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ON THE STRUCTURE OF THE SOLUTION SET OF EVOLUTION INCLUSIONS WITH FRÉCHET SUBDIFFERENTIALS

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In this paper we consider a Cauchy problem in which is present an evolution inclusion driven by the Fréchet subdifferential $\partial^- f$ of a function $f\colon \Omega \rightarrow R \cup \{+\infty\}$ (Ω is an open subset of a real separable Hilbert space) having a φ-monotone subdifferential of order two and a perturbation $F\colon I \times \Omega \rightarrow P_{fc}(H)$ with nonempty, closed and convex values.

First we show that the Cauchy problem has a nonempty solution set which is an R_δ-set in $C(I,H)$, in particular, compact and acyclic. Moreover, we obtain a Kneser-type theorem. In addition, we establish a continuity result about the solution-multifunction $x \rightarrow S(x)$. We also produce a continuous selector for the multifunction $x \rightarrow S(x)$. As an application of this result, we obtain the existence of solutions for a periodic problem.

Key words: Upper Semicontinuity, Hausdorff Metric, Fréchet Subdifferential, Evolution Inclusion, φ-Monotone Subdifferential of Order Two, R_δ-Set, Continuous Selector, Periodic Problem.

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1. Introduction

The topological property that the solution set of a differential equation is an R_δ-set in $C(I,R^n)$, $I=[0,T]$, (in particular, nonempty, compact and connected) has been an object of investigation by many authors. It is known that the solution set of the Cauchy problem

$$x'(t) = f(t,x(t)) \text{ a.e. on } I, \quad x(0) = x_0,$$

where $f(\cdot,\cdot)$ is a bounded, continuous function on $I \times \mathbb{R}^n$, is an R_δ-set (see [2]). This result was extended recently to differential inclusions by C.J. Himmelberg-F.S. Van Vleck (cf. [13]) for autonomous systems and by F.S. DeBlasi-J. Myjak (cf. [10]) for non-autonomous systems. In a recent paper, N.S. Papageorgiou and F. Papalini

[18] established that the solution set of an evolution inclusion, driven by a time-dependent subdifferential $\partial\varphi(t,x)$ and by a convex valued perturbation term $F(t,x)$ satisfying a continuity hypothesis in the x-variable, is R_δ in $C(I,H)$, where H is a separable Hilbert space.

The purpose of this paper is to study the mentioned topological property of the following Cauchy problem for evolution inclusion:

$$\begin{cases} x' \in -\partial^- f(x) + F(t,x), & \text{a.e. on } I, \\ x(0) = x_0, & x_0 \in \mathrm{dom}(f), \end{cases} \tag{1}$$

where $\partial^- f$ is the Fréchet subdifferential of a function $f:\Omega \to R \cup \{+\infty\}$ (Ω is an open subset of a real separable Hilbert space) having a φ-monotone subdifferential of order two, while $F: I \times \Omega \to P_{fc}(H)$ is a multifunction with nonempty, closed and convex values. About the problem (1), the recent existence theorems obtained in [19] and in [6] are known. In [19], Papalini also proves that the solution set for the problem (1) is path-connected. But, we want to observe that a path-connected set need not be an R_δ set.

In this note, we first prove two existence theorems (cf. Theorem 1 and Theorem 2) for the problem (1). Moreover, in Section 3, we obtain that the solution set $S(x_0)$ is a R_δ (cf. Theorem 3) if "f" is a function with the properties:

(i) f has a φ-monotone subdifferential of order two;

(ii) $\exists r > 0$ such that $clB(x_0,r) \subset \Omega$ and the set $L(c) = \{x \in ClB(x_0,r): \|x\|^2 + f(x) \leq c\}$ is compact in H, $\forall c \in R$,

(iii) $\exists \tilde{k}, \tilde{r} > 0$ such that $f(x) \leq \tilde{k}$, $\forall x \in \mathrm{dom}(f) \cap B(x_0, \tilde{r})$;

(iv) $\exists N > 0$ and $r' > 0$ such that $\|\mathrm{grad}^- f(x)\| \leq N$, $\forall x \in \mathrm{dom}(\partial^- f) \cap B(x_0, r')$.

and $F: I \times \Omega \to P_{fc}(H)$ is a multifunction such that

(j) $\forall x \in \Omega$, $t \to F(t,x)$ is measurable;

(jj)'' $\forall t \in I$, $x \to F(t,x)$ is $(\text{u.s.c.})_m$ on Ω;

(jjj) $\exists \gamma \in L^2(I,R^+)$: $\|F(t,x)\| = \sup\{\|z\|: z \in F(t,x)\} \leq \gamma(t)$, a.e. in I, $\forall x \in \Omega$.

As an immediate consequence of Theorem 3, we deduce a Kneser-type theorem (cf. Corollary 1). In addition, if $F: I \times \Omega \to P_{fc}(H)$ is a multifunction satisfying conditions (j), (jjj) and the following hypothesis:

(jj) for a.e. $t \in I$, $GrF(t,\cdot)$ is sequentially closed in $\Omega \times H_w$ (here H_w stands for the Hilbert space H equipped with the weak topology),

we establish a continuity result about the solution-multifunction $x \to S(x)$ (cf. Theorem 4). Moreover, in order to generate a continuous selector for this solution-multifunction, we are forced to strengthen the hypothesis (jj) on the orientor field "F" (cf. Theorem 5) with the following condition:

(jj)' $\exists K \in L^1(I,R^+)$: $h(F(t,x),F(t,y)) \leq k(t)\|x-y\|$, a.e. in I, $\forall x,y \in \Omega$.

Finally, we present an application of these results in the study of the existence of periodic solutions for a class of evolution inclusions involving the Fréchet subdifferential.

We want to observe that the class of proper, convex and lower semicontinuous functions is included in the class of lower semicontinuous functions with a φ-monotone subdifferential of order two (cf. [19]).

Therefore, we can say that our theorems extend the similar results obtained in [18].

2. Mathematical Preliminaries

Let X be a separable Banach space. We will be using the following notations:

$P_{f(c)}(X) = \{A \subseteq X\colon A$ nonempty, closed (and convex)$\}$,

If (Ω, Σ) is a measurable space, a multifunction $F\colon \Omega \to P_f(X)$ is said to be measurable, if for all $x \in X$, the function $\omega \to d(x, F(\omega)) = \inf\{ \| x - z \| : z \in F(\omega)\}$ is measurable. If $F(\cdot)$ is measurable, then $GrF = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times B(X)$, with $B(X)$ being the Borel σ-field of X (graph measurability), while the converse is true if we assume that there exists a complete, σ-finite measure μ defined on Σ. By S_F^p $(1 \leq p \leq \infty)$ we will denote the set of all measurable selectors of $F(\cdot)$ that belong in the Lebesgue-Bochner space $L^p(\Omega, X)$; i.e., $S_F^p = \{f \in L^p(\Omega, X) : f(\omega) \in F(\omega)\ \mu\text{-a.e.}\}$. In general, this set may be empty. It is easy to check using Aumann's selection theorem (cf. [21], Theorem 5.10), that for a graph measurable multifunction $F\colon \Omega \to 2^X\{\emptyset\}$, S_F^p is nonempty if and only if the function $\omega \to \inf\{ \| z \| : z \in F(\omega)\}$ belongs to $L^p(\Omega, R^+)$. Recall that a subset K of $L^p(\Omega, X)$ is decomposable if for every triple $(f, g, A) \in K \times K \times \Sigma$, we have $f\chi_A + g\chi_{A^c} \in K$, where χ_A denotes the characteristic function of the set A. Clearly, S_F^p is decomposable.

A subset A of X is said to be an absolute retract if, given any metric space Y and a closed $B \subseteq Y$ and a continuous function $f\colon B \to A$, there exists a continuous extension $f\colon Y \to A$ of f. Then A is said to be a R_δ-set if $A = \bigcap_{n \in \mathbb{N}} A_n$ for a decreasing sequence of compact absolute retracts A_n of X (cf. [15]).

Moreover, if $(A_n)_n$ are nonempty subsets of X, we define

$$s\text{-}\underline{\lim}_{n \to +\infty} A_n = \{x \in X : \exists (x_n)_n, x_n \in A_n, \forall n \geq 1 : x = s - \lim_{n \to +\infty} x_n\}$$

and

$$w\text{-}\overline{\lim}_{n \to +\infty} A_n = \{x \in X : \exists (x_k)_k, x_k \in A_{n_k}, \forall k \geq 1 : x = w - \lim_{k \to +\infty} x_k\},$$

(here w-denotes the weak topology). We say that A_n's converge to A in the Kuratowski-Mosco sense (denoted by $A_n \to A$) if and only if $s\text{-}\underline{\lim}_{n \to +\infty} A_n = w\text{-}\overline{\lim}_{n \to +\infty} A_n$. Suppose T is a topological space. A multifunction $F\colon T \to P(X)$ is said to be upper semicontinuous, $(\text{u.s.c.})_t$, if for $C \subseteq X$ nonempty closed, we have that $F^-(C) = \{x \in X : F(x) \cap C \neq \emptyset\}$ is closed in X. Also F is lower semicontinuous, $(\text{l.s.c.})_t$, if $F^+(C) = \{x \in X : F(x) \subset C\}$ is closed in X. This definition of $(\text{l.s.c.})_t$ is equivalent to saying that if $t_n \to t$ in T then $F(t) = s\text{-}\underline{\lim}_{n \to +\infty} F(t_n) = \{x \in X : \lim_{n \to +\infty} d(x, F(t_n)) = 0\}$.

Recall that on $P_f(X)$ we can define a generalized metric, known in the literature as the "Hausdorff metric", by setting for $A, B \in P_f(X)$,

$$h(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(b, A) : b \in B\}\}$$

(where $d(a, B) = \inf\{ \| a - b \| : b \in B\}$, similarly for $d(b, A)$). A multifunction F: $T \to P_f(X)$ is said to be Hausdorff continuous (H-continuous) if it is continuous from T into the metric space $(P_f(X), h)$. Moreover, F is said to be Hausdorff upper semicontinuous, $(\text{u.s.c.})_m$, if for every $t \in T$ and for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|t - t'| < \delta \Rightarrow F(t') \subseteq F(t) + \varepsilon B_1$, where B_1 is the unit ball in X.

