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ANALYSIS OF AN N/G/1 FINITE QUEUE WITH THE SUPPLEMENTARY VARIABLE METHOD

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In this paper, we suggest a new approach to the analysis of an N/G/1 finite queue with the supplementary variable method. Compared to the conventional approach, our approach yields a simpler formula for the queue length distribution, which in turn gives a more efficient computational algorithm. Also, the new approach enables us to derive the joint density of the queue length and the elapsed service time.

Key words: N-process, Elapsed Service Time, Imbedded Markov Chain, Supplementary Variable Method, Schur-Banachiewicz Formula.

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1. Introduction

Very complex input flows often occur in integrated service communication systems. As an approximation to such a stream, Neuts [4] introduced the N-*process*. This N-process is analytically tractable and can appropriately represent the correlation and burstiness of the stream. Many familiar arrival processes are special cases of the N-process.

To investigate the performance of the service facility with finite resources, Blondia [1] considered an N/G/1 finite queue, i.e., a single server queue with K waiting rooms in which customers arrive according to an N-process. For the analysis, he used the imbedded Markov chain technique upon service completion epochs. He also gave a computational algorithm for the queue length distribution of the system by using the Schur-Banachiewicz formula [3] for the inverse of the block matrices.

However, the computational algorithm suggested by Blondia [1] needs a large amount of work. This motivated us to study an N/G/1 finite queue. Our aim is to obtain a more efficient computational algorithm. To this end, we employ the supplementary variable method originated by Cox [2].

This paper is organized as follows. In Section 2, we define N-process as originally

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introduced in Neuts [4]. Section 3 consists of the joint density of the queue length and the elapsed service time and the distribution of the queue length of the N/G/1 finite queue.

2. N-Process

Consider a continuous-time Markov process with m transient states and a single absorbing state. Then the infinitesimal generator of this Markov chain has the form

$$Q = \begin{pmatrix} T & \boldsymbol{T}^0 \\ \mathbf{0} & 0 \end{pmatrix},$$

where T is an $m \times m$ non-singular matrix with $T_{i,i} < 0$, $T_{i,j} \geq 0$ for $i \neq j$. The vector $\boldsymbol{T}^0$ is non-negative and satisfied $Te + \boldsymbol{T}^0 = \mathbf{0}$, with $e = (1, \ldots, 1)^t$. Let $(\boldsymbol{\alpha}, \alpha_{m+1})$ be a vector of initial state probabilities of the Markov process. In what follows, we shall assume that $\alpha_{m+1} = 0$.

Now, construct a continuous-time process by restarting the above Markov process Q instantaneously after each absorption through a multinomial trial with probability $\boldsymbol{\alpha}$ and outcomes $1, \ldots, m$. Then this process is also a Markov process with the state space $\{1, 2, \ldots, m\}$ and the infinitesimal generator

$$Q^* = T + T^0 A^0,$$

where T^0 is an $m \times m$ matrix whose columns are all $\boldsymbol{T}^0$ and $A^0 = diag(\alpha_1, \ldots, \alpha_m)$. A transition from the state i to the state j in the Markov process Q^*, which does not involve absorptionis, will be called an (i,j)-*transition*, while the others are called (i,j)-*renewal transition*. Then the N-process is an arrival process defined in the following way [4].

(1) During any sojourn in the state i, there are Poisson group arrivals of rate λ_i and group size of densities $P\phi_i(k), k \geq 0\}$. We shall denote $\phi_i^*(z)$ the *p.g.f.* of $\{\phi_i(k), k \geq 0\}$, and define $\Lambda = diag(\lambda_1, \ldots, \lambda_m)$ and $\Phi(z) = diag(\phi_1^*(z), \ldots, \phi_m^*(z))$.

(2) At (i,j)-renewal transitions, there are group arrivals with probability density $\{\Phi_{i,j}(k): k \geq 0\}$ whose *p.g.f.* is $\Psi_{i,j}(z)$. Let us denote the $m \times m$ matrix $(\Psi_{i,j}(z))$ by $\Phi(z)$.

