Journal of Applied Mathematics and Stochastic Analysis, **12**:3 (1999), 253-260.

UNIQUENESS OF SOLUTIONS TO INTEGRODIFFERENTIAL AND FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS

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(Received April, 1998; Revised August, 1998)

In this paper we study a class of integrodifferential and functional integrodifferential equations with infinite delay. These problems are reformulated as abstract integrodifferential and functional integrodifferential equations. We use Nagumo type conditions to establish the uniqueness of solutions to these abstract equations.

Key words: Integrodifferential Equation, Mild Solution, Nagumo Condition, Finite and Infinite Delays.

AMS subject classifications: 34G20, 24K15.

1. Introduction

In the present work we are concerned with the following integrodifferential and functional integrodifferential equations considered in a real Banach space X:

$$\frac{du(t)}{dt} = f[t, u(t), \int_{t_0}^{t} k_1(s, u(s))ds], \quad t > t_0 \tag{1.1}$$

$$u(t_0) = x$$

$$\frac{du(t)}{dt} = g[t, u_t, \int_{t_0}^{t} k_2(s, u_s)ds], \quad t > t_0, \tag{1.2}$$

$$u_{t_0} = \phi$$

where in (1.1) the nonlinear map f is defined from $\bar{J} \times X \times X$ into X, $J = (t_0, t_0 + T]$, $\bar{J} = [t_0, t_0 + T]$, $0 < T < \infty$, the nonlinear map k_1 is defined from $\bar{J} \times X$ into X, $x \in X$, and in (1.2), g is defined from $\bar{J} \times C_X \times C_X$ into X, $C_X = BUC((-\infty, 0]; X)$ ($BUC(I; X)$ denotes the space of bounded uniformly continuous functions from the interval I into a Banach space X endowed with the supremum norm), for any $u \in BUC((-\infty, t_0 + T]; X)$ and $t \in [t_0, t_0 + T]$ the map $u_t \in C_X$ is defined by

$$u_t(s) = u(t+s), \quad -\infty, s \leq 0,$$

the nonlinear map k_2 is defined from $\bar{J} \times C_X$ into C_X and $\phi \in C_X$.

Particular cases of (1.1) and (1.2), in which $k_1 \equiv k_2 \equiv 0$, have been considered by many authors, see for instance, Rogers [10], Kotta [4, 5]. For the case $X = \boldsymbol{R}^n$, we refer to Kappel and Schappacher [3].

In the present work, we shall be concerned with the uniqueness of solutions only. Proving the existence of solutions to (1.1) and (1.2) will be our next concern. For the existence of solutions for the particular cases mentioned above, we refer to Hale [1], Ladas and Lakshmikantham [6], Martin and Smith [8], and Martin [9].

2. Preliminaries

We shall establish the uniqueness of *mild solutions* to (1.1) and (1.2), which would also establish the uniqueness in the case of classical solutions.

By a mild solution to (1.1) we mean a continuous function $u \in C(\bar{J}, X)$ such that

$$u(t) = x + \int_{t_0}^{t} f[s, u(s), \int_{t_0}^{s} k_1(\tau, u(\tau))d\tau]ds, \quad t_0 \leq t \leq t_0 + T, \tag{2.1}$$

where $x \in X$. By a mild solution to (1.2) we mean a continuous function $u \in BUC((-\infty, t_0 + T], X)$ such that

$$u(t) = \begin{cases} \phi(t - t_0), & -\infty \leq t \leq t_0, \\ \phi(0) + \int_{t_0}^{t} f[s, u_s, \int_{t_0}^{s} k_1(\tau, u_\tau)d\tau]ds, & t_0 \leq t \leq t_0 + T, \end{cases} \tag{2.2}$$

where $\phi \in C_X$.

We consider a function $h \in C(J, \boldsymbol{R}^+)$ satisfying the following condition

(H)

$$\lim_{t \to t_0^+} \int h(t)dt = -\infty.$$

For, instance, we may take

$$h(t) = \frac{1}{(t - t_0)^2} \quad t \in J.$$

Then h satisfies condition **(H)**.