Let $I = [0,T]$ be furnished with the σ-finite and complete Lebesgue measure, H a (real) separable Hilbert space, Ω is an open subset of H and a function $f: \Omega \to R \cup \{+\infty\}$, the multifunction $\partial^- f: \Omega \to 2^H$, defined as follows

$$\partial^- f(x) = \begin{cases} \emptyset, & \text{if } f(x) = +\infty, \\ \left\{\alpha \in H: \liminf\limits_{y \to x} \dfrac{f(y) - f(x) - \langle \alpha, y - x\rangle}{\| y - x \|} \geq 0\right\}, & \text{if } f(x) < +\infty, \end{cases}$$

where x is a fixed element of Ω, is called the *Fréchet subdifferential* of f. The sets $\text{dom}(f) = \{x \in \Omega: f(x) < +\infty\}$ and $\text{dom}(\partial^- f) = \{x \in \Omega: \partial^- f(x) \neq \emptyset\}$ are the domains of f and $\partial^- f$ respectively. For every $x \in \text{dom}(\partial^- f)$, we denote by $\text{grad}^- f(x)$ the element of minimal norm of $\partial^- f(x)$. Recall that the values of $\partial^- f$ are closed and convex.

If $f: \Omega \to R \cup \{+\infty\}$ is a lower semicontinuous function, we say that f has a *φ-monotone subdifferential of order two* if there exists a continuous map $\varphi: [\text{dom}(f)]^2 \times R^2 \to R^+$ such that

for every $x, y \in \text{dom}(\partial^- f)$ and for every $\alpha \in \partial^- f(x)$ and $\beta \in \partial^- f(y)$, (2.1)

we have $\langle \alpha - \beta, x - y\rangle \geq -\varphi(x, y, f(x), f(y))(1 + \|\alpha\|^2 + \|\beta\|^2)\|x - y\|^2$.

In the following, we consider the multivalued Cauchy problem:

$$\begin{cases} x'(t) \in -\partial^- f(x(t)) + F(t, x(t)), & \text{a.e. on } I, \\ x(0) = x_0, & x_0 \in \text{dom}(f). \end{cases} \tag{1}$$

By a "strong solution" of (1), we mean a function $x \in C(I, \Omega)$ such that $x(\cdot)$ is absolutely continuous on any compact subset of $(0,T)$ and with the property

(1) $x(t) \in \text{dom}(f)$, a.e. on I;

(2) $\exists h \in L^2(I, H)$ such that

$$h(t) \in F(t, x(t)) \text{ and } x'(t) \in -\partial^- f(x(t)) + h(t) \text{ a.e. on } I;$$

(3) $x(0) = x_0$.

Recall that an absolutely continuous function $x: [0,T] \to H$ is differentiable almost everywhere (see [3], Theorem 2.1) and so in problem (1) the derivative $x'(\cdot)$ is a strong derivative.

We make the following hypothesis on the function f:

$H(f)_0$: $f: \Omega \to R \cup \{+\infty\}$ is a function with the properties:

(*i*) f has a φ-monotone subdifferential of order two;

(*ii*) $\exists r > 0$ such that $clB(x_0, r) \subset \Omega$ and the set $L(c) = \{x \in clB(x_0, r): \|x\|^2 + f(x) \leq c\}$ is compact in H, $\forall c \in R$,

(*iii*) $\exists \tilde{k}, \tilde{r} > 0$ such that $f(x) \leq \tilde{k}$, $\forall x \in \text{dom}(f) \cap B(x_0, \tilde{r})$;

(*iv*) $\exists N > 0$ and $r' > 0$ such that $\|\text{grad}^- f(x)\| \leq N$, $\forall x \in \text{dom}(\partial^- f) \cap B(x_0, r')$.

Moreover, we will need the following hypothesis on the orientor field F:

$H(F)_0$: $F: I \times \Omega \to P_{fc}(H)$ is a multifunction such that

- (j) $\forall x \in \Omega$, $t \to F(t,x)$ is measurable;
- (jj) for a.e. $t \in I$, $GrF(t,\cdot)$ is sequentially closed in $\Omega \times H_w$ (here H_w stands for the Hilbert space H equipped with the weak topology);
- (jjj) $\exists \gamma \in L^2(I, R^+)$: $\| F(t,x) \| = \sup\{ \| z \| : z \in F(t,x)\} \leq \gamma(t)$, a.e. in I, $\forall x \in \Omega$.

$H(F)_1$: $F: I \times \Omega \to P_f(H)$ is a multifunction with properties (j), (jjj) and the following hypothesis

- $(jj)'$ $\exists K \in L^1(I, R^+)$: $h(F(t,x), F(t,y)) \leq k(t) \| x - y \|$, a.e. in I, $\forall x, y \in \Omega$.

$H(F)_2$: $FI \times \Omega \to P_{fc}(H)$ is a multifunction with properties (j), (jjj) and the following hypothesis

- $(jj)''$ $\forall t \in I$, $x \to F(t,x)$ is $(\text{u.s.c.})_m$ on Ω.

$H(F)_3$: $FI \times \Omega \to P_f(H)$ is a multifunction with property (jjj) and the following hypotheses:

- $(j)'$ $(t,x) \to F(t,x)$ has measurable graph,
- $(jj)'''$ $\forall t \in I$, $x \to F(t,x)$ is $(\text{l.s.c.})_t$ on Ω.

First, if

$$k = \max\{ \| \gamma \|_2, \widetilde{k} \}, \tag{2.2}$$

we observe that from our hypotheses on f, it is possible go find the following positive numbers:

$$\exists\, \overline{r} > 0 \text{ such that } f(x) \geq f(x_0) - 1, \ \ \forall x \in clB(x_0, \overline{r}), \tag{2.3}$$

$\exists r^* > 0$ such that φ is bounded in the set

$$\{clB(x_0, r^*)^2 \cap \operatorname{dom}(f)^2\} \times [f(x_0) - 1, k + k^2/2]; \tag{2.4}$$

so, we put

$$R^* = \min\{r, \widetilde{r}, r^*, \overline{r}, r'\} \tag{2.5}$$

(cf. $H(f)$, (2.3) and (2.4)). Now, we take the following version of Theorem 3.6 of Tosques (cf. [20], p. 82) into account:

Proposition 1: *Let $f: \Omega \to R \cup \{+\infty\}$ be a function with a φ-monotone subdifferential of order two. Then, $\forall x_0 \in dom(f)$, $\forall M \geq 0$ $\exists \overline{T} > 0$, $\exists \widehat{r} > 0$ with the property:*

$$\forall u \in clB(x_0, \widehat{r}) \cap dom(f) \text{ with } f(u) \leq M,$$

$\forall T^* > 0$ and $\forall h \in L^2([0,T^*], H)$ with $\| h \|_2 \leq M$, there exists a unique function $u_h: [0, \widehat{T}] \to dom(f)$, where $\widehat{T} = min\{T^*, \overline{T}\}$, that is a strong solution of the Cauchy problem

$$(P_u)_h \begin{cases} x' \in -\partial^- f(x) + h \\ x(0) = u, \end{cases}$$

with the properties

- (i) *$u_h \in H^{1,2}([0,T], H)$ is continuous on $[0, \widehat{T}]$ and absolutely continuous on the compact subsets of $(0, \widehat{T})$;*
- (ii) *$u_h(t) \in dom(\partial^- f)$ and $u_h'(t) \in -\partial^- f(u_h(t)) + h(t)$ a.e. in $[0, \widehat{T}]$;*

(*iii*) $u_h(0) = u$;
(*iv*) $u_h \in L^2([0,\widehat{T}], H)$;
(*v*) $\int_0^t \| u_h'(s) \|^2 ds \leq 2(f(u))) - (f \circ u_h)(t) + \int_0^t \| h(s) \|^2 ds$, $\forall t \in [0,\widehat{T}]$;
(*vi*) $f \circ u_h$ *is absolutely continuous on* $[0,\widehat{T}]$;
(*vii*) $(f \circ u_h)'(t) = \langle h(t) - u_h'(t), u_h(t)\rangle$ *a.e. in* $[0,\widehat{T}]$;
(*viii*) $(f \circ u_h)(t_2) - (f \circ u_h)(t_1) = \int_{t_2}^{t_1} \langle grad^- f(u_h(t)), u_h'(t)\rangle dt$, $\forall t_1, t_2 \in [0,\widehat{T}]$.

Remark 1: In [20], the author observed that, if $h_1, h_2 \in L^2([0,T],H)$ and u_1, u_2 are strong solutions of $u_i' \in -\partial^- f(u) + h_i$ $(i = 1,2$ respectively) on $[0,T]$, then $\forall t_0$, $t \in [0,T]$ with $t_0 \leq t$ we have

(*ix*)
$$\| u_1(t) - u_2(t) \| \leq (\| u_1(t_0) - u_2(t_0) \| + \int_{t_0}^t \| h_1(s) - h_2(s) \| ds)$$
$$\cdot \exp \int_{t_0}^{t} \varphi(u_1(s), u_2(s), (f \circ u_1)(s), (f \circ u_2)(s))(1 + \| u_1'(s) - h_1(s) \|^2$$
$$+ \| u_2'(s) - h_2(s) \|^2) ds \text{ (where we put } 0.\infty = \infty).$$

Fixed $x_0 \in \mathrm{dom}(f)$, k as in (2.2) and $T^* = T$, we can say that there exist

$$\bar{T} = \bar{T}(x_0, k) > 0 \text{ and } \rho = \rho(x_0, k) > 0$$

such that $\forall u \in clB(x_0, \rho) \cap \mathrm{dom}(f)$ with $f(u) \leq k$, and $\forall h \in L^2([0,T],H)$ with $\| h \|_2 \leq k$, there exists a unique function u_h: $[0,\widehat{T}] \to \mathrm{dom}(f)$, where $\widehat{T} = \min\{T, \bar{T}\}$, that is a strong solution of the Cauchy problem $(P_u)_h$ with the properties (*i*)-(*viii*) of Proposition 1.

