EI\frac{\partial^4\eta}{\partial x^4} + m\frac{\partial^2\eta}{\partial t^2} + \gamma\eta = f(x,t), \tag{8}$$

analysed by Timoshenko and others (cf. Squire *et al.* [13]), is the one-dimensional analogy of the plate equation (1) without viscoelasticity. The corresponding dispersion relation for free waves in an elastic beam on a Winkler foundation is

$$\omega^2 = \frac{EIk^4+\gamma}{m} \tag{9}$$

analogous to (7), and the phase speed $c(k)$ has similar behaviour. Timoshenko [16] calculated the critical load speed where the theoretical steady state deflexion of the elastic beam is infinite, which is identical with the minimum phase speed $c_{min} = (4\gamma EI/m^2)^{1/4}$ at wave number $k_{min} = (\gamma/(EI))^{1/4}$, to be about 2,000 km/hour!

Railway engineers have consequently tended to disregard the critical speed phenomenon, based on this model with the fundamental assumptions that the rail is elastic and the foundation reaction is directly proportional to its displacement, and subsequent investigations have not as yet much altered this common perception. However, one notable model variation which has been considered is the inclusion of compressive stress in the beam (Timoshenko [16]), a suggestion which has become more relevant given the increasing use of continuously-welded rails. Thus if the additional term $N\ \partial^2\eta/\partial x^2$ is included on the left-hand side of (8), the critical speed reduces to $(1 - N/N_{cr})^{1/2}c_{min}$ where $N_{cr} = 2\sqrt{\gamma EI}$ is the buckling stress coefficient, for a steadily moving localized load (Kerr [6], [7]).

5. Ladder Sleeper Rail Tracks

Experiments with longitudinal rather than transverse railway sleepers have continued over the years, although longitudinal sleepers require some mechanism to maintain the track gauge and the sleeper components may be more expensive to construct (Wakui *et al.* [18]). Indeed, rail tracks with so-called ladder sleepers may well emerge in North America and Japan, for heavy haul and fast rail systems. Potential advantages are lower rail track maintenance and a smoother ride.

A typical ladder sleeper component consists of two prestressed concrete longitudinal beams from 6 to 12 or 13 metres in length with transverse connecting rods (steel pipes) acting as gauge ties, inserted every 3 metres between the prestressing strands which are the main reinforcement for the beams. These components laid end to end produce a structure which not only provides continuous support to the rails but also a much more even pressure distribution (with lower peak pressure) on the foundation, in comparison with conventional transverse sleepers. Rail fasteners to the ladder sleepers every 75 centimetres or so, four to every gauge tie say, means that the rail and longitudinal sleeper tend to act as a composite. Consequently, the rate of ballast and subgrade settlement is usually much lower than with conventional transverse sleepers, and the composite ladder structure may also bridge across weak spots where partial subsidence has occurred. Decreased track irregularity due to lateral loading is also envisaged, given the greater transverse resistance provided by the continuous longitudinal structure. Continuous rubber buffers can be placed between the rails and the sleepers too, and possibly other buffers underneath the sleepers, with the objective to give a smoother ride.

The flexural rigidity EI of the composite rail and longitudinal sleeper may be of order 10^6Nm^2, and the combined mass per unit length of the rail and longitudinal sleeper about 300 kg/m, values quite similar to the parameters for a conventional cross-tie railway. Thus the critical speed c_{min} predicted by modelling the composite structure as an elastic beam on a continuous Winkler foundation is more than

500 km/hour, higher than the maximum speed attained on all but magnetically-levitated rail systems, for a foundation stiffness coefficient γ of order $10^7 \mathrm{N}m^2$ or more. The corresponding wavelength is only a few metres however, comparable with the length of a ladder sleeper component.

6. Summary

Mathematical modelling of the response of a continuously-supported flexible plate or beam to a moving load has been applied to predict and interpret important transport system features, in parametrically diverse cold region operations and railway engineering. (There also have been railways built over frozen waterways however, not discussed here – see for example, Squire *et al.* [13].)

Developed theory for a floating ice plate has defined various significant phenomena, such as the dependence of the wave pattern on the load speed and the pronounced resonant response at the critical load speed coincident with the minimum phase speed of generated flexural-gravity waves, consistent with extensive field observations. Recent time-dependent analysis for an elastic plate predicts an eventual steady state response except at this critical load speed; and the response is pronounced but also steady state at that load speed, when viscoelasticity is included in the thin plate equation. An important observation is that the plate acceleration is negligible relative to the fluid inertia contribution from the foundation in this context, where the horizontal wavelength of the surface deflexion is generally much larger than the plate thickness.

Earlier theory in the railway engineering context assumed a Winkler foundation, where the plate or beam acceleration term is retained and the horizontal wavelength of the response is much smaller, for typical parameters. The critical load speed for an elastic plate or beam on a Winkler foundation comfortably exceeds the highest operational speed of most fast rail systems, based on stiffness estimates for typical ballast and substrate. Other theoretical models have been proposed, and their further investigation may be warranted. Thus there may be foundation features which are not represented satisfactorily by any isotropic elastic model, in addition to any intrinsic remodelling of the plate or beam.

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