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ON A THIRD ORDER PARABOLIC EQUATION WITH A NONLOCAL BOUNDARY CONDITION

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In this paper we demonstrate the existence, uniqueness and continuous dependence of a strong solution upon the data, for a mixed problem which combine classical boundary conditions and an integral condition, such as the total mass, flux or energy, for a third order parabolic equation. We present a functional analysis method based on an a priori estimate and on the density of the range of the operator generated by the studied problem.

Key words: Integral Condition, Third Order Parabolic Equation, A Priori Estimate, Strong Solution.

AMS subject classifications: 35K25, 35K50.

1. Introduction

In the rectangle $Q = (0,l) \times (0,T)$, with $l < \infty$ and $T < \infty$, we consider the one-dimensional third order parabolic equation

$$\mathcal{L}v = \frac{\partial v}{\partial t} - \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial v}{\partial x}\right) = \mathrm{f}(x,t). \tag{1.1}$$

Assumption A: We shall assume that

$$c_0 \leq a(x,t) \leq c_1, \frac{\partial a(x,t)}{\partial t} \leq c_2,$$

where $c_i > 0$, $(i = 0,1,2)$.

We pose the following problem for equation (1.1): to determine its solution v in Q satisfying *the initial condition*

$$\ell u = v(x,0) = \Phi(x), \quad x \in (0,l), \tag{1.2}$$

and *the boundary conditions*

$$\frac{\partial v(0,t)}{\partial x} = \chi(t), \quad t \in (0,T), \tag{1.3}$$

$$\frac{\partial^2 v(0,t)}{\partial x^2} = \vartheta(t), \quad t \in (0,T), \tag{1.4}$$

$$\int_0^l v(x,t)dx = m(t), \ \ t \in (0,T), \tag{1.5}$$

where $\Phi(x)$, $\chi(t)$, $\vartheta(t)$, $m(t)$, $a(x,t)$ and $\mathrm{f}(x,t)$ are known functions.

The data satisfies the following compatibility conditions:

$$\frac{\partial \Phi(0)}{\partial x} = \chi(0), \ \frac{\partial^2 \Phi(0)}{\partial x^2} = \vartheta(0), \ \int_0^l \Phi(x)dx = m(0).$$

The first investigation of problems of this type goes back to Cannon [12] and Batten [2] independently in 1963. The author of [12] proved, with the aid of an integral equation, the existence and uniqueness of the solution for a mixed problem which combine Dirichlet and integral conditions for the homogeneous heat equation. Kamynin [21] extended the result of [12] to the general linear second order parabolic equation in 1964, by using a system of integral equations.

Along a different line, mixed problems for second order parabolic equations, which combine classical and integral conditions, were considered by Ionkin [17], Cannon-van der Hoek [13, 14], Benouar-Yurchuk [3], Yurchuk [25], Cahlon-Kulkarni-Shi [11], Cannon-Esteva-van der Hoek [15], and Shi [23]. A mixed problem with integral condition for second order pluriparabolic equation has been investigated in Bouziani [7]. Mixed problems with only integral conditions for a $2m$-parabolic equation was studied in Bouziani [6], and for second order parabolic and hyperbolic equations in Bouziani-Benouar [8, 9].

In this paper, we demonstrate that problem (1.1)-(1.5) possesses a unique strong solution that depends continuously upon the data. We present a functional analysis method which is an elaboration of that in Bouziani [4, 5] and Bouziani-Benouar [10].

To achieve the purpose, we reduce the nonhomogeneous boundary conditions (1.3)-(1.5) to homogeneous conditions, by introducing a new, unknown function u defined as:

$$u(x,t) = v(x,t) - \mathfrak{U}(x,t),$$

where

$$\mathfrak{U}(x,t) = x\left(1 - \frac{2x^2}{l^2}\right)\chi(t) + \frac{1}{2}\left(x^2 - \frac{l^2}{3}\right)\vartheta(t) + \frac{4x^3}{l^4}m(t).$$

Then, the problem can be formulated as follows:

$$\mathcal{L}u = \mathrm{f} - \mathcal{L}\mathfrak{U} = f, \tag{1.6}$$

$$\ell u = u(x,0) = \Phi(x) - \ell\mathfrak{U} = \varphi(x), \tag{1.7}$$

$$\frac{\partial u(0,t)}{\partial x} = 0, \tag{1.8}$$

$$\frac{\partial^2 u(0,t)}{\partial x^2} = 0, \tag{1.9}$$

$$\int_0^l u(x,t)dx = 0. \tag{1.10}$$

Here, we assume that the function φ, satisfies conditions of the form (1.8)-(1.10), i.e.,

$$\frac{\partial \varphi(0)}{\partial x} = 0, \ \frac{\partial^2 \varphi(0)}{\partial x^2} = 0 \text{ and } \int_0^l \varphi(x)dx = 0. \tag{1.11}$$