Therefore, if we fix (cf. (2.5))

$$R = \min\left\{\rho, \frac{R^*}{2}\right\}, \tag{2.7}$$

we can say that, $\forall u \in clB(x_0, R) \cap \mathrm{dom}(f)$ and $\forall h \in L^2([0,T],H)$ with $\| h \|_2 \leq k$, there exists a unique strong solution of the problem $(P_u)_h$, with the properties mentioned in Proposition 1 and, moreover, we have that the following conditions are satisfied:

$$f(x) \geq f(x_0) - 1, \quad \forall x \in clB(x_0, 2R) \cap \mathrm{dom}(f), \tag{2.8}$$

$$L(c) = \left\{x \in clB(x_0, 2R) \colon \| x \|^2 + f(x) \leq c\right\} \text{ is compact in } H, \ \forall c \in R, \tag{2.9}$$

$$\begin{aligned} L = \sup\{\varphi(x_1, x_2, y_1, y_2) \colon x_1, x_2 \in clB(x_0, 2R) \cap \mathrm{dom}(f), \\ y_1, y_2 \in [f(x_0) - 1, k + k^2/2]\}, \end{aligned} \tag{2.10}$$

$$f(x) \leq k, \quad \forall x \in clB(x_0, R) \cap \mathrm{dom}(f), \tag{2.11}$$

$$\| \mathrm{grad}^- f(x) \| \leq N, \quad \forall x \in clB(x_0, 2R) \cap \mathrm{dom}(\partial^- f), \tag{2.12}$$

(where φ: $[\mathrm{dom}(f)]^2 \times R^2 \to R^+$ is the mentioned continuous map that verifies (2.1)).

Now, we choose T' such that $T' = \frac{R^2}{2(k+1-f(x_0))+k^2}$ and we define $T_0 = \min\{\widehat{T}, T'\}$. Then we can consider the solution of the problem $(P_u)_h$, $\forall u \in clB(x_0, R) \cap \mathrm{dom}(f)$, defined in $[0, T_0]$ (cf. $H(f)$).

Next let V be the following set:

$$V = \{h \in L^2([0,T_0], H): \| h(t) \| \leq \gamma(t) \text{ a.e. in } [0, T_0]\} \tag{2.13}$$

and, $\forall u \in clB(x_0, R) \cap \mathrm{dom}(f)$, we consider the function:

$$\begin{gathered} s_u: V \to C([0,T_0], \Omega) \\ h \to s_u(h) = u_h \end{gathered} \tag{2.14}$$

where u_h is the unique solution of the problem $(P_u)_h$.

3. Topological Structure of the Solution Set

In order to prove the nonemptiness and the topological structure of the solution set $S(x_0) \subseteq C(I, \Omega)$ of (1), we will need some auxiliary results. The first is an approximation lemma which can be proved as Lemma 1 of [9] with some appropriate modifications (here the multifunction is defined in $I \times \Omega$, where Ω is an open subset of the Hilbert space H).

Lemma 1: *Let $F: I \times \Omega \to P_{fc}(H)$ be a multifunction satisfying the hypothesis $H(F)_2$, then there exists a sequence of multifunction $F_n: I \times \Omega \to P_{fc}(H)$, $n \geq 1$, with the properties:*

(i) *$\forall n \geq 1$ and $\forall x \in \Omega$ there exist $k_n(x) > 0$ and $\varepsilon_n = \varepsilon_n(x) > 0$ such that if $x_1, x_2 \in clB(x, \varepsilon_n) = \{y \in \Omega: \| x - y \| \leq \varepsilon_n\}$, then $h(F_n(t, x_1), F_n(t, x_2)) \leq k_n(x)\gamma(t) \| x_1 - x_2 \|$, a.e. on I (i.e., a.e. on I, $F_n(t, \cdot)$ is locally h-Lipschitz);*

(ii) *$F(t,x) \subseteq \ldots \subseteq F_n(t,x) \subseteq F_{n+1}(t,x) \subseteq \ldots$, $\forall (t,x) \in I \times \Omega$;*

(iii) *$\| F_n(t,x) \| \leq \gamma(t)$ a.e. in I, $\forall x \in \Omega$;*

(iv) *$F_n(t,x) \to F(t,x)$ as $n \to \infty$ for every $(t,x) \in I \times \Omega$*

and finally there exist maps $u_n: I \times \Omega \to H$, $n \geq 1$, measurable in I, locally-Lipschitz in x (as $F_n(t, \cdot)$) and $u_n(t,x) \in F_n(t,x)$ for every $(t,x) \in I \times \Omega$.

Now, we prove the following result concerning the solution map $p: (u,h) \mapsto p(u,h)$, where $p(u,h)$ is the unique strong solution of the above Cauchy problem $(P_u)_h$.

Lemma 2: *If hypothesis $H(f)$ holds and R is the positive number defined in (2.7), then the solution map $p: [clB(x_0, R) \cap dom(f)] \times V \to C([0,T_0], \Omega)$ is sequentially continuous by considering on $L^2([0,T_0], H)$ the weak topology.*

Proof: First, we will show that if $u \in clB(x_0, R) \cap \mathrm{dom}(f)$, then $\forall h \in V$, we have (cf. (2.13))

$$p(u,h)(t) = u_h(t) \in clB(u, R), \; \forall t \in [0, T_0]. \tag{3.1}$$

Indeed, if $t \in [0, T_0]$, from (v) of Proposition 1, it follows

$$\| u_h(t) - u \| \leq \int_0^t \| u_h'(s) \| \, ds \leq \sqrt{t} \left(\int_0^t \| u_h'(s) \|^2 ds \right)^{1/2} \tag{3.2}$$

$$\leq \sqrt{t}[2(f(u)) - f(u_h(t))] + k^2)^{1/2},$$

where k is defined as in (2.2).

Let $\widetilde{T} = \sup\{t \in [0, T_0]: \| u_h(s) - u \| \leq R, \forall s \in [0, t]\}$, by the continuity of u_h we have that $\widetilde{T} > 0$. Therefore, to obtain (3.1) it is sufficient to prove that $\widetilde{T} = T_0$. Indeed, if $\widetilde{T} < T_0$, by (2.14), (2.8) and by recalling that $T_0 \leq \frac{R^2}{2(k+1-f(x_0)) + k^2}$, it follows that $\| u_h(\widetilde{T}) - u \| \leq \sqrt{T_0}[2(k+1-f(x_0)) + k^2]^{1/2} \leq R$.

But this inequality, being u_h continuous in $\widetilde{T}$, contradicts the definition of $\widetilde{T}$.

Now, by (3.1) and by (*ii*) of Proposition 1, we deduce that $u_h(t) \in clB(x_0, 2R) \cap \operatorname{dom}(f)$, and therefore, by using (2.8), (3.2) and (2.11), we obtain that

$$f(u_h(t)) \in [f(x_0) - 1, M_2], \forall t \in [0, T_0], \ \forall h \in V, \tag{3.3}$$

where $M_2 = k + \frac{k^2}{2}$.

Moreover, using (3.1), we claim that $s_u(V)$ (cf. (2.14)) is a subset of the closed and convex set

$$K_u = \{x \in C([0, T_0], \Omega): x(t) \in clB(u, R), \forall t \in [0, T_0]\}. \tag{3.4}$$

Next, we are able to prove that p is sequentially continuous by considering on $L^2([0, T_0], H)$ the weak topology. Indeed, we consider a sequence $(u_n, h_n)_n \subseteq [clB(x_0, R) \cap \operatorname{dom}(f)] \times V$ such that $(u_n)_n$ converges to u in H (where $u \in clB(x_0, R) \cap \operatorname{dom}(f)$) and $(h_n)_n$ weakly converges to h in $L^2([0, T_0], H)$. In order to prove that $p(u_n, h_n)$ converges to $p(u, h)$ in $C([0, T_0], \Omega)$, we set $x_n = p(u_n, h_n)$, $\forall n \in N$, and $x = p(u, h)$. Taking Remark 1 into account, we have

$$\begin{aligned} \| x_n(t) - x(t) \| \leq & (\| x_n(0) - x(0) \| + \int_0^t \| h_n(s) - h(s) \| ds) \\ & \cdot \exp \int_0^t \varphi(x_n(s), x(s), (f \circ x_n)(s), (f \circ x)(s))) \\ & \cdot (1 + \| x_n'(s) - h_n(s) \|^2 \mid \| x'(s) - h(s) \|^2) ds \end{aligned} \tag{3.5}$$

and so, being $x_n(t)$, $x(t) \in B(x_0, 2R) \cap \operatorname{dom}(\partial^- f)$, from (*vii*), (*viii*) and (2.12), we can say that

$$\| x_n'(t) \| \leq \| \operatorname{grad}^- f(x_n(t)) \| + \| h_n(t) \| \leq N + \gamma(t), \text{ a.e. in } [0, T_0], \tag{3.6}$$

$$\| x'(t) \| \leq \| \operatorname{grad}^- f(x(t)) \| + \| h(t) \| \leq N + \gamma(t), \text{ a.e. on } [0, T_0]. \tag{3.6$'$}$$

Then, we have

$$\| x_n'(t) - h_n(t) \|^2 \leq N^2 + 4\gamma^2(t) + 4N\gamma(t)$$

and, therefore taking (2.2) and (2.10) into account, we can deduce that

$$\exp \int_0^t \varphi(x_n(s), x(s), (f \circ x_n)(s), (f \circ x)(s))(1 + \| x_n'(s) - h_n(s) \|^2$$

$$+ \| x'(s) - h(s) \|^2)ds \leq \exp(L(T_0 + 2N^2T_0 + 8k^2 + 8Nk\sqrt{T_0})) = C.$$

Then, it follows that

$$\| x_n(t) \| \leq \| x \|_\infty + 2C(R + k\sqrt{T_0}) = M_1 < \infty, \ \forall t \in [0, T_0], \tag{3.7}$$

taking the properties (3.3) and (3.7) into account, we deduce that

$$\{x_n(t)\}_{n \geq 1} \subseteq \{y \in H: \| y \|^2 + f(y) \leq M_1^2 + M_2 = M_3\} = L(M_3) \quad \forall t \in [0, T_0]. \tag{3.8}$$

Also, if $s, t \in [0, T_0]$, $s \leq t$, $n \in N$, we have (cf. (3.6))

$$\| x_n(t) - x_n(s) \| \leq \int_s^t \| x_n'(\tau) \| d\tau \leq \sqrt{t-s}(N^2T_0 + k^2 + 2Nk\sqrt{T_0}) = M_4\sqrt{t-s}$$

from which we can say that $(x_n)_n$ is equi-Hölder continuous in $[0, T_0]$. Moreover, by (3.8), we have that $\{x_n(t)\}_{n \geq 1}$ is included in the compact set $L(M_3)$, $\forall t \in [0, T_0]$. Thus, by using the Arzelà-Ascoli Theorem, it follows that the set $\{x_n\}_{n \geq 1}$ is relatively compact in $C([0, T_0], \Omega)$. Therefore, we may assume that, by passing to a subsequence if it is necessary, $x_n \rightarrow y$ in $C([0, T_0], \Omega)$. Finally, applying the lemma of [7], we have that $y = p(u, h)$, so we can conclude that the solution map p is sequentially continuous by considering on $L^2([0, T_0], H)$, the weak topology.

