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A CLASS OF NONLINEAR VARIATIONAL INEQUALITIES INVOLVING PSEUDOMONOTONE OPERATORS

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We present the solvability of a class of nonlinear variational inequalities involving pseudomonotone operators in a locally convex Hausdorff topological vector spaces setting. The obtained result generalizes similar variational inequality problems on monotone operators.

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1. Introduction

Recently, the author [4] studied a class of nonlinear variational inequalities in a locally convex Hausdorff topological vector space setting and extended a class of nonlinear variational inequalities involving monotone operators by utilizing a fixed point theorem [5]. The aim of this note is to present a class of generalized nonlinear variational inequalities involving pseudomonotone operators in a locally convex Hausdorff topological vector space setting. The obtained result generalizes the variational inequality results involving monotone operators in a similar setting, especially [4, 5]. For more details on variational and hemivariational inequalities in a Hilbert space setting, we refer to [3].

Let K be a subset of a real locally convex Hausdorff topological vector space X with its dual X' and let $T: K \rightarrow X'$ be a pseudomonotone nonlinear mapping. Let $\langle w, x \rangle$ denote the duality pairing between the elements w in X' and elements x in X. We consider the nonlinear variational inequality (NVI) problem: Find an element u_0 in K such that

$$\langle Tu_0, v - u_0 \rangle \geq 0 \text{ for all } v \text{ in } K. \tag{1}$$

A mapping $T: K \rightarrow X'$ is said to be:

(i) pseudomonotone [1] if

$$\langle Tu, v-u\rangle \geq 0 \Rightarrow \langle Tv, v-u\rangle \geq 0 \text{ for all } u,v \text{ in } K;$$

(*ii*) quasimonotone [1] if

$$\langle Tu, v-u\rangle > 0 = \Rightarrow \langle Tv, v-u\rangle \geq 0 \text{ for all } u,v \text{ in } K;$$

(*iii*) monotone if for all u,v in K, we have

$$\langle Tv - Tu, v-u\rangle \geq 0.$$

We note that the following implication holds:

$$\text{monotone} \Rightarrow \text{pseudomonotone} \Rightarrow \text{quasimonotone}.$$

Before we consider our main result, we need to recall an auxiliary result (a variant of [5, Proposition 9.9] and similar to [2, Theorem 1] crucial to the NVI problem (1) at hand.

Lemma 1.1: *Let K be a nonempty, compact and convex subset of a locally convex space X and let $S: K \rightarrow P(K)$ be a multivalued mapping such that the following assumptions hold:*

(*i*) *The set $S(x)$ is nonempty and convex for all $x \in K$.*

(*ii*) *The preimages $S^{-1}(y)$ contain relatively open subsets U_y with respect to K for all y in K.*

(*iii*) $K = \bigcup \{U_y : y \in K\}$.

Then there exists an element x_0 in K such that $x_0 \in S(x_0)$.

2. The Main Result

This section deals with the main result on the solvability of the NVI problem (1).

Theorem 2.1: *Let K be a nonempty, compact and convex subset of a locally convex Hausdorff topological vector space X and let $T: K \rightarrow X'$ be a pseudomonotone mapping. Then the NVI problem* (1) *has a solution u_0 in K.*

Proof: Assume the NVI problem (1) has no solution. Then for each $u \in K$, the set $\{v \text{ in } K: \langle Tu, v-u\rangle < 0\}$ is nonempty. As a result, if we define a multivalued mapping $F: K \rightarrow P(K)$ by

$$F(u) = \{v \text{ in } K: \langle Tu, v-u\rangle < 0\},$$

then clearly $F(u)$ is nonempty and convex for each u in K. It follows that

$$F^{-1}(v) = \{u \in K: v \in F(u)\} = \{u \in K: \langle Tu, v-u\rangle < 0\}.$$

It suffices to show that $F^{-1}(v)$ is relatively open with respect to K. For each v in K, the complement of $F^{-1}(v)$ in K,

$$c_K(F^{-1}(v)) = \{u \in K: \langle Tu, v-u\rangle \geq 0\}$$

$$\subset \{u \in K : \langle Tv, v-u\} \geq 0\},$$

by the pseudomonotonicity of T, denoted by $G(v)$. We need to show that $G(v)$ is convex and closed in K. To show $G(v)$ is convex, if $u_1, u_2 \in G(v)$, $0 < t < 1$ and $u_t = tu_1 + (1-t)u_2$, then $u_t \in K$ and, $\langle Tv, v-u_1\rangle \geq 0$ and $\langle Tv, v-u_2\rangle \geq 0$. As a result of this, we have

$$\langle Tv, v-u_t\rangle = \langle Tv, v - tu_1 - (1-t)u_2\rangle = \langle Tv, t(v-u_1) + (1-t)(v-u_2)\rangle$$

$$= t\langle Tv, v-u_1\rangle + (1-t)\langle Tv, v-u_2\rangle \geq 0.$$

This implies $u_t \in G(v)$. Next, to show $G(v)$ is relatively closed with respect to K, let $\{u_a\}$ be a Moore-Smith sequence in $G(v)$. Then $\langle Tv, v-u_a\rangle \geq 0$. Assume $u_a \rightarrow u$ in K. Now all we need is to show $u \in G(v)$. We can now express

$$\langle Tv, v-u\rangle = \langle Tv, v-u_a\rangle + \langle Tv, u_a - u\rangle$$

$$\geq 0 + \langle Tv, u_a - u\rangle = 0.$$

Thus, $u \in G(v)$. Since $G(v)$ is a closed convex subset of K, it implies $c_K(G(v)) = \{u \in K\colon \langle Tv, v-u\rangle < 0\}$ is open in K. This results in

$$c_K(G(v)) \subset F^{-1}(v) \text{ since any } u \notin F^{-1}(v) \text{ implies } u \notin c_K(G(v)).$$

Hence, for each v in K, $F^{-1}(v)$ contains an open subset $c_K(G(v))$. Thus for all u in K, there exists an element v in K such that $\langle Tv, v-u\rangle < 0$. Therefore, $K = \bigcup\{c_K(G(v)) : v \in K\}$. Now by Lemma 1.1, there exists an element x_0 in K such that $x_0 \in F(x_0)$, that means $0 > \langle Tx_0, x_0 - x_0\rangle = 0$, a contradiction. This completes the proof.

Corollary 2.1: *Let X be a locally convex Hausdorff topological vector space and K a nonempty, compact and convex subset of X. Suppose that $T: K \rightarrow X'$ (dual of X) is a monotone mapping. Then the NVI problem (1) has a solution.*

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