Instead of searching for the function v, we search for the function u. So, the strong solution of problem (1.1)-(1.5) will be given by: $v(x,t) = u(x,t) + \mathcal{U}(x,t)$. □

2. Preliminaries

We employ certain function spaces to investigate our problem. Let $L^2(0,l)$, $L^2(0,T;L^2(0,l)) = L^2(Q)$ be the standard functional spaces, $\| \cdot \|_{0,Q}$ and $(\cdot,\cdot)_{0,Q}$ denote the norm and the scalar product in $L^2(Q)$, $L^2_\sigma(0,l)$ be the weighted space of square integrable functions on $(0,l)$ with the finite norm

$$\| u \|^2_{L^2_\sigma(0,l)} := \int_0^l (l-x)u^2 dx,$$

$B^1_2(0,l)$ be the Hilbert space defined, for the first time in [6], by

$$B^1_2(0,l) := \{u / \mathcal{T}_x u \in L^2(0,l)\},$$

where $\mathcal{T}_x u := \int_x^l u(\xi,t)d\xi$, and let $L^2(0,T;B^1_2(0,l))$ be the space of all functions which are square integrable on $(0,T)$ in the Bochner sense, i.e., Bochner integrable and satisfying

$$\int_0^T \| u \|^2_{B^1_2(0,l)} dt < \infty.$$

Problem (1.6)-(1.10) is equivalent to the operator equation

$$Lu = \mathcal{F},$$

where $\mathcal{F} = (f,\varphi), L = (\mathcal{L},\ell)$ with the domain $D(L)$ consisting of all functions $u \in L^2(Q)$ with $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^3 u}{\partial x^3}, \frac{\partial^2 u}{\partial t \partial x}, \frac{\partial^3 u}{\partial t \partial x^2} \in L^2(Q)$ and (u) satisfying conditions (1.8)-(1.10); the operator L is on B into F; B is the Banach space obtained by the completion of $D(L)$ in the form

$$\| u \|^2_B = \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(0,T;B^1_2(0,l))} + \sup_{0 \le \tau \le T} \left\| \frac{\partial u(x,\tau)}{\partial x} \right\|^2_{L^2_\sigma(0,l)}$$

and F is the Hilbert space of the vector-valued functions $\mathcal{F} = (f,\varphi)$ with the norm

$$\| \mathfrak{F} \|_F^2 = \| f \|_{0,Q}^2 + \left\| \frac{\partial \varphi}{\partial x} \right\|_{L^2_\sigma(0,l)}^2 .$$

Let $\overline{L}$ be the closure of the operator L with the domain $D(\overline{L})$.

Definition: A solution of the operator equation

$$\overline{L} u = \mathfrak{F}$$

is called a *strong solution* of the problem (1.6)-(1.10).

We now introduce the family of operators $\rho_\epsilon^{-1}\theta$ and $\left(\rho_\epsilon^{-1}\right)^*\theta$ defined by the formulas

$$\rho_\epsilon^{-1}\theta = \frac{1}{\epsilon}\int_0^t e^{\frac{1}{\epsilon}(\tau - t)}\theta(x,\tau)d\tau, \quad \epsilon > 0,$$

$$\left(\rho_\epsilon^{-1}\right)^*\theta = -\frac{1}{\epsilon}\int_t^T e^{\frac{1}{\epsilon}(t - \tau)}\theta(x,\tau)d\tau, \quad \epsilon > 0,$$

which we use as smoothing operators with respect to t. These operators provide the solutions of the problems

$$\begin{cases} \epsilon\dfrac{\partial \rho_\epsilon^{-1}\theta}{\partial t} + \rho_\epsilon^{-1}\theta = \theta, & (2.1) \\ \rho_\epsilon^{-1}\theta(x,0) = 0 & (2.2) \end{cases}$$

and

$$\begin{cases} -\epsilon\dfrac{\partial\left(\rho_\epsilon^{-1}\right)^*\theta}{\partial t} + \left(\rho_\epsilon^{-1}\right)^*\theta = \theta, & (2.3) \\ \left(\rho_\epsilon^{-1}\right)^*\theta(x,T) = 0 & (2.4) \end{cases},$$

respectively. They have the following properties.