We also need the following two existence theorems for the problem (1).

Theorem 1: *Let Ω be an open subset of a (real) separable Hilbert space H, f: $\Omega \rightarrow R \cup \{+\infty\}$ be a function satisfying $H(f)$ and $F: I \times \Omega \rightarrow P_{fc}(H)$ be a multifunction satisfying the hypothesis $H(F)_0$, then we can choose $T_0 > 0$ and $R > 0$ as in Remark 1 such that $\forall u \in clB(x_0, R) \cap dom(f)$ the set $S(u)$ is nonempty and compact in $C([0, T_0], \Omega)$.*

Proof: First, we observe that the set

$$V = \{h \in L^2([0, T_0], H): \| h(t) \| \leq \gamma(t) \text{ a.e. in } [0, T_0]\}$$

is bounded, convex, closed and weakly compact in $L^2([0, T_0], H)$. Using analogous considerations made in the first part of the proof of Lemma 2, we can say:

$$u_h(t) \in clB(u, R), \ \forall t \in [0, T_0], \ \forall h \in V, \tag{3.9}$$

by which we deduce that $u_h(t) \in clB(x_0, 2R) \cap \text{dom}(f)$, $\forall t \in [0, T_0]$, $\forall h \in V$, and so we can write that $f(u_h(t)) \in [f(x_0) - 1, k + \frac{k^2}{2}]$, $\forall h \in V$, $\forall t \in [0, T_0]$.

Now, denoted with

$$K_u = \{x \in C([0, T_0], \Omega): x(t) \in clB(u, R), \forall t \in [0, T_0]\},$$

we have that K_u is closed and convex and $s_u(V) \subseteq K_u$. Moreover, we claim that $s_u(V)$ is relatively compact. Indeed, if $u_h \in s_u(V)$, taking the property (v) of Proposition 1 into account, we obtain that

$$\| u_h' \|_2 \leq [2(k+1-f(x_0)) + k^2]^{\frac{1}{2}}.$$

Therefore, it is possible to write the following inequality

$$\| u_h(t^*) - u_h(t) \| \leq [2(k+1-f(x_0)) + k^2]^{\frac{1}{2}} \mid t^* - t \mid^{\frac{1}{2}}, \ \forall t^*, t \in [0, T_0],$$

from which it is easy to see that $s_u(V)$ is uniformly equicontinuous. Moreover, for every $t \in [0, T_0]$, the set $W(t) = \{u_h(t) \in \Omega : u_h \in s_u(V)\}$ is relatively compact in H. Therefore, from the Arzelà-Ascoli Theorem, it follows that $s_u(V)$ is relatively compact in $C([0, T_0], \Omega)$.

Now, we want to show that function s_u is continuous from V with the weak topology on $L^2([0, T_0], H)$ into K_u with the topology of $C([0, T_0], \Omega)$.

First we observe that, since $s_u(V)$ is relatively compact in the space $C([0, T_0], \Omega)$, it is sufficient to prove that $Gr(s_u)$ is sequentially closed in $V_w \times s_u(V)$. Then let $(h_n, x_n) \in Gr(s_u)$, $(h_n, x_n) \rightarrow (h, x)$ in $L^2([0, T_0]H)_w \times C([0, T_0], \Omega)$. By the lemma of [7] we deduce that $x = s_u(h)$. Therefore, the function s_u is continuous. Then, by Mazur's Theorem we can say that the set $M = cl\,\mathrm{conv}\, s_u(V)$ is convex and compact.

Now, let $L: V \rightarrow 2^{L^2([0, T_0], H)}$ be the multifunction defined by

$$L(h) = S^2_{F(\cdot, s_u(h)(\cdot))}, \quad \forall h \in V.$$

It is possible to prove that the multifunction L has nonempty values. To this end with fixed h, there exists $(v_n)_n$, where v_n is a simple function, such that $v_n(t) \rightarrow s_u(h)(t)$ a.e. in $[0, T_0]$ as $n \rightarrow \infty$. Since, for a.e. $t \in [0, T_0]$, $GrF(t, \cdot)$ is sequentially closed in $\Omega \times H_w$, we have that

$$w\text{-}\overline{\lim}_{n \rightarrow +\infty} F(t, v_n(t)) \subset F(t, s_u(h)(t)).$$

Since for every $n > 1$, $v_n(\cdot)$ is a simple function and $F(\cdot, x)$ is measurable, we can easily check that $t \rightarrow F(t, v_n(t))$ is measurable. Applying Aumann's selection theorem we can get a measurable selection $g_n(\cdot)$ of $F(\cdot, v_n(\cdot))$ such that $\| g_n(t) \| \leq \gamma(t)$ a.e. on $[0, T_0]$, for $n > 1$. By passing to a subsequence if necessary, we may assume that $g_n \rightarrow g$ weakly in $L^2([0, T_0], H)$.

From Theorem 3.1 of [16] we have that $g \in S^2_{F(\cdot, s_u(h)(\cdot))}$. Then we can say that L has nonempty values. Moreover, it is easy to see that for every $h \in V$, we have $L(h)$ is a closed and convex subset of V. Now, if we consider in V the weak topology, we claim that $L: V_w \rightarrow P_{fc}(V_w)$ is (u.s.c.)$_t$. Recalling that V_w is compact and metrizable, to show the upper semicontinuity of $L(\cdot)$, it is enough to show that GrL is sequentially closed in $V_w \times V_w$. To this end, let $(h_n, f_n)_n$, $(h_n, f_n) \in GrL$, with the property $(h_n, f_n) \rightarrow (h, f)$ in $V_w \times V_w$. Then, for each $n > 1$, we have $s_u(h_n) \in M$ and we know that M is compact in $C([0, T_0], H)$. So by passing to a subsequence if necessary, we may assume that $s_u(h_n) \rightarrow q$ in $C([0, T_0], \Omega)$. Now taking that $h_n \rightarrow h$ in V_w into account, from the proposition of [7], we conclude that $q = s_u(h)$. Using again Theorem 3.1 of [16], we have that $f(t) \in cl\,\mathrm{conv}\, w\text{-}\overline{\lim}_{n \rightarrow +\infty} F(t, s_u(h_n)(t)) \subseteq F(t, s_u(h)(t))$, a.e. on $[0, T_0]$. Therefore, $f \in S^2_{F(\cdot, s_u(h)(\cdot))}$. Then we can conclude that $(h, f) \in GrL$. So L is (u.s.c.)$_t$ as claimed.

By the fixed point theorem (see [11]), we obtain there exists $h \in V$ such that

$h \in L(h)$. Clearly, u_h is the desired solution of the Cauchy problem (1) (satisfying the mentioned properties of Proposition 1).

From the above proof, we see that the solution set of (1) is a subset of M, which we know is a compact subset of $C([0,T_0],\Omega)$. So to prove the compactness of the solution set of (1) in $C([0,T_0],\Omega)$, it is enough to show that it is closed in $C([0,T_0],\Omega)$. So let $(x_n)_n$, where x_n is the solution of (1) such that $x_n \to x$ in $C([0,T_0],\Omega)$. We have:

$$\begin{cases} x_n'(t) \in -\partial^- f(x_n(t)) + h_n(t) \text{ a.e.} \\ x_n(0) = u, \end{cases}$$

with $h_n \in V$. By passing to a subsequence if necessary, we may assume that $h_n \to h$ weakly in V. Again from the lemma in [7], we deduce that $x = s_u(h)$. Now, using Theorem 3.1 of [16], we have that $h \in S^2_{F(\cdot,x(\cdot))}$. Therefore, the solution set $S(u)$ is compact in $C([0,T_0],\Omega)$.

Theorem 2: *Let Ω be an open subset of a (real) separable Hilbert space H, $f:\Omega \to R \cup \{+\infty\}$ be a function satisfying $H(f)$ and $F: I \times \Omega \to P_f(H)$ be a multifunction satisfying the hypothesis $H(F)_3$, then we can choose $T_0 > 0$ and $R > 0$ as in Remark 1 such that $\forall u \in clB(x_0,R) \cap dom(f)$, $S(u)$ is a nonempty subset of $C([0,T_0],\Omega)$.*

Proof: Using analogous considerations made in the first part of the proof of Theorem 1, we can say that the set

$$V = \{h \in L^2([0,T_0],H): \| h(t) \| \le \gamma(t) \text{ a.e. in } [0,T_0]\}$$

is bounded, convex, closed and weakly compact in $L^2([0,T_0],H)$. Moreover, if u_h is the unique solution of the problem $(P_u)_h$, we have:

$$u_h(t) \in clB(u,R), \ \forall t \in [0,T_0], \ \ \forall h \in V,$$

$$u_h(t) \in clB(x_0,2R) \cap \mathrm{dom}(f), \ \ \forall t \in [0,T_0], \ \forall h \in V,$$

$$f(u_h(t)) \in \left[f(x_0) - 1, k + \frac{k^2}{2}\right], \ \ \forall h \in V, \ \ \forall t \in [0,T_0]$$

and additionally, the set $s_u(V)$ is compact in $C([0,T_0],\Omega)$ and the set $M = cl\,\mathrm{conv}\, s_u(V)$ is convex and compact in $C([0,T_0],\Omega)$. Now, let $R: M \to P_f(L^1(H))$ be defined by $R(x) = S^1_{F(\cdot,x(\cdot))}$, $\forall x \in M$. From Theorem 4.1 of [16], we have that $R(\cdot)$ is (l.s.c.)$_t$. Since the values of R are nonempty (cf. Theorem 4.1 of [21]), closed and decomposable, we can apply Fryszkowski's selection theorem [12] and we get a continuous function $r: M \to L^1(H)$ such that $r(x) \in R(x)$ for every $x \in M$. Then we can observe that the function $\eta = s_u \circ r: M \to M$ has a fixed point, i.e. there exists $x \in M$ such that $\eta(x) = x$. Then x is solution for the following problem

$$\begin{cases} x'(t) \in -\partial^- f(x) + r(x) \\ x(0) = u, \end{cases}$$

and so, being $r(x)(t) \in F(t,x(t))$ a.e. on $[0,T_0]$, we can conclude that $x \in S(u)$.