Remark: If $h \in C(J, \boldsymbol{R}^+)$ satisfies condition **(H)**, then the function $\tilde{h} \in C(J, \boldsymbol{R}^+)$ given by

$$\tilde{h}(t) = h(t) + C, \quad t \in J$$

for any positive constant C also satisfies condition **(H)**.

The main tool for proving the uniqueness of solutions is the following lemma due to Kotta [5]. This result is a generalization of an analogous result of Rogers [10]. For the sake of completeness, we shall give a proof of the lemma here.

Lemma 2.1: *Let $u \in C(\bar{J}, \boldsymbol{R}^+)$ and suppose that*

(*i*) $u(t) \leq \int_{t_0}^{t} h(s)u(s)ds$,

(*ii*) $u(t) = o(e^{\int h(t)dt})$, *as* $t \to t_0^+$,

where $h \in C(J, \boldsymbol{R}^+)$ satisfying condition **(H)**. *Then $u \equiv 0$ on $\bar{J}$.*

Proof: Let

$$F(t) = \int_{t_0}^{t} h(s)u(s)ds, \quad t \in J. \tag{2.3}$$

Using (*i*) and (2.3) we obtain

$$\frac{d}{dt}F(t) = h(t)u(t) \leq h(t)F(t).$$

Therefore,

$$\frac{d}{dt}(e^{-\int h(t)dt}F(t)) \leq 0. \tag{2.4}$$

Inequality (2.4) implies that $F(t)e^{-\int h(t)dt}$ is a nonincreasing function on J.

From (*ii*) it follows that for every $\epsilon > 0$, there exists a $\delta > 0$ such that for $t \in J$ with $0 < t - t_0 < \delta$, we have

$$e^{-\int h(t)dt}F(t) = e^{-\int h(t)dt}\int_{t_0}^{t} h(s)u(s)ds$$

$$\leq \epsilon e^{-\int h(t)dt}\int_{t_0}^{t} h(s)e^{\int h(s)ds}ds. \tag{2.5}$$

Now,

$$\frac{d}{dt}e^{\int h(t)dt} = h(t)e^{\int h(t)dt}. \tag{2.6}$$

Integrating (2.6) over the interval (t_0, t), we get

$$e^{\int h(t)dt}\Big|_{t_0}^{t} = \int_{t_0}^{t} h(s)e^{\int h(s)ds}ds. \tag{2.7}$$

Using condition **(H)** in (2.7), we get

$$e^{\int h(t)dt} = \int_{t_0}^{t} h(s)e^{\int h(s)ds}ds. \tag{2.8}$$

Using (2.8) in (2.5), we get

$$e^{-\int h(t)dt}F(t) \leq \epsilon. \tag{2.9}$$

From (2.9), we have that

$$\lim_{t \to t_0^+} e^{-\int h(t)dt}F(t) = 0.$$

Hence, $F(t) \equiv 0$ on $\bar{J}$, which in turn implies that $u \equiv 0$ on $\bar{J}$.

3. Main Result

We first state and prove the following uniqueness theorem for (1.1).

Theorem 3.1: *Suppose that* $f: \bar{J} \times X \times X \to X$ *is a continuous function and satisfies the conditions*

$$\| f(t, x_1, y_1) - f(t, x_2, y_2) \| \leq h(t)[\| x_1 - x_2 \| + \| y_1 - y_2 \|] \tag{3.1}$$

for any $t \in \bar{J}$, $x_i, y_i \in X$, $i = 1,2$; *and*

$$\| f(t, x(t), y(t)) \| = o(h(t)e^{\int h(t)dt}) \text{ as } t \to t_0^+, \tag{3.2}$$

for any $t \in \bar{J}$, $x, y \in C(\bar{J}; X)$, *where* $h \in C(J, \boldsymbol{R}^+)$ *satisfies condition* **(H)**. *Further, suppose that* $k_1: \bar{J} \times X \to X$ *satisfies*

$$\| k_1(t, x_1) - k_1(t, x_2) \| \leq C(t) \| x_1 - x_2 \| \tag{3.3}$$

for any $t \in \bar{J}, x_i \in X$, $i = 1,2$; *where* $C(t)$ *is a nonnegative integrable function on* J. *Then* (1.1) *has at most one solution.*