(3) At (i,j)-transitions $(i \neq j)$, there are group arrivals with probability densities $\{\Omega_{i,j}(k), k \geq 0\}$, whose *p.g.f.* is $\Omega_{i,j}(z)$. For notational convenience, we set $\Omega_{i,i}(z) \equiv 1$ for all i and define $(\Omega_{i,j}(z))_{1 \leq i,j \leq m}$ by $\Omega(z)$.

Define the conditional probabilities

$$P_{i,j}(n,t) = Pr(J(t) = j,\ N(t) = n \mid J(0) = i, N(0) = 0),$$

where $N(t)$ and $J(t)$ denote the number of arrivals during $(0,t]$ and the state of the underlying Markov process $\boldsymbol{Q}^*$ at time t, respectively. We also define conditional probability matrices $\boldsymbol{P}(n,t) = (P_{i,j}(n,t))_{1 \leq i,j < m}, n \geq 0$. It was shown in [4] that

$$\sum_{n=0}^{\infty} z^n \boldsymbol{P}(n,t) = exp(\boldsymbol{R}(z)t), 0 \leq z \leq 1, \tag{1}$$

with $\boldsymbol{R}(z) = \sum_{n=0}^{\infty} z^n \boldsymbol{R}_n$ and

$$\boldsymbol{R}_0 = \boldsymbol{\Lambda\Phi}(0) - \boldsymbol{\Lambda} + \boldsymbol{T}^0\boldsymbol{A}^0 \circ \boldsymbol{\Psi}(0) + \boldsymbol{T} \circ \boldsymbol{\Omega}(0)$$

$$\boldsymbol{R}_n = \boldsymbol{\Lambda\Phi}(n) + \boldsymbol{T}^0\boldsymbol{A}^0 \circ \boldsymbol{\Psi}(n) + \boldsymbol{T} \circ \boldsymbol{\Omega}(n), n \geq 1,$$

where $\circ$ denotes the Schur (entrywise) product of two matrices. For the upcoming analysis, we shall assume that the matrix $\boldsymbol{R}_0^{-1}$ exists.

3. Analysis of an N/G/1 Finite Queue

In this section, we will analyze the N/G?1 finite queue with the supplementary variable method. The queue size is assumed to be K. When describing the N-process, we will use the same notations as in Section 2. The successive service times are independent and identically distributed according to $H(x)$. Also the hazard rate function and the mean of $H(x)$ are denoted by $r(x)$ and μ respectively.

3.1 Supplementary Variable Method

Let $X(t)$ denote the number of customers in the system at time t. We define the *elapsed service time* $S(t)$ as follows: If $X(t) > 0$, $S(t)$ denotes the amount of service already received by a customer in service. Otherwise, $S(t)$ denotes the amount of time elapsed after the last service completion. Then, the triplet $(J(t), X(t), S(t))$ is a three-dimensional Markov process with state space $\{1,\ldots,m\} \times \{0,\ldots,K\} \times [0,\infty)$.

Suppose that

$$\pi(i,n,x)dx = \lim_{t\to\infty} Pr(J(t) = i, X(t) = n, x \leq S(t) < x + dx)$$

exists for all states and define $\boldsymbol{\pi}(n,x) = (\pi(1,n,x),\ldots,\pi(m,n,x))$. Then the Kolmogorov differential equations of the joint density $\boldsymbol{\pi}(n,x)$ can be written down as follows:

$$\frac{d}{dx}\boldsymbol{\pi}(0,x) = \boldsymbol{\pi}(0,x)\boldsymbol{R}_0, \tag{2}$$

$$\frac{d}{dx}\boldsymbol{\pi}(n,x) = -\boldsymbol{\pi}(n,x)r(x) + \sum_{k=1}^{n} \boldsymbol{\pi}(k,x)\boldsymbol{R}_{n-k}, \; 0 < n < K, \tag{3}$$

$$\frac{d}{dx}\boldsymbol{\pi}(K,x) = -\boldsymbol{\pi}(K,x)r(x) + \sum_{k=1}^{K} \sum_{l=K-k}^{\infty} \boldsymbol{\pi}(k,x)\boldsymbol{R}_l. \tag{4}$$