Moreover, we need to prove the following:

Lemma 3: *Let Ω be an open subset of a (real) separable Hilbert space H, $f:\Omega \to R \cup \{+\infty\}$ be a function satisfying $H(f)$, $x_0 \in dom(f)$, and $g: I \times \Omega \to H$ be a* *function such that:*

(α) $\forall x \in \Omega$, $t \mapsto g(t,x)$ *is measurable in* I;

($\alpha\alpha$) $\exists \gamma \in L^2(I, R^+)$: $\| g(t,x) \| \leq \gamma(t)$, *a.e. on* I; $\forall x \in \Omega$,

($\alpha\alpha\alpha$) *a.e.* $t \in I$, $x \mapsto g(t,x)$ *is locally Lipschitz (i.e.,* $\forall x \in \Omega$, *there exists* $k(x) > 0$ *and* $\epsilon_x > 0$ *such that if* $x_1, x_2 \in clB(x, \varepsilon_x)$, *then* $\| g(t,x_1) - g(t,x_2) \| \leq k(x)\gamma(t) \| x_1 - x_2 \|$, *a.e. on* I).

Under these assumptions we can say that $\forall u \in clB(x_0, R) \cap dom(f)$ *the following Cauchy problem:*

$$(P_u)_g \begin{cases} x' \in -\partial^- f(x) + g(t,x) \\ x(0) = u \end{cases}$$

has a unique strong solution defined on $[0, T_0]$.

Proof: Fix $u \in clB(x_0, R) \cap \mathrm{dom}(f)$. From ($\alpha\alpha$), we deduce that there exists a Lebesgue null-set $M \subseteq I$ such that $\forall t \in I \backslash M$ the function $x \mapsto g(t,x)$ is locally Lipschitz.

Now, we consider the multifunction $F: I \times \Omega \rightarrow P_{fc}(H)$, where

$$F(t,x) = \begin{cases} \{g(t,x)\}, & (t,x) \in (I - M) \times \Omega \\ \{0\}, & \text{otherwise.} \end{cases}$$

Using Theorem 2, we have that there exists a solution of the problem $(P_u)_g$ defined on $[0, T_0]$. Now, we prove that there exists an interval $[0, b]$, where $b \in]0, T_0]$, such that the solution of the problem $(P_u)_g$ is unique in this interval. In fact, if $x_1: [0, T_0] \rightarrow \Omega$ and $x_2: [0, T_0] \rightarrow \Omega$ are two strong solutions of the problem $(P_u)_g$, then we find two functions β_1 and β_2 with the properties:

$$\beta_1(t) \in \partial^- f(x_1(t)),\ \beta_2(t) \in \partial^- f(x_2(t)) \text{ a.e. on } [0, T_0]$$

$$x_1'(t) \in -\beta_1(t) + g(t, x_1(t)), \text{ a.e. on } [0, T_0],$$

$$x_2'(t) \in -\beta_2(t) + g(t, x_2(t)), \text{ a.e. on } [0, T_0].$$

Therefore, taking $H(f)(i)$ into account, we have

$$\begin{aligned} \langle x_1'(t) - x_2'(t), x_1(t) - x_2(t) \rangle &= -\langle \beta_1(t) - \beta_2(t), x_1(t) - x_2(t) \rangle \\ &\quad + \langle g(t, x_1(t)) - g(t, x_2(t)), x_1(t) - x_2(t) \rangle \\ &\leq L(1 + \| \beta_1(t) \|^2 + \| \beta_2(t) \|^2) \| x_1(t) - x_2(t) \|^2 \\ &\quad + \langle g(t, x_1(t)) - g(t, x_2(t)), x_1(t) - x_2(t) \rangle. \end{aligned} \tag{3.10}$$

Now, using analogous considerations made previously in Lemma 2, we can say

$$\| \beta_i(t) \|^2 \leq N^2 + 4\gamma^2(t) + 4N\gamma(t), \text{ a.e. on } [0, T_0],\ i = 1, 2,$$

and so, by (3.10), let $\delta(\cdot) = L(1 + 2N^2 + 8\gamma^2(\cdot) + 8N\gamma(\cdot)) \in L^1([0, T_0], R^+)$, we get

$$\int_0^t \langle x_1'(s) - x_2'(s), x_1(s) - x_2(s)\rangle ds$$

$$\leq \int_0^t \delta(s) \| x_1(s) - x_2(s) \|^2 ds$$

$$+ \int_0^t \langle g(s, x_1(s)) - g(s, x_2(s)), x_1(s) - x_2(s)\rangle ds, \ \forall t \in [0, T_0]. \tag{3.11}$$

Now, we can observe that $x_1(t), x_2(t) \in clB(u, R)$, $\forall t \in [0, T_0]$ and, recalling the hypothesis $(\alpha\alpha\alpha)$, we can find $k(u) > 0$ and $\varepsilon_u > 0$ such that then $\| g(t, y_1) - g(t, y_2) \| \leq k(u)\gamma(t) \| y_1 - y_2 \|$, $\forall y_1, y_2 \in clB(u, \varepsilon_u)$, a.e. on I. By continuity of functions x_1, x_2 in the point 0, we can say that there exists $b \in]0, T_0]$ with the property $x_1(t), x_2(t) \in clB(u, \varepsilon_u)$, $\forall t \in [0, b]$. Therefore, we deduce that

$$\| g(t, x_1(t)) - g(t, x_2(t)) \| \leq k(u)\gamma(t) \| x_1(t) - x_2(t) \|, \ \forall t \in [0, b]. \tag{3.12}$$

Then, by (3.11) and (3.12) we obtain

$$\| x_1(t) - x_2(t) \|^2 \leq 2 \int_0^t [\delta(s) + k(u)\gamma(s)] \| x_1(s) - x_2(s) \|^2 ds, \ \forall t \in [0, b], \tag{3.13}$$

and thus, using Gronwall's inequality, we deduce that $x_1 = x_2$ in $[0, b]$. In order to prove the uniqueness in the interval $[0, T_0]$, we set $T^* = \sup\{t \in [0, T_0]: x_1(s) = x_2(s) \ \forall s \in [0, t]\}$. Clearly, $T^* > 0$ and $T^* = \max\{t \in [0, T_0]: x_1(s) = x_2(s) \ \ \forall s \in [0, t]\}$. Finally, we can show that $T^* = T_0$. In fact, if by contradiction, $T^* < T_0$ we can say that the functions $x_1: [0, T_0] \to \Omega$ and $x_2: [0, T_0] \to \Omega$ are solutions of the following problem

$$\begin{cases} x \in -\partial^- f(x) + g(t, x) \\ x(T^*) = \widetilde{x} \quad (\text{where } \widetilde{x} = x_1(T^*) = x_2(T^*)). \end{cases}$$

Then, thanks to the analogous argument made in order to obtain (3.13), we can show that there exists $\alpha > 0$ such that

$$\| x_1(t) - x_2(t) \|^2$$

$$\leq 2 \int_{T^*}^t [\delta(s) + k(x)\gamma(s)] \| x_1(s) - x_2(s) \|^2 ds \ \ \forall t \in [T^*, T^* + \alpha]. \tag{3.14}$$

Therefore we can deduce that $x_1 = x_2$ in $[T^*, T^* + \alpha]$, which is absurd by the definition of T^*. Consequently, we have that $T^* = T_0$. Hence the problem $(P_u)_g$ has a unique strong solution defined $[0, T_0]$.

Now, we are ready for the result on the topological structure of the solution set $S(x_0)$.

Theorem 3: *Let Ω be an open subset of a (real) separable Hilbert space H, $f: \Omega \to R \cup \{+\infty\}$ be a function satisfying $H(f)$ and $F: I \times \Omega \to P_{fc}(H)$ be a multi-*

function satisfying the hypothesis $H(F)_2$, *then we can choose* $T_0 > 0$ *as in Remark* 1 *such that* $S(x_0)$ *is nonempty and it is an* R_δ*-set in* $C([0,T_0],\Omega)$.

Proof: First, by applying Lemma 1, we can choose a sequence of multifunction $(F_n)_n$, $F_n: I\times\Omega\to P_{fc}(H)$, $n\geq 1$ be as postulated by Lemma 1. For fixed $n\geq 1$, we consider the following multivalued Cauchy problem

$$\begin{cases} x'(t)\in -\partial^- f(x(t))+F_n(t,x(t)), \text{ a.e. on } I, \\ x(0)=x_0. \end{cases} \tag{3.15}$$

Moreover, by Lemma 1, we can see that, for every $n\in N$, there exists a selection u_n: $I\times\Omega\to H$ of the multifunction F_n with the properties mentioned in the Lemma 1. So, using analogous considerations made in the first part of the proof of Lemma 3, we deduce that the problem, similar to the problem (3.15), which possesses the perturbation u_n, has a solution (cf. Lemma 3). Hence, we can say that the set of solutions $S_n(x_0)$ of the problem (3.15) is nonempty. Now, we note that $S_n(x_0)\subseteq s_{x_0}(V)$, where V is defined as in (2.13), and $s_{x_0}=p(x_0,\cdot): V\to C([0,T_0],\Omega)$. Easily (cf. Theorem 1), we can observe that the set $s_{x_0}(V)$ is compact in $C([0,T_0],\Omega)$. Hence, in order to prove that $S_n(x_0)$ is compact, we consider $(x_m)_m$, $x_m\in S_n(x_0)$, $m\geq 1$, and $x_m\to x$ in $C([0,T_0],\Omega)$ as $m\to\infty$. We have that $x_m=s_{x_0}(f_m)$ with $f_m\in S^2_{F_n(\cdot,x_m(\cdot))}$. We may assume that $f_m\to f^*$ weakly in $L^2([0,T_0],H)$ and $f^*\in V$, being V weakly closed in $L^2([0,T_0],H)$. Moreover, $f^*\in S^2_{F_n(\cdot,x(\cdot))}$. In fact, Lemma 2 provides that $x_m\to s_{x_0}(f^*)=x$ in $C([0,T_0],\Omega)$, so applying the convergence theorem (cf. [1]), we have that $f^*(t)\in F_n(t,x(t))$, a.e. in $[0,T_0]$. Thus $S_n(x_0)$ is closed, hence compact in $C([0,T_0],\Omega)$.