Lemma 1: *For all $\theta \in L^2(0,T)$, we have*

(*i*) $\rho_\epsilon^{-1}\theta(x,t) \in H^1(0,T)$ *and* $\rho_\epsilon^{-1}\theta(x,0) = 0$*;*

(*ii*) $\left(\rho_\epsilon^{-1}\right)^*\theta(x,t) \in H^1(0,T)$ *and* $\left(\rho_\epsilon^{-1}\right)^*\theta(x,T) = 0$.

Lemma 2: *For all θ and all $\hbar$ in $L^2(Q)$, we have*

$$\int_Q \rho_\epsilon^{-1}\theta\hbar\, dx dt = \int_Q \theta\left(\rho_\epsilon^{-1}\right)^*\hbar\, dx dt.$$

This lemma states that the operators $\left(\rho_\epsilon^{-1}\right)^*$ are conjugate to ρ_ϵ^{-1}. □

Lemmas 1 and 2 are proved directly by using the definitions of operators ρ_ϵ^{-1} and $\left(\rho_\epsilon^{-1}\right)^*$. □

Lemma 3: *For all $\theta \in L^2(0,T)$, we have*

$$\rho_\epsilon^{-1}\frac{\partial\theta}{\partial\tau} = \frac{\partial}{\partial t}\rho_\epsilon^{-1}\theta + \frac{1}{\epsilon}e^{-t/\epsilon}\theta(x,0).$$

For the proof of the above lemma, it suffices to integrate by parts the expression $\rho_\epsilon^{-1}\frac{\partial\theta}{\partial\tau}$. □

Lemma 4: *For all $\theta \in L^2(0,T)$, we have*

(i)

$$\int_0^T \left\| \rho_\epsilon^{-1}\theta \right\|_{0,(0,l)} dt \leq \int_0^T \| \theta \|_{0,(0,l)} dt$$

and

$$\int_0^T \left\| \rho_\epsilon^{-1}\theta - \theta \right\|_{0,(0,l)} dt \to 0 \quad \textit{for } \epsilon \to 0;$$

(ii)

$$\int_0^T \left\| \left(\rho_\epsilon^{-1}\right)^* \theta \right\|_{0,(0,l)}^2 dt \leq \int_0^T \| \theta \|_{0,(0,l)}^2 dt$$

and

$$\int_0^T \left\| \left(\rho_\epsilon^{-1}\right)^* \theta - \theta \right\|_{0,(0,l)}^2 dt \to 0 \quad \textit{for } \epsilon \to 0.$$

Proof of Lemma 4 is similar to the proof of the lemma of Section 2.18 in [1]. □

We easily get the following lemma.

Lemma 5: *If*

$$A(t)u = \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial u}{\partial x}\right) \tag{2.5}$$

then

$$A(t)\rho_\epsilon^{-1} = \rho_\epsilon^{-1}A(\tau) + \epsilon\rho_\epsilon^{-1}A'(\tau)\rho_\epsilon^{-1},$$

where $A'(t)$ is the operator of form (2.5) whose coefficient is the first derivative with respect to t of the corresponding coefficient of $A(t)$.

3. A Priori Estimate and Its Consequences

Theorem 1: *Under Assumption A, there exists a positive constant c, independent of u, such that*

$$\| u \|_B \leq c \| Lu \|_F. \tag{3.1}$$

Proof: We multiply equation (1.6) by an integro-differential operator

$$Mu = (l-x)\mathfrak{T}_x\frac{\partial u}{\partial t} - 2\mathfrak{T}_x^2\frac{\partial u}{\partial t}$$

and integrate over Q^τ, where $Q^\tau = (0,l)\times(0,\tau)$. Consequently,

$$\int\int_{Q^\tau} \mathfrak{L}u \cdot Mu\,dx\,dt = \int\int_{Q^\tau} \frac{\partial u}{\partial t}(l-x)\mathfrak{T}_x\frac{\partial u}{\partial t}dx\,dt - 2\int\int_{Q^\tau}\frac{\partial u}{\partial t}\mathfrak{T}_x^2\frac{\partial u}{\partial t}dx\,dt \tag{3.2}$$

$$-\int\limits_{Q^\tau}\!\!\int \frac{\partial^2}{\partial x^2}\Big(a(x,t)\frac{\partial u}{\partial x}\Big)(l-x)\Im_x\frac{\partial u}{\partial t}dxdt+2\int\limits_{Q^\tau}\!\!\int \frac{\partial^2}{\partial x^2}\Big(a(x,t)\frac{\partial u}{\partial x}\Big)\Im_x^2\frac{\partial u}{\partial t}dxdt.$$