Proof: Let $x(t)$ and $y(t)$ be solutions of (1.1). Then we have

$$\| x(t) - y(t) \| \leq \int_{t_0}^{t} \| f[(s, x(s), \int_{t_0}^{s} k_1(\tau, x(\tau))d\tau]$$

$$- f[(s, y(s), \int_{t_0}^{s} k_1(\tau, y(\tau))d\tau] \| ds. \tag{3.4}$$

Using (3.1) and (3.3) in (3.4), we get

$$\| x(t) - y(t) \| \leq \int_{t_0}^{t} h(s)[\| x(s) - y(s) \| + \int_{t_0}^{s} \| k_1(\tau, x(\tau)) - k_1(\tau, y(\tau)) \| d\tau]ds$$

$$\leq \int_{t_0}^{t} h(s)[\, \| x(s) - y(s) \| + \int_{t_0}^{s} C(\tau) \| x(\tau) - y(\tau) \| d\tau] ds$$

$$\leq \int_{t_0}^{t} (h(s) + C_T) \left\{ \sup_{t_0 \leq \tau \leq s} \| x(\tau) - y(\tau) \| \right\} ds$$

$$\leq \int_{t_0}^{t} \tilde{h}(s) \left\{ \sup_{t_0 \leq \tau \leq s} \| x(\tau) - y(\tau) \| \right\} ds, \tag{3.5}$$

where

$$C_T = \int_{t_0}^{t_0 + T} C(s) ds$$

and

$$\tilde{h}(t) = h(t) + C_T.$$

Now, (3.5) implies that for every $t_0 \leq \eta \leq t$, we have

$$\| x(\eta) - y(\eta) \| \leq \int_{t_0}^{\eta} \tilde{h}(s) \left\{ \sup_{t_0 \leq \tau \leq s} \| x(\tau) - y(t) \| \right\} ds$$

$$\leq \int_{t_0}^{t} \tilde{h}(s) \left\{ \sup_{t_0 \leq \tau \leq s} \| x(\tau) - y(\tau) \| \right\} ds. \tag{3.6}$$

From (3.6) we get

$$\sup_{t_0 \leq \eta \leq t} \| x(\eta) - y(\eta) \| \leq \int_{t_0}^{t} \tilde{h}(s) \left\{ \sup_{t_0 \leq \tau \leq s} \| x(\tau) - y(\tau) \| \right\} ds. \tag{3.7}$$

Replacing the dummy variable η on the left-hand side of inequality (3.7) by τ, we finally get

$$\sup_{t_0 \leq \tau \leq t} \| x(\tau) - y(\tau) \| \leq \int_{t_0}^{t} \tilde{h}(s) \left\{ \sup_{t_0 \leq \tau \leq s} \| x(\tau) - Y(\tau) \| \right\} ds. \tag{3.8}$$

Now, using (3.2) in inequality (3.4) we have that for $\epsilon > 0$, there exists $\delta > 0$ such that for $0 < t - t_0 < \delta$,

$$\| x(t) - y(t) \| \leq \epsilon \int_{t_0}^{t} h(s) e^{\int h(s) ds} ds$$

$$\leq \epsilon \int_{t_0}^{t} (h(s) + C_T) e^{C_T s} e^{\int h(s) ds} ds$$

$$= \epsilon \int_{t_0}^{t} \widetilde{h}(s) e^{\int \widetilde{h}(s)ds} ds. \tag{3.9}$$

Again, for $t_0 \le \tau \le t$, we have

$$\| x(\tau) - y(\tau) \| \le \epsilon \int_{t_0}^{\tau} \widetilde{h}(s) e^{\int \widetilde{h}(s)ds} ds$$

$$\le \epsilon \int_{t_0}^{t} \widetilde{h}(s) e^{\int \widetilde{h}(s)ds} ds. \tag{3.10}$$

Taking supremum in (3.10) we get

$$\sup_{t_0 \le \tau \le t} \| x(\tau) - y(\tau) \| \le \epsilon \int_{t_0}^{t} \widetilde{h}(s) e^{\int \widetilde{h}(s)ds} ds = \epsilon e^{\int \widetilde{h}(t)dt}. \tag{3.11}$$

We obtained the desired result using Lemma 2.1. This completes the proof of Theorem 3.1. □

Next, we state and prove below a similar uniqueness result for (1.2).

Theorem 3.2: *Suppose that* $g: \bar{J} \times C_X \times C_X \to C_X$ *is continuous and satisfies the condition*

$$\| g(t, \phi_1, \psi_1) - g(t, \phi_2, \psi_2) \|_{C_X} \le h(t)[\, \| \phi_1 - \phi_2 \|_{C_X} + \| \psi_1 - \psi_2 \|_{C_X}], \tag{3.12}$$

for any $t \in \bar{J}$, $\phi_i, \psi_i \in C_X$, $i = 1,2$; *and*

$$\| g(t, \phi_t, \psi_t) \|_{C_X} = o(h(t) e^{\int h(t)dt}), \text{ as } t \to t_0^+, \tag{3.13}$$

for any $t \in \bar{J}$, $\phi, \psi \in C(-\infty, t_0 + T]; X)$, *where* $h \in C(J, \boldsymbol{R}^+)$ *satisfies condition* **(H)**. *Suppose that* $k_2: \bar{J} \times C_X \to C_X$ *is continuous and satisfies*

$$\| k_2(t, \phi_1) - k_2(t, \phi_2) \|_{C_X} \le D(t) \| \phi_1 - \phi_2 \|_{C_X}, \tag{3.14}$$

where $D(t)$ *is a nonnegative integrable function on* J. *Then* (1.2) *has at most one solution.*

Proof: Suppose that x and y are two solutions to (1.2). Then $x_{t_0} = y_{t_0} = \phi$ implies that

$$x \equiv y, \text{ on } (-\infty, t_0].$$

Therefore,

$$\| x_t - y_t \|_{C_X} = \sup_{s \in (-\infty, 0]} \| x(t+s) - yt + s) \| = \max_{\theta \in [t_0, t]} \| x(\theta) - y(t) \|. \tag{3.15}$$

From definition (2.2) of mild solutions to (1.2) and (3.15) we have

$$\| x_t - y_t \|_{C_X} \leq \int_{t_0}^{t} \| g[s, x_s, \int_{t_0}^{s} k_2(\tau, x_\tau) dt] - g[s, y_s \int_{t_0}^{s} k_2(\tau, y_\tau) d\tau] \|_{C_X} ds. \tag{3.16}$$

Using (3.12) and (3.14) in (3.16), we obtain

$$\begin{aligned} \| x_t - y_t \|_{C_X} &\leq \int_{t_0}^{t} h(s)[\| x_s - y_s \|_{C_X} + \int_{t_0}^{s} D(\tau) \| x_\tau - y_\tau \|_{C_X} d\tau] ds \\ &\leq \int_{t_0}^{t} (h(s) + \int_{t_0}^{s} D(\tau) d\tau)[\| x_s - y_s \|_{C_X} ds \\ &\leq \int_{t_0}^{t} (h(s) + D_T) \| x_s - y_s \|_{C_X} ds \\ &\leq \int_{t_0}^{t} \widetilde{h}(s) \| x_s - y_s \|_{C_X} ds, \end{aligned} \tag{3.17}$$

where

$$D_T = \int_{t_0}^{t_0 + T} D(s) ds$$

and

$$\widetilde{h}(t) = h(t) + D_T, \quad t \in J.$$

Now, using (3.13) in (3.16), we have that for $\epsilon > 0$, there exists $\delta > 0$ such that for $0 < t - t_0 < \delta$,

$$\begin{aligned} \| x_t - y_t \|_{C_X} &\leq \epsilon \int_{t_0}^{t} h(s) e^{\int h(s) ds} ds \\ &\leq \epsilon \int_{t_0}^{t} (h(s) + D_T) e^{D_T s} e^{\int h(s) ds} ds \\ &\leq \epsilon \int_{t_0}^{t} \widetilde{h}(s) e^{\int \widetilde{h}(s) ds} ds \\ &= \epsilon e^{\int \widetilde{h}(t) dt}. \end{aligned} \tag{3.18}$$

Again, we apply Lemma 2.1 to get the desired result. This completes the proof of Theorem 3.2. □

Acknowledgement

The first author would like to acknowledge the financial support provided by the National Board for Higher Mathematics under its research project No. 48/2/97-R&D-II/2003 to carry out this work.

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