We also claim that, for every $n\geq 1$, $S_n(x_0)$ is contractible. To check this property, we consider, given $r\in[0,T_0)$ and $x\in S_n(x_0)$, the following problem

$$\begin{cases} z'\in -\partial^- f(z)+u_n(t,z), \text{ a.e. on } [r,T], \\ z(r)=x(r), \end{cases} \tag{3.16}$$

where u_n is the mentioned selector of F_n. The problem (3.16) has a unique solution. In fact, by denoting with $g_n: I\times\Omega\to H$ the function defined $g_n(\tau,z)=u_n\ (\tau+r,z)$ $(=u_n(t,z))$, $\forall\tau\in[0,T-r]$, $\forall z\in\Omega$, the problem (3.16) can be rewritten in this way:

$$\begin{cases} z'\in -\partial^- f(z)+g_n(\tau,z), \text{ a.e. on } [0,T-r], \\ z(0)=x(0)=x_0. \end{cases} \tag{3.16$'$}$$

By Lemma 3, we can deduce that this problem has a unique solution defined in the interval $[0,\widetilde{T}_0]$, where $\widetilde{T}_0=\min\{T-r,\overline{T},T'\}$. Therefore, since $T_0-r<\widetilde{T}_0$, the problem (3.16)$'$ has a unique solution $\overline{x}$ defined in $[0,T_0-r]$. Now setting $\widetilde{x}(t)=\overline{x}(t-r)=\overline{x}(\tau)$, $t\in[r,T_0]$ $(\tau\in[0,T_0-r])$, we have that $\widetilde{x}:[r,T_0]\to\Omega$ is the unique solution of the problem (3.16) such that

$$u_n(t,\widetilde{x}(t))\in F_n(t,\widetilde{x}(t)), \text{ a.e. in } [r,T_0]; \tag{3.17}$$

$$\widetilde{x} \text{ is continuous on } [r, T_0] \text{ and absolutely continuous on the compact subsets of }]r, T_0[; \tag{3.18}$$

$$\widetilde{x}\,(t) \in clB(x_0, 2R), \ \ \forall t \in [r, T_0] \tag{3.19}$$

(where R is chosen as in Remark 1)

$$f(\widetilde{x}\,(t)) \in \left[f(x_0) - 1, k + \frac{k^2}{2}\right], \ \ \forall t \in [r, T_0]; \tag{3.20}$$

$$\| \widetilde{x}'(t) \|^2 \leq \| \operatorname{grad}^- f(\overline{x}\,(t-r)) \| + \| u_n(t-r, \overline{x}\,(t-r)) \|$$

$$\leq N + \gamma(t), \ \ \forall t \in [r, T_0]. \tag{3.21}$$

Now, we denote with $z(r,x)(\,\cdot\,) \in C([r,T_0],\Omega)$ the unique solution of the problem (3.16). For $r = T_0$, we set $z(T_0, x) = x$. So, we can define the function $h: [0, T_0] \times S_n(x_0) \to S_n(x_0)$ by

$$h(r,x)(t) = \begin{cases} x(t) & \text{for } 0 \leq t \leq r; \\ z(r,x)(t) & \text{for } r \leq t \leq T_0. \end{cases}$$

Evidently, $h(r,x)(0) = x(0) = x_0$. On the other hand, $h(0,x) = z(0,x) = z_0$, with $z_0 \in C([0,T_0],\Omega)$ being the unique solution of

$$\begin{cases} z' \in -\partial^- f(z) + u_n(t,z) \text{ a.e. on } [0, T_0], \\ z(0) = x(0) = x_0. \end{cases}$$

Moreover, $h(T_0, x) = z(T_0, x) = x$.

If we can show that $h(\,\cdot\,,\,\cdot\,)$ is continuous, we will have established the contractibility of $S_n(x_0)$ in $C([0,T_0],\Omega)$. To this end, let $\{(r_m, x_m)\}_m \subseteq [0,T_0] \times S_n(x_0)$, with $(r_m, x_m) \to (r, x)$ in $[0, T_0] \times S_n(x_0)$. We consider two distinct cases:

Case 1: $r_m > r$, for every $m \geq 1$:

Let

$$h(r_m, x_m)(t) = \begin{cases} x_m(t) & \text{for } 0 \leq t \leq r_m \\ z(r_m, x_m)(t) & \text{for } r_m \leq t \leq T_0, \end{cases}$$

we put $v_m = h(r_m, x_m)$. Evidently, $v_m \in S_n(x_0)$, $m \geq 1$, and so by passing to a subsequence if necessary, ew may assume that $v_m \to v$ in $C([0,T_0],\Omega)$. From the definition of $h(\,\cdot\,,\,\cdot\,)$, we see that, for $t \in [0, r]$, we have $v(t) = x(t)$. Let $y \in C([0,T_0],\Omega)$ be the unique solution of

$$\begin{cases} y'(t) \in -\partial^- f(y(t)) + u_n(t, v(t)), \text{ a.e. on } [r, T_0], \\ y(r) = x(r) \ (= v(r)). \end{cases}$$

Let $N \geq 1$, we can say that there exists $m_N \in N$ such that for all $m \geq m_N$ we have $r_m < r_N$. Then, for all $m \geq m_N$ we obtain that $v_m'(t) \in -\partial^- f(v_m(t)) +$

$u_n(t, v_m(t))$ a.e. on $[r_N, T_0]$ (where $[r_N, T_0] \subseteq [r_m, T_0]$). Then, we find two functions α_m and α with the properties:

$$\alpha_m(t) \in \partial^- f(v_m(t)),\ \alpha(t) \in \partial^- f(y(t)), \text{ a.e. on } [r_N, T_0]$$

$$\alpha_m(t) = -v_m'(t) + u(t, v_m(t)),\ \alpha(t) = -y'(t) + u_n(t, v(t)), \text{ a.e. on } [r_N, T_0].$$

By (ix) of Remark 1, we deduce

$$\| y(t) - v_m(t) \|$$

$$\leq \left(\| y(r_N) - v_m(r_N) \| + \int_{r_N}^{t} \| u_n(s, v(s)) - u_n(s, v_m(s)) \| ds \right)$$

$$\exp \int_{r_N}^{t} \varphi(y(s), v_m(s), (f \circ y)(s), (f \circ v_m(s)))(1 + \| \alpha(s) \|^2 + \| \alpha_m(s) \|^2) ds.$$

Now, taking $v_m(t), y(t) \in clB(x_0, 2R)$, and $f(v_m(t)),\ f(y(t)) \in [f(x_0) - 1, k + \frac{k^2}{2}]$, $\forall t \in [r_N, T_0]$ into account, using analogous considerations made previously in Lemma 2, we have

$$\exp \int_{r_N}^{t} \varphi(y(s), v_m(s), (f \circ y)(s), (f \circ v_m(s)))(1 + \| \alpha(s) \|^2 + \| \alpha_m(s) \|^2) ds$$

$$\leq \exp(LT_0(1 + 2N^2) + 8k^2L + 8NLk\sqrt{T_0}) = C,$$

then, we can say that

$$\| y(t) - v_m(t) \|$$

$$\leq C \left[\| y(r_N) - v_m(r_N) \| + \int_{r_N}^{t} \| u(s, v(s)) - u_n(s, v_m(s)) \| ds \right], \ \forall t \in [r_N, T_0].$$

Applying the limit $m \to \infty$, we get that

$$\| y(t) - v(t) \| \leq C \| y(r_N) - v(r_N) \|, \text{ for } t \in [r_N, T_0].$$

Note that as $N \to \infty$ we have $y(r_N) \to x(r)$ and $v(r_N) \to v(r) = x(r)$. Since $N \geq 1$ was arbitrary, we conclude that $y(t) = v(t)$ for $t \in [r, T_0]$. Hence $v = h(r, x)$ and so $h(r_m, x_m) \to h(r, x)$ in $C([0, T_0], \Omega)$ as $m \to \infty$.

Case 2: $r_m \leq r$, for every $m \geq 1$. Now, for all $t \in [0, r[$, there exists a natural number $\bar{m}$ such that for all $m \geq \bar{m}$ we have $r_m > t$. Keeping the notation introduced in the analysis of Case 1, we see that $v(t) = x(t)$ for $t \in [0, r]$.

Moreover, the same arguments as in Case 1 give us that

$$\| y(t) - v_m(t) \| \leq C \left[\| y(r) - v_m(r) \| + \int_{r}^{t} \| u_n(s, v(s)) - u_n(s, v_m(s)) \| ds \right]$$

$$\text{for } t \in [r, T_0],$$

and by applying the limit $m \to \infty$, we obtain

$$\| y(t) - v(t) \| \leq C \| y(r) - v(r) \| \text{ for } t \in [r, T_0].$$

But $y(r) = x(r) = v(r)$. So $y(t) = v(t)$ for $t \in [r, T_0]$. Hence, $v = h(r, x)$ and so again we have $h(r_m, x_m) \rightarrow h(r, x)$ in $C([0, T_0], \Omega)$ as $m \rightarrow \infty$.