We know from the integration by parts that

$$\int\limits_{Q^\tau}\!\!\int \frac{\partial u}{\partial t}(l-x)\Im_x\frac{\partial u}{\partial t}dxdt=-\frac{1}{2}\int\limits_{Q^\tau}\!\!\int\Big(\Im_x\frac{\partial u}{\partial t}\Big)^2dxdt, \tag{3.3}$$

$$-2\int\limits_{Q^\tau}\!\!\int \frac{\partial u}{\partial t}\Im_x^2\frac{\partial u}{\partial t}dxdt=2\int\limits_{Q^\tau}\!\!\int\Big(\Im_x\frac{\partial u}{\partial t}\Big)^2dxdt, \tag{3.4}$$

$$-\int\limits_{Q^\tau}\!\!\int \frac{\partial^2}{\partial x^2}\Big(a(x,t)\frac{\partial u}{\partial x}\Big)(l-x)\Im_x\frac{\partial u}{\partial t}dxdt=\frac{1}{2}\int_0^l(l-x)a(x,\tau)\Big(\frac{\partial u(x,\tau)}{\partial x}\Big)^2dx$$

$$-\frac{1}{2}\int_0^l(l-x)a(x,0)\Big(\frac{\partial\varphi}{\partial x}\Big)^2dx-\frac{1}{2}\int\limits_{Q^\tau}(l-x)\frac{\partial a(x,t)}{\partial t}\Big(\frac{\partial u}{\partial x}\Big)^2dxdt \tag{3.5}$$

$$-2\int\limits_{Q^\tau}\!\!\int a(x,t)\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}dxdt,$$

$$2\int\limits_{Q^\tau}\!\!\int \frac{\partial^2}{\partial x^2}\Big(a(x,t)\frac{\partial u}{\partial x}\Big)\Im_x^2\frac{\partial u}{\partial t}dxdt=2\int\limits_{Q^\tau}\!\!\int a(x,t)\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}dxdt. \tag{3.6}$$

Substituting (3.3)-(3.6) into (3.2), we obtain

$$\frac{3}{2}\int\limits_{Q^\tau}\!\!\int\Big(\Im_x\frac{\partial u}{\partial t}\Big)^2dxdt+\frac{1}{2}\int_0^l(l-x)a(x,\tau)\Big(\frac{\partial u(x,\tau)}{\partial x}\Big)^2dx$$

$$=\int\limits_{Q^\tau}\!\!\int f\Big((l-x)\Im_x\frac{\partial u}{\partial t}-2\Im_x^2\frac{\partial u}{\partial t}\Big)dxdt+\frac{1}{2}\int_0^l(l-x)a(x,0)\Big(\frac{\partial\varphi}{\partial x}\Big)^2dx \tag{3.7}$$

$$+\frac{1}{2}\int\limits_{Q^\tau}\!\!\int(l-x)\frac{\partial a(x,t)}{\partial t}\Big(\frac{\partial u}{\partial x}\Big)^2dxdt.$$

Further, by virtue of inequality (2.2) in [6] and the Cauchy inequality, the first integral on the right-hand side of (3.7) is estimated as follows

$$\int\limits_{Q^\tau}\!\!\int f\Big((l-x)\Im_x\frac{\partial u}{\partial t}-2\Im_x^2\frac{\partial u}{\partial}\Big)dxdt$$

$$\le\frac{3l^2}{2}\int\limits_{Q^\tau}\!\!\int f^2dxdt+\int\limits_{Q^\tau}\!\!\int\Big(\Im_x\frac{\partial u}{\partial t}\Big)^2dxdt. \tag{3.8}$$

Substituting (3.8) in (3.7) and using Assumption A, we get

$$\left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(0,\tau;B^1_2(0,l))} + \left\| \frac{\partial u(x,\tau)}{\partial x} \right\|^2_{L^2_\sigma(0,l)} \tag{3.9}$$

$$\leq c_3 \left(\| f \|^2_{0,Q^\tau} + \left\| \frac{\partial \varphi}{\partial x} \right\|^2_{L^2_\sigma(0,l)} \right) + c_4 \left\| \frac{\partial u}{\partial x} \right\|_{L^2(0,\tau;L^2_\sigma(0,l))},$$

where

$$c_3 = \frac{\max(3l^2, c_1)}{\min(1, c_0)}$$

and

$$c_4 = \frac{c_2}{\min(1, c_0)}.$$

We eliminate the last term on the right-hand side of (3.9). To do this we use the following lemma.