The joint density $\boldsymbol{\pi}(n,x)$ should satisfy the boundary conditions

$$\boldsymbol{\pi}(0,0) = \int_0^{\infty} \boldsymbol{\pi}(1,x)r(x)dx, \tag{5}$$

$$\pi(n,0) = \int_0^\infty \pi(n+1,x)r(x)dx + \int_0^\infty \pi(0,x)R_n dx,\ 0 < n < K, \tag{6}$$

$$\pi(K,0) = \sum_{i=0}^{\infty} \int_0^\infty \pi(0,x)R_i dx, \tag{7}$$

and the normalization condition

$$\sum_{n=0}^{K} \int_0^\infty \pi(n,x)dx e = 1, \tag{8}$$

where $e = (1,\ldots,1)^t$.

Now, we shall find the joint density $\pi(n,x)$ of the queue length and the elapsed service time. From equation (1), we obtain

$$\frac{d}{dx}P(n,x) = \sum_{k=0}^{n} P(k,x)R_{n-k},\ \ n \geq 0.$$

With this and equations (2)-(8), we get

$$\pi(0,x) = \pi(0,0)P(0,x), \tag{9}$$

$$\pi(n,x) = \sum_{k=1}^{n} \pi(k,0)P(n-k,x)(1-H(x)), 0 < n < K, \tag{10}$$

$$\pi(K,x) = \sum_{k=1}^{K} \sum_{i=K-k}^{\infty} \pi(k,0)P(i,x)(1-H(x)). \tag{11}$$

We may also derive the above solutions by conditioning on the state of the system time x back.

Before finding the coefficients $\pi(n,0)$, we consider the embedded Markov chain $\{J(\tau_n), X(\tau_n)\}$, where $\{\tau_n n \geq 0\}$ is the n^{th} epoch of service or idle completion. Then the transition probability matrix of $\{J(\tau_n), X(\tau_n)\}$ is

$$Q_E = \begin{pmatrix} 0 & U_1 & \cdots & U_{K-2} & U_{K-1} & \sum_{n=K}^{\infty} U_n \\ A_0 & A_1 & \cdots & A_{K-2} & \sum_{n=K-1}^{\infty} A_n & 0 \\ 0 & A_0 & \cdots & A_{K-3} & \sum_{n=K-2}^{\infty} A_n & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \sum_{n=0}^{\infty} A_n & 0 \end{pmatrix},$$

where

$$U_n = \int_0^\infty P(0,x)R_n dx = [-R_0^{-1}]R_n,\ n \geq 1,$$

$$\boldsymbol{A}_n = \int_0^\infty \boldsymbol{P}(n,x)dH(x), \quad n \geq 0.$$

The matrix $\boldsymbol{U}_n$ (or $\boldsymbol{A}_n$) is the probability that n customers arrive during an idle time (or a service time).

Theorem 1: *The coefficients* $\boldsymbol{\pi}(n,0)$ *of the joint density* $\boldsymbol{\pi}(n,x)$ *are given by*

$$(\boldsymbol{\pi}(0,0),\ldots,\boldsymbol{\pi}(K,0) = \frac{1}{\mu - \boldsymbol{x}_0[\mu \boldsymbol{I} + \boldsymbol{R}^{-1}(0)]\boldsymbol{e}}(\boldsymbol{x}_0,\ldots,\boldsymbol{x}_K),$$

where $(\boldsymbol{x}_0,\ldots,\boldsymbol{x}_K)$ *is the stationary vector of the transition probability matrix* $\boldsymbol{Q}_E$. I *is an identity matrix of size* m.

Proof: By inserting (9)-(11) into the boundary conditions, we show that $(\boldsymbol{\pi}(0,0),\ldots,\boldsymbol{\pi}(K,0))$ is a positive invariant vector of $\boldsymbol{Q}_E$, that is,

$$(\boldsymbol{\pi}(0,0),\ldots,\boldsymbol{\pi}(K,0)) = c(\boldsymbol{x}_0,\boldsymbol{x}_K) \text{ for some constant } c > 0.$$

Applying (9)-(11) to the normalization condition, we have

$$\boldsymbol{\pi}(0,0)[-\boldsymbol{R}_0^{-1}]\boldsymbol{e} + \mu \sum_{n=1}^{\infty} \boldsymbol{\pi}(n,0)\boldsymbol{e} = 1.$$

Therefore, we have

$$c = \frac{1}{\mu - \boldsymbol{x}_0[\mu I + \boldsymbol{R}_0^{-1}]\boldsymbol{e}}.$$

So the proof is complete.

The matrices $\boldsymbol{P}(n,x)$ and $\boldsymbol{A}_n$ can be efficiently evaluated by means of an iterative procedure in [3]. Therefore, we can compute $\boldsymbol{\pi}(n,x)$ by deriving the stationary vector of the transition probability matrix $\boldsymbol{Q}_E$. As Blondia [1] did, we can also reduce the complexity of the computation for the stationary vector with the Schur-Banachiewicz formula for the inverse of block matrices.

3.2 Queue Length Distribution

In this subsection, we shall consider two computational algorithms to obtain the queue length distribution $\boldsymbol{\pi}(n)$ using the coefficients $\boldsymbol{\pi}(n,0)$ derived in the previous subsection.

Let us define

$$\boldsymbol{M}_n = \int_0^\infty \boldsymbol{P}(n,x)(1 - H(x))dx, \quad n \geq 0$$

and let $\boldsymbol{M}(z)$ be the generating function of $\{\boldsymbol{M}_n, n \geq 0\}$. Then equations (9)-(10) yield

$$\boldsymbol{\pi}(0) = \boldsymbol{\pi}(0,0)[-\boldsymbol{R}_0^{-1}],$$

$$\boldsymbol{\pi}(n) = \sum_{k=1}^{n} \boldsymbol{\pi}(k,0)\boldsymbol{M}_{n-k}, \quad 1 \leq n \leq K-1. \tag{12}$$

Since $\sum_{n=0}^{K}\boldsymbol{\pi}(n)$ is the stationary vector $\boldsymbol{\theta}$ of the underlying Markov process $\boldsymbol{Q}^*$, we get

$$\boldsymbol{\pi}(K) = \boldsymbol{\theta} - \sum_{n=0}^{K-1}\boldsymbol{\pi}(n). \tag{13}$$

Using the fact that $(\boldsymbol{\pi}(0,0),\ldots,\boldsymbol{\pi}(K,0))$ is an invariant vector of the transition probability matrix $\boldsymbol{Q}_E$ and $\boldsymbol{A}(z) = \boldsymbol{M}(z)\boldsymbol{R}(z) + \boldsymbol{I}$, we have

$$\begin{aligned}\boldsymbol{\pi}(n) &= \left(\sum_{k=1}^{n-1}\boldsymbol{\pi}(k)\boldsymbol{R}_{n-k} = \boldsymbol{\pi}(0)\boldsymbol{R}_{n-1}\right)[-\boldsymbol{R}_0^{-1}]\\ &- (\mathbf{1}_{(n\geq 2)}\boldsymbol{\pi}(n-1,0) - \boldsymbol{\pi}(n,0))[-\boldsymbol{R}_0^{-1}],\ 1 \leq n \leq K-1,\end{aligned} \tag{14}$$

where $\mathbf{1}_{(\cdot)}$ is the indicator function.

The above equation (14) for the queue length distribution has a simpler form than on derived by Blondia [1], since Blondia's formulas require additional computation of matrices $\{\boldsymbol{R}_n(s), n \geq 0\}$ satisfying $[\boldsymbol{R}(z) + s\boldsymbol{I}]^{-1} = \sum_{n=0}^{\infty}\boldsymbol{R}_n(s)z^n$. Consequently, we can obtain a more efficient computational algorithm for the queue length distribution.

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