In general, we can always find a subsequence $\{r_m\}_{m \geq 1}$ satisfying Case 1 or Case 2. Thus we have proved the continuity of the map $h(\cdot, \cdot)$. So, for every $n \geq 1$, $S_n(x_0)$ is compact and contractible in $C([0, T_0], \Omega)$. To finish the proof we show that $S(x_0) = \bigcap_{n \in N} S_n(x_0)$. Clearly $S(x_0) \subseteq \bigcap_{n \in N} S_n(x_0)$. Let $x \in \bigcap_{n \in N} S_n(x_0)$. Then by definition $x = s_{x_0}(h_n)$ with $h_n \in S^2_{F_n(\cdot, x(\cdot))}, n \geq 1$. Evidently $\{h_n\}_{n \geq 1}$ is bounded in $L^2([0, T_0], H)$. So, by passing to a subsequence if necessary, we may assume that $h_n \rightarrow h$ weakly in $L^2([0, T_0], H)$. From Theorem 3.1 of [16], we have that

$$h(t) \in \overline{\text{conv}}\ w\text{-}\overline{\lim}_{n \rightarrow +\infty} F_n(t, x(t)) \subseteq F(t, x(t)), \text{ a.e. on } [0, T_0],$$

and therefore we can say that $h \in S^2_{F(\cdot, x(\cdot))}$. Finally, by Lemma 2 we conclude that $x = p(x_0, h)$. So $x \in S(x_0)$ and therefore we have $S(x_0) = \bigcap_{n \in N} S_n(x_0)$. Using a result of [15], we conclude that $S(x_0)$ is a R_δ-set in $C([0, T_0], \Omega)$.

Being the evolution map "e" defined by $e(x, t) = x(t)$, $\forall (x, t) \in C([0, T_0], \Omega) \times [0, T_0]$, a continuous function, we have as an immediate consequence of Theorem 3 above, the following Kneser-type theorem for (1).

Corollary 1: *If hypotheses $H(f)$ and $H(F)_2$ hold, then for every $t \in [0, T_0]$, the set $R(t) = S(x_0)(t) = \{x(t) : x \in S(x_0)\}$ (the reachable set at time $t \in I$) is compact and connected in H.*

In what follows we obtain, as a consequence of Theorem 1 and Lemma 2, a continuity result about the solution-multifunction $x \mapsto S(x)$.

Theorem 4: *If hypotheses $H(f)$ and $H(F)_0$ hold, then there exist $T_0 > 0$ and $R > 0$, chosen as in Remark 1, such that the multifunction $S: dom(f) \cap clB(x_0, R) \rightarrow P_k(C([0, T_0], \Omega))$ is $(u.s.c.)_t$.*

Proof: The set $S(x)$ is nonempty and compact in $C([0, T_0], \Omega)$ for every $x \in \text{dom}(f) \cap clB(x_0, R)$ (see, Theorem 1). Now we need to show that given $C \subseteq C([0, T_0], \Omega))$ nonempty and closed the set, $S^-(C) = \{x \in \text{dom}(f) \cap clB(x_0, R): S(x) \cap C \neq \emptyset\}$ is closed in $\text{dom}(f) \cap clB(x_0, R) \subseteq \Omega$. To this end, let $u_n \in S^-(C)$, $n \geq 1$. For each $n \geq 1$, let $x_n = p(u_n, h_n)$, $h_n \in S^2_{F(\cdot, x_n(\cdot))}$. Since $\{h_n\}_n$ is bounded in $L^2([0, T_0], H)$ (cf. hypothesis (jjj)), by passing to a subsequence if necessary, we may assume that $h_n \rightarrow h$ weakly in $V \subseteq L^2([0, T_0], H)$. From Lemma 2 we have that $p(u_n, h_n) \rightarrow p(u, h)$ in $C([0, T_0], \Omega))$. Now let $x = p(u, h)$, from hypothesis $H(F)_0(jj)$ and (jjj) and Theorem 3.1 (cf. [16]), we have that $h \in S^2_{F(\cdot, x(\cdot))}$. So $x \in S(u) \cap C$. Since $u \in \text{dom}(f) \cap clB(x_0, R)$, we can conclude that $u \in S^-(C)$. Therefore $S(\cdot)$ is $(\text{u.s.c.})_t$.

Next we will generate a continuous selector for the multifunction $x \mapsto S(x)$. For this we will need the hypothesis $H(F)_1$ on the orientor field F.

Theorem 5: *If hypotheses $H(f)$ and $H(F)_1$ hold and the set $dom(f)$ is a closed subset of Ω, then, for all $\eta > 0$ such that $clB(x_0, \eta) \subseteq B(x_0, R)$ there exists $u: dom(f) \cap clB(x_0, \eta) \rightarrow C([0, T_0], \Omega)$ a continuous map such that $u(\xi) \in S(\xi)$ for every $\xi \in dom(f) \cap clB(x_0, \eta)$.*

Proof: Let $x_0(\xi)(\cdot) \in C([0, T_0], \Omega)$ be the unique solution of the evolution equation (cf. Proposition 1)

$$\begin{cases} x' \in -\partial^- f(x), \\ x(0) = \xi \in \mathrm{dom}(f) \cap clB(x_0, \eta). \end{cases}$$

Let $R_0: \mathrm{dom}(f) \cap clB(x_0, \eta) \to P_f(L^1([0,T_0],H))$ be defined $R_0(\xi) = S^1_{F(\cdot, x_0(\xi)(\cdot))}$, for every $\xi \in \mathrm{dom}(f) \cap clB(x_0, \eta)$. Now we can observe, using analogous considerations made in the first part of Lemma 2, that $R_0(\cdot)$ is h-Lipschitz and, moreover, that its values are nonempty, closed and decomposable. So we can apply Theorem 3 of [4] and we obtain $r_0: \mathrm{dom}(f) \cap clB(x_0, \eta) \to L^1([0,T_0],H)$ a continuous map such that $r_0(\xi) \in R_0(\xi)$, for every $\xi \in \mathrm{dom}(f) \cap clB(x_0, \eta)$.

Let $x_1(\xi)(\cdot) \in C([0,T_0],\Omega)$ be the unique solution of

$$\begin{cases} x' \in -\partial^- f(x) + r_0(\xi), \\ x(0) = \xi \in \mathrm{dom}(f) \cap clB(x_0, \eta). \end{cases}$$

We claim that, for every fixed $\xi \in \mathrm{dom}(f) \cap clB(x_0, \eta)$, by induction we can generate two sequences $\{x_n(\xi)(\cdot)\}_{n \geq 0} \subseteq C([0,T_0],\Omega)$ and $\{r_n(\xi)(\cdot)\}_{n \geq 0} \subseteq L^2([0,T_0],H)$ satisfying the following properties:

(a) $x_n(\xi)(\cdot) \in C([0,T_0],\Omega)$ is the unique solution of

$$\begin{cases} x'(t) \in -\partial^- f(x(t)) + r_{n-1}(\xi)(t) \text{ a.e. in } [0,T_0], \\ x(0) = \xi, \ \forall n \geq 1; \end{cases}$$

(b) $\xi \to r_n(\xi)$ is continuous from $\mathrm{dom}(f) \cap clB(x_0, \eta)$ into $L^1([0,T_0],H)$, $\forall n \geq 0$;

(c) $r_n(\xi)(t) \in F(t, x_n(\xi)(t))$ a.e. on $[0,T_0]$, for every $\xi \in \mathrm{dom}(f) \cap clB(x_0, \eta)$, $\forall n \geq 0$;

(d) $\| r_n(\xi)(t) - r_{n-1}(\xi)(t) \| \leq k(t)\beta_n(t)$ a.e. on $[0,T_0]$, $\forall n \geq 1$,

with

$$\beta_n(t) = \int_0^t C^n \gamma(s) \frac{(\theta(t) - \theta(s))^{n-1}}{(n-1)!} ds + T_0 \left(\sum_{k=0}^{n} \frac{\varepsilon}{2^{k+1}} \right) \frac{(\theta(t))^{n-1}}{(n-1)!}$$

with $\varepsilon > 0$ and $\theta(t) = \int_0^t k(s)ds$.

Our first purpose is to prove that we are able to find the functions $x_1(\xi)(\cdot)$ and $r_0(\xi)(\cdot)$ satisfying the properties (a)-(d). Evidently the fixed function $x_1(\xi)(\cdot)$ has the property (a). Moreover, we can define the multifunction $R_1: \mathrm{dom}(f) \cap clB(x_0, \eta) \to 2^{L^1([0,T_0],H)}$ as

$$R_1(\xi) = \left\{ z \in S^1_{F(\cdot, x_1(\xi)(\cdot))} : \| z(t) - r_0(\xi)(t) \| \leq k(t)\beta_1(t) \text{ a.e. on } [0,T_0] \right\}$$

$$\text{for every } \xi \in \mathrm{dom}(f) \cap clB(x_0, \eta),$$

where

$$\beta_1(t) = C \int_0^t \gamma(s)ds + T_0\left(\frac{\varepsilon}{2} + \frac{\varepsilon}{4}\right), \ \forall t \in [0,T_0].$$

First, we observe that $R_1(\xi) \neq \emptyset$, $\forall \xi \in \mathrm{dom}(f) \cap clB(x_0, \eta)$. In fact, using hypothesis

$H(F)_1(jj)'$ and the property (ix) of Remark 1, we have

$$d(r_0(\xi)(t), F(t, x_1(\xi)(t))) \leq k(t)\beta_1(t), \text{ a.e. on } [0, T_0].$$

Therefore, we can say that the multifunction $\Gamma_1(\xi):[0,T_0] \to 2^H$, defined by

$$\Gamma_1(\xi)(t) = \{v \in F(t, x_1(\xi)(t)) : \| v - r_0(\xi)(t) \| \leq k(t)\beta_1(t)\}, \ \forall t \in [0, T_0],$$

has, for a.e. $t \in [0, T_0]$, nonempty values and thus, without any loss of generality, we can assume that Γ_1 has nonempty values. Moreover, $\Gamma_1(\xi)(\cdot)$ has measurable graph and so we are ready to apply theorem of [21] and we can deduce that there exists a measurable selection $z:[0,T_0] \to H$ of the multifunction $\Gamma_1(\xi)(\cdot)$. So, we conclude that $z \in R_1(\xi)$.

On the other hand, with analogous considerations made in order to obtain the function $r_0(\xi)(\cdot)$, we can get the function $r_1(\xi)(\cdot)$, selection of the multifunction

$$t \to clR_1(\xi)(t) = \left\{z \in S^1_{F(\cdot, x_1(\xi)(\cdot))} : \| z(t) - r_0(\xi)(t) \| \leq k(t)\beta_1(t)\right\}.$$

Easily, we can observe that $r_1(\xi)(\cdot)$ also satisfies the properties (b), (c) and (d).

Now, we suppose that we were able to produce $\{x_k(\xi)\}_{k=0}^n$ and $\{r_k(\xi)\}_{k=0}^n$ satisfying $(a) \to (d)$ above.

Let $x_{n+1}(\xi)(\cdot) \in C([0,T_0], \Omega)$ be the unique solution of

$$\begin{cases} x' \in -\partial^- f(x) + r_n(\xi), \text{ a.e. } [0, T_0] \\ x(0) = \xi \in \mathrm{dom}(f) \cap clB(x_0, \eta). \end{cases}$$

Therefore, obviously, this function satisfies the condition (a). As before, we get

$$\begin{aligned}
&\| x_{n+1}(\xi)(t) - x_n(\xi)(t) \| \\
&\leq C \int_0^t \| r_n(\xi)(s) - r_{n-1}(\xi)(s) \| ds \leq C \int_0^t k(s)\beta_n(s) ds \\
&= C \int_0^t k(s) \int_0^s \gamma(\tau) C^n \frac{(\theta(s) - \theta(\tau))^{n-1}}{(n-1)!} d\tau ds \\
&\quad + T_0 \left(\sum_{k=0}^n \frac{\varepsilon}{2^{k+1}} \right) \int_0^t k(s) \frac{(\theta(s))^{n-1}}{(n-1)!} ds \qquad (3.22) \\
&= \int_0^t C^{n+1} k(s) \left(\int_0^s \gamma(\tau) \frac{(\theta(s) - \theta(\tau))^{n-1}}{(n-1)!} d\tau \right) ds \\
&\quad + T_0 \left(\sum_{k=1}^n \frac{\varepsilon}{2^{k+1}} \right) \int_0^t k(s) \frac{(\theta(s))^n}{(n-1)!} ds \\
&< \int_0^t C^{n+1} \gamma(\tau) \frac{(\theta(t) - \theta(\tau))^n}{n!} d\tau + T_0 \left(\sum_{k=0}^{n+1} \frac{\varepsilon}{2^{k+1}} \right) \frac{(\theta(t))^n}{n!} = \beta_{n+1}(t)
\end{aligned}$$

a.e. in $[0, T_0]$.

Therefore, using hypothesis $H(F)_1(jj)'$, we have

$$\begin{aligned} d(r_n(\xi)(t), F(t, x_{n+1}(\xi)(t)) &\leq k(t) \| x_n(\xi)(t) - x_{n+1}(\xi)(t) \| \\ &\leq k(t)\beta_{n+1}(t), \text{ a.e. on } [0, T_0]. \end{aligned} \tag{3.23}$$

Now let $R_{n+1}: \mathrm{dom}(f) \cap clB(x_0, \eta) \to 2^{L^1([0,T_0],H)}$ be the multifunction defined by

$$R_{n+1}(\xi)$$

$$= \left\{ z \in S^1_{F(\cdot, x_{n+1}(\xi)(\cdot))} : \| z(t) - r_n(\xi)(t) \| \leq k(t)\beta_{n+1}(t) \text{ a.e. in } [0, T_0] \right\}.$$

From (3.23) above, we know that the multifunction $\Gamma_{n+1}(\xi): [0, T_0] \to 2^H$, defined by

$$\Gamma_{n+1}(\xi)(t) = \{ v \in F(t, x_{n+1}(\xi)(t)) : \| v - r_n(\xi)(t) \| < k(t)\beta_{n+1}(t) \}$$

is such that $\Gamma_{n+1}(\xi)(t) \neq \emptyset$ a.e. on $[0, T_0]$.

By modifying the above multifunction on a Lebesgue-null subset of $[0, T_0]$, we may assume without any loss of generality that $\Gamma_{n+1}(\xi)(t) \neq \emptyset$ for every $t \in [0, T_0]$. Also from Theorem 3.3 of [17], we know that $t \to F(t, x_{n+1}(\xi)(t))$ is measurable (hence graph measurable), while $(t, v) \to \| v - r_n(\xi)(t) \| - k(t)\beta_{n+1}(t) = \gamma_{n+1}(\xi)(t, v)$ is clearly jointly measurable. So,

$$Gr\Gamma_{n+1}(\xi) = \{ (v, t) \in GrF(\cdot, x_{n+1}(\xi)(\cdot) : \gamma_{n+1}(\xi)(t, v) < 0 \} \in \mathcal{L}([0, T_0] \times B(H)$$

with $\mathcal{L}([0, T_0])$ being the Lebesgue σ-field of $[0, T_0]$.

Applying Aumann's selection theorem, we get a measurable function $\tilde{z}: [0, T_0] \to H$ such that $\tilde{z}(t) \in \Gamma_{n+1}(\xi)(t)$, $\forall t \in [0, T_0]$, so $\tilde{z} \in R_{n+1}(\xi)$, for every $\xi \in \mathrm{dom}(f) \cap clB(x_0, \eta)$. Moreover, $\xi \mapsto R_{n+1}(\xi)$ is (l.s.c.)$_t$ with decomposable values. Therefore, applying Theorem 3 of [4] to the multifunction $clR_{n+1}(\cdot)$, we can get a continuous map $r_{n+1}: \mathrm{dom}(f) \cap clB(x_0, \eta) \to L^1([0, T_0], H)$ such that $r_{n+1}(\xi) \in clR_{n+1}(\xi)$, for every $\xi \in \mathrm{dom}(f) \cap clB(x_0, \eta)$. Hence,e $r_{n+1}(\xi)(t) \in F(t, x_{n+1}(\xi)(t))$ a.e. on $[0, T_0]$ and $\| r_{n+1}(\xi)(t) - r_n(\xi)(t) \| \leq k(t)\beta_{n+1}(t)$ a.e. on $[0, T_0]$. Thus, by induction, we have produced the two sequences

$$\{x_n(\xi)\}_{n \geq 0} \subseteq C([0, T_0], \Omega) \text{ and } \{r_n(\xi)\}_{n \geq 0} \subseteq L^2([0, T_0], H)$$

satisfying $(a) \to (d)$ above.

Then, using (3.22) we have

$$\begin{aligned} \int_0^{T_0} \| r_n(\xi)(t) - r_{n-1}(\xi)(t) \| dt &\leq \int_0^{T_0} k(t)\beta_n(t) dt \\ &\leq \int_0^{T_0} C^n \gamma(s) \frac{(\theta(T_0))^n}{n!} ds + T_0 \varepsilon \frac{(\theta(T_0))^n}{n!} \\ &\leq \frac{(\theta(T_0))^n}{n!} [C^n \| \gamma \|_1 + T_0 \varepsilon]. \end{aligned}$$

So, from the above inequality, we deduce that $\{r_n(\xi)\}_{n \geq 0}$ is a $L^1([0,T_0],H)$-Cauchy sequence, uniformly in $\xi \in \mathrm{dom}(f) \cap clB(x_0,\eta)$. Also from (a) we have

$$\| x_{n+1}(\xi) - x_n(\xi) \|_{C([0,T_0],\Omega)} \leq C \| r_{n+1}(\xi) - r_n(\xi) \|_{L^1([0,T_0],H)}$$

and, therefore, we can say that $\{x_n(\xi)\}_{n \geq 0}$ is a Cauchy sequence in $C([0,T_0],\Omega)$, uniformly in $\xi \in \mathrm{dom}(f) \cap clB(x_0,\eta)$.

Let $n \to \infty$ we have $x_{n+1}(\xi) \to x(\xi)$ in $C([0,T_0],\Omega)$, $r_n(\xi) \to r(\xi)$ in $L^1([0,T_0],H)$ and both limits are continuous in $\mathrm{dom}(f) \cap clB(x_0,\eta)$. Now let $y(\xi) \in C([0,T_0],\Omega)$ be the unique solution of the problem

$$\begin{cases} z' \in -\partial^- f(z) + r(\xi) \text{ a.e. in } [0,T_0] \\ z(0) = \xi \in \mathrm{dom}(f) \cap clB(x_0,\eta). \end{cases}$$

By the property (c), we have $r(\xi)(t) \in F(t, x(\xi)(t))$ a.e. As before, we have

$$\| x_n(\xi)(t) - y(\xi)(t) \| \leq C \int_0^t \| r_{n-1}(\xi)(s) - r(\xi)(s) \| \, ds, \quad \forall t \in [0,T_0],$$

by which we deduce that $x_n(\xi) \to y(\xi)$ in $C([0,T_0],\Omega)$. Hence we have $x(\xi) = y(\xi)$, $\forall \xi \in \mathrm{dom}(f) \cap clB(x_0,\eta)$.

Therefore, the function $u: \xi \mapsto x(\xi)$ is the desired selector of the multifunction $\xi \mapsto S(\xi)$.

4. An Application: Existence of Periodic Solutions

An immediate consequence of Theorem 5 is the following corollary:

Corollary 2: *If hypotheses $H(f)$, $H(F)_1$ hold, dom(f) is a closed subset of Ω and if there exists a compact and convex subset of dom$(f) \cap clB(x_0,\eta)$ such that $S(K)(T_0) \subseteq K$, where η is fixed as in Theorem 5, then there exists a solution $x(\cdot) \in C([0,T_0],\Omega)$ for the problem*

$$\begin{cases} x' \in -\partial^- f(x) + F(t,x) \ \ a.e. \ on \ [0,T_0], \\ \qquad x(0) = x(T_0). \end{cases}$$

Proof: Let $u: \mathrm{dom}(f) \cap clB(x_0,\eta) \to C([0,T_0],\Omega)$ be the continuous selector of the multifunction $\xi \mapsto S(\xi)$ guaranteed by Theorem 5. Let $e_{T_0}: C([0,T_0],\Omega) \to H$ be the evaluation map, i.e. $e_{T_0}(x) = x(T_0)$.

Let $\widehat{u} = e_{T_0} \circ u: \mathrm{dom}(f) \cap clB(x_0,\eta) \to H$. we observe that the restriction of $\widehat{u}$ to the set K assumes values in the set K, and moreover, $\widehat{u}: K \to K$ is a continuous and compact map. So Schauder's fixed point theorem gives us $\xi \in K$ such that $\xi = \widehat{u}(\xi)$.

Then $u(\xi)(\cdot) \in C([0,T_0],\Omega)$ is the desired periodic trajectory.

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