Lemma 6: *If $f_i(\tau)$ $(i = 1,2,3)$ are nonnegative functions on $(0,T)$, $f_1(\tau)$ and $f_2(\tau)$ are integrable on $(0,T)$, and $f_3(\tau)$ is nondecreasing on $(0,T)$ then it follows, from*

$$\mathfrak{T}_\tau f_1 + f_2 \leq f_3 + c\mathfrak{T}_\tau f_2,$$

that

$$\mathfrak{T}_\tau f_1 + f_2 \leq \exp(c\tau).f_3,$$

where

$$\mathfrak{T}_\tau f_i = \int_0^\tau f_i(t)dt, \quad (i = 1,2).$$

The proof of the above lemma is similar to that of Lemma 7.1 in [16]. □

Returning to the proof of Theorem 1, we denote the first term on the left-hand side of (3.9) by $f_1(\tau)$, the remaining term on the same side on (3.9) by $f_2(\tau)$, and the sum of two first terms on the right-hand side of (3.9) by $f_3(\tau)$. Consequently, Lemma 6 implies the inequality

$$\left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(0,\tau;B^1_2(0,l))} + \left\| \frac{\partial u(x,\tau)}{\partial x} \right\|^2_{L^2_\sigma(0,l)} \tag{3.10}$$

$$\leq c_3 e^{c_4\tau} \left(\| f \|^2_{0,Q^\tau} + \left\| \frac{\partial \varphi}{\partial x} \right\|^2_{L^2_\sigma(0,l)} \right)$$

$$\leq c_5 \left(\| f \|^2_{0,Q} + \left\| \frac{\partial \varphi}{\partial x} \right\|^2_{L^2_\sigma(0,l)} \right),$$

where

$$c_5 = c_3 \exp(c_4 T).$$

Since the right-hand side of the above inequality does not depend on τ, in the left-hand side we take the upper bound with respect to τ from 0 to T. Therefore, we obtain inequality (3.1), where $c = c_5^{1/2}$.

Proposition 1: *The operator L from B into F is closable.*

The proof of this proposition is analogous to the proof of the proposition in [7]. □

Since the points of the graph of $\overline{L}$ are limits of the sequences of points of the graph of L, we can extend (3.1) to apply to strong solutions by taking the limits.

Corollary 1: *Under Assumption A, there is a constant $c > 0$, independent of u, such that*

$$\| u \|_B \leq c \| \overline{L} u \|_F, \quad \forall u \in D(\overline{L}). \tag{3.11}$$

Let $R(L)$ and $R(\overline{L})$ denote the set of values taken by L and $\overline{L}$, respectively.

Inequality (3.11) implies the following corollary.

Corollary 2: *The range $R(\overline{L})$ is closed in F, $\overline{R(L)} = R(\overline{L})$ and $\overline{L}^{-1} = \overline{L^{-1}}$, where $\overline{L^{-1}}$ is the extension of L^{-1} by continuity from $R(L)$ to $\overline{R(L)}$.*

4. Solvability of the Problem

Theorem 2: *Let Assumption A be satisfied and let $\frac{\partial a}{\partial x}$ and $\frac{\partial^2 a}{\partial x \partial t}$ be bounded. Then for arbitrary $f \in L^2(Q)$ and $\frac{\partial \varphi}{\partial x} \in L^2_\sigma(0,l)$, problem (1.6)-(1.10) admits a unique strong solution $u = \overline{L}^{-1}\mathfrak{F} = \overline{L^{-1}}\mathfrak{F}$.*

Proof: Corollary 1 asserts that, if a strong solution exists, it is unique and depends continuously on $\mathfrak{F}$. (If u is considered in the topology of B and $\mathfrak{F}$ is considered in the topology of F.) Corollary 2 states that, to prove that (1.6)-(1.10) has a strong solution for an arbitrary $\mathfrak{F} = (f, \varphi) \in F$ it is sufficient to show the equality $\overline{R(L)} = F$. To this end, we need the following proposition.

Proposition 2: *Let the assumptions of Theorem 2 hold and let $D_0(L)$ be the set of all $u \in D(L)$ vanishing in a neighborhood of $t = 0$. If, for $\hbar \in L^2(Q)$ and for all $u \in D_0(L)$, we have*

$$(\mathcal{L}u, \hbar)_{L^2(Q)} = 0, \tag{4.1}$$

then $\hbar$ vanishes almost everywhere in Q.

Proof of the proposition: We can write (4.1) as follows

$$\int_Q \int \frac{\partial u}{\partial t} \cdot \hbar dx dt = \int_Q \int A(t) u \cdot \hbar dx dt. \tag{4.2}$$

Replacing u by the smooth function $\rho_\epsilon^{-1} u$ in (4.2), this yields, from Lemma 5, that

$$\int_Q \int \frac{\partial \rho_\epsilon^{-1}}{\partial t} \cdot \hbar dx dt = \int_Q \int \rho_\epsilon^{-} A u \cdot \hbar dx dt + \epsilon \int_Q \int \rho_\epsilon^{-1} A' \rho_\epsilon^{-1} u \cdot \hbar dx dt. \tag{4.3}$$

Applying Lemma 3 to the left-hand side of (4.3), and Lemma 2 to the obtained equality, we obtain

$$\begin{aligned} &\int_Q \int \frac{\partial u}{\partial t} \cdot \left(\rho_\epsilon^{-1}\right)^* \hbar dx dt \\ &= \int_Q \int A u \cdot \left(\rho_\epsilon^{-1}\right)^* \hbar dx dt + \epsilon \int_Q \int A' \rho_\epsilon^{-1} u \cdot \left(\rho_\epsilon^{-1}\right)^* \hbar dx dt. \end{aligned} \tag{4.4}$$

The standard integration by parts with respect to t in the left-hand side of (4.4) leads to

$$\int\int_{Q} u\cdot\frac{\partial\left(\rho_\epsilon^{-1}\right)^*}{dt}dxdt=\int\int_{A} Au\cdot\left(\rho_\epsilon^{-1}\right)^*\hbar dxdt$$
$$+\epsilon\int\int_{Q} A'\rho_\epsilon^{-1}u\cdot\left(\rho_\epsilon^{-1}\right)^*\hbar dxdt. \tag{4.5}$$

The operator $A(t)$ with boundary conditions (1.8)-(1.10) has, on $L^2(0,l)$, the continuous inverse. Hence,

$$A'\rho_\epsilon^{-1}u=A'\rho_\epsilon^{-1}A^{-1}Au=\Lambda_\epsilon Au. \tag{4.6}$$

Thus, from (4.5) and (4.6), we obtain

$$\begin{aligned}\int\int_{Q} u\cdot\frac{\partial\left(\rho_\epsilon^{-1}\right)^*\hbar}{\partial t}dxdt&=\int\int_{Q} Au\cdot\left(\rho_\epsilon^{-1}\right)^*\hbar dxdt+\epsilon\int\int_{Q}\Lambda_\epsilon Au\cdot\left(\rho_\epsilon^{-1}\right)^*\hbar dxdt\\&=\int\int_{Q} Au\cdot\left(I+\epsilon\Lambda_\epsilon^*\right)\left(\rho_\epsilon^{-1}\right)^*\hbar dxdt.\end{aligned} \tag{4.7}$$

Defining $A^{-1}(t)$, we apply operator $\Im_x^2$ to both sides of $A(t)u=g$. After this operation, we get

$$\frac{\partial u}{\partial x}=\frac{1}{a(x,t)}\int_0^x(x-\xi)g(\xi,t)d\xi. \tag{4.8}$$

We now integrate each term of (4.8) over $[0,x]$ with respect to ξ. Consequently,

$$A^{-1}(t)g=\int_0^x\frac{d\xi}{a(\xi,t)}\int_0^\xi(\xi-\eta)g(\eta,t)d\eta+c_6. \tag{4.9}$$

To compute the constant c_6 in (4.9), we multiply (4.8) by $(l-x)$ and integrate the obtained equation over $[0,l]$. Therefore,

$$\int_0^l(l-x)\frac{\partial u}{\partial x}dx=\int_0^l\frac{(l-x)dx}{a(x,t)}\int_0^x(x-\xi)g(\xi,t)d\xi. \tag{4.10}$$

Integration by parts of the left-hand side of (4.10), gives

$$c_6=-\frac{1}{l}\int_0^l\frac{(l-x)dx}{a(x,t)}\int_0^x(x-\xi)g(\xi,t)d\xi.$$

Note that for the determination of Λ_ϵ and Λ_ϵ^*, the corresponding calculations are not difficult, but they are long. Therefore, we only give the final results of the computations:

$$\Lambda_\epsilon Au = \Bigg(\frac{\partial^3 a(x,t)}{\partial x^2 \partial t}\rho_\epsilon^{-1} - 2\frac{\partial^2 a(x,t)}{\partial x \partial t}\rho_\epsilon^{-1}\frac{1}{a(x,\tau)}\frac{\partial a(x,\tau)}{\partial x} + \frac{\partial a(x,t)}{\partial t}\rho_\epsilon^{-1}\frac{1}{(a(x,\tau))^2}$$

$$\times\left(\frac{\partial a(x,\tau)}{\partial x}\right)^2 - \frac{\partial a(x,t)}{\partial t}\rho_\epsilon^{-1}\frac{1}{a(x,\tau)}\frac{\partial^2 a(x,\tau)}{\partial x^2}\Bigg)\frac{1}{a(x,\tau)}\left(\int_0^x (x-\xi)Au(\xi,\tau)d\xi\right) \quad (4.11)$$

$$+2\left(\frac{\partial^2 a(x,t)}{\partial x \partial t}\rho_\epsilon^{-1} - \frac{\partial a(x,t)}{\partial t}\rho_\epsilon^{-1}\frac{\partial a(x,\tau)}{\partial x}\frac{1}{a(x,\tau)}\right)\frac{1}{a(x,\tau)}\left(\int_0^x Au(\xi,\tau)d\xi\right)$$

$$+\frac{\partial a(x,t)}{\partial t}\rho_\epsilon^{-1}\frac{1}{a(x,\tau)}Au;$$

$$\Lambda_\epsilon^*\left(\rho_\epsilon^{-1}\right)^*\hbar = \frac{1}{a(x,t)}\left(\rho_\epsilon^{-1}\right)^*\frac{\partial a(x,\tau)}{\partial \tau}\left(\rho_\epsilon^{-1}\right)^*\hbar + \int_x^l \frac{(x-\xi)}{a(\xi,t)}\Bigg\{\left(\rho_\epsilon^{-1}\right)^*\frac{\partial^3 a(\xi,\tau)}{\partial \tau \partial \xi^2}$$

$$-2\frac{1}{a(\xi,t)}\frac{\partial a(\xi,t)}{\partial \xi}\left(\rho_\epsilon^{-1}\right)^*\frac{\partial^2 a(\xi,\tau)}{\partial \tau \partial \xi} + 2\frac{1}{(a(\xi,t))^2}\left(\frac{\partial a(\xi,t)}{\partial x}\right)^2\left(\rho_\epsilon^{-1}\right)^*\frac{\partial a(\xi,\tau)}{\partial \tau}$$

$$+\frac{1}{a(\xi,t)}\frac{\partial a(\xi,t)}{\partial \xi}\left(\rho_\epsilon^{-1}\right)^*\frac{\partial a(\xi,\tau)}{\partial \tau}\Bigg\}\left(\rho_\epsilon^{-1}\right)^*\hbar(\xi,\tau)d\xi \quad (4.12)$$

$$+2\int_x^l \frac{1}{a(\xi,t)}\left(\left(\rho_\epsilon^{-1}\right)^*\frac{\partial^2 a(\xi,\tau)}{\partial \tau \partial \xi} - \frac{1}{a(\xi,t)}\frac{\partial a(\xi,t)}{\partial \xi}\left(\rho_\epsilon^{-1}\right)^*\frac{\partial a(\xi,\tau)}{\partial \tau}\right)\left(\rho_\epsilon^{-1}\right)^*\hbar(\xi,\tau)d\xi.$$

The left-hand side of (4.7) shows that the mapping $\int\int_Q Au \cdot K_\epsilon\left(\rho_\epsilon^{-1}\right)^*\hbar dxdt$ is a continuous linear functional of u, where

$$K_\epsilon\left(\rho_\epsilon^{-1}\right)^*\hbar = \left(I+\epsilon\Lambda_\epsilon^*\right)\left(\rho_\epsilon^{-1}\right)^*\hbar. \quad (4.13)$$

Consequently, this assertion holds true, if the function K_ϵ has the following properties

$$\frac{\partial K_\epsilon}{\partial x} \in L^2(Q),\ \frac{\partial^2 K_\epsilon}{\partial x^2} \in L^2(Q) \text{ and } \frac{\partial^3 K_\epsilon}{\partial x^3} \in L^2(Q),$$

and satisfies the following conditions:

$$K_\epsilon\Big|_{x=l} = 0,\ \frac{\partial K_\epsilon}{\partial x}\Big|_{x=l} = 0,\ \frac{\partial^2 K_\epsilon}{\partial x^2}\Big|_{x=0} = 0 \text{ and } \frac{\partial^2 K_\epsilon}{\partial x^2}\Big|_{x=l} = 0. \quad (4.14)$$

From (4.12), we deduce that the operator Λ_ϵ^* is bounded on $L^2(Q)$. Hence, the norm of $\epsilon\Lambda_\epsilon^*$ on $L^2(Q)$ is smaller than 1 for sufficiently small ϵ. So, the operator K_ϵ has the continuous inverse operator in $L^2(Q)$.

From (4.12) and (4.14), we deduce that

$$\left(I+\epsilon\frac{1}{a(x,t)}\left(\rho_\epsilon^{-1}\right)^*\frac{\partial a(x,\tau)}{\partial \tau}\right)\left(\rho_\epsilon^{-1}\right)^*\hbar\Big|_{x=l} = 0, \quad (4.15)$$

$$\left(I+\epsilon\frac{1}{a(x,t)}\left(\rho_\epsilon^{-1}\right)^*\frac{\partial a(x,\tau)}{\partial\tau}\right)\frac{\partial\left(\rho_\epsilon^{-1}\right)^*\hbar}{\partial x}\Bigg|_{x=l}=0, \tag{4.16}$$

$$\left(I+\epsilon\frac{1}{a(x,t)}\left(\rho_\epsilon^{-1}\right)^*\frac{\partial a(x,\tau)}{\partial\tau}\right)\frac{\partial^2\left(\rho_\epsilon^{-1}\right)^*\hbar}{\partial x^2}\Bigg|_{x=0}=0, \tag{4.17}$$

$$\left(I+\epsilon\frac{1}{a(x,t)}\left(\rho_\epsilon^{-1}\right)^*\frac{\partial a(x,\tau)}{\partial\tau}\right)\frac{\partial^2\left(\rho_\epsilon^{-1}\right)^*\hbar}{\partial x^2}\Bigg|_{x=l}=0. \tag{4.18}$$

For each fixed $x\in[0,l]$ and sufficiently small ϵ, the operator

$$\left(I+\epsilon\frac{1}{a(x,t)}\left(\rho_\epsilon^{-1}\right)^*\frac{\partial a(x,\tau)}{\partial\tau}\right)\left(\rho_\epsilon^{-1}\right)^*$$

has the continuous inverse operator on $L^2(0,T)$. Hence, (4.15)-(4.18) imply that

$$\left(\rho_\epsilon^{-1}\right)^*\hbar\,|_{x=l}=0,\ \frac{\partial\left(\rho_\epsilon^{-1}\right)^*\hbar}{\partial x}\Bigg|_{x=l}=0,\ \frac{\partial^2\left(\rho_\epsilon^{-1}\right)^*\hbar}{\partial x^2}\Bigg|_{x=0}=0,$$

$$\frac{\partial^2\left(\rho_\epsilon^{-1}\right)^*\hbar}{\partial x^2}\Bigg|_{x=l}=0.$$

In other words, (4.15)-(4.18) imply that

$$\hbar\,|_{x=l}=0,\ \frac{\partial\hbar}{\partial x}\Big|_{x=l}=0,\ \frac{\partial^2\hbar}{\partial x^2}\Big|_{x=0}=0,\ \frac{\partial^2\hbar}{\partial x^2}\Big|_{x=l}=0. \tag{4.19}$$

Set

$$\hbar=\left((l-x)\mathfrak{T}_x z-2\mathfrak{T}_x^2 z\right). \tag{4.20}$$

Differentiating (4.20) with respect to x, we obtain

$$\begin{cases}\dfrac{\partial\hbar}{\partial x}=(\mathfrak{T}_x z-(l-x)z)\in L^2(Q),\\[2mm] \dfrac{\partial^2\hbar}{\partial x^2}=-(l-x)\dfrac{\partial z}{\partial x}\in L^2(Q),\\[2mm] \dfrac{\partial^3\hbar}{\partial x^3}=-\dfrac{\partial}{\partial x}\left((l-x)\dfrac{\partial z}{\partial x}\right)\in L^2(Q).\end{cases} \tag{4.21}$$

From (4.20), (4.21), and (4.19), we deduce that the conditions

$$\mathfrak{T}_l z=0,\mathfrak{T}_l^2 z=0,(l-x)\frac{\partial z}{\partial x}\Big|_{x=0}=0,(l-x)\frac{\partial z}{\partial x}\Big|_{x=l}=0 \tag{4.22}$$

are met.

In (4.2), we replace $\hbar$ by its representation (4.20). Consequently,

$$\int_Q\int \frac{\partial u}{\partial t}((l-x)\mathfrak{T}_x z - 2\mathfrak{T}_x^2 z)dxdt = \int_Q\int A(t)u((l-x)\mathfrak{T}_x z - 2\mathfrak{T}_x^2 z)dxdt$$
$$= -\int_Q\int a(x,t)\frac{\partial u}{\partial x}(l-x)\frac{\partial z}{\partial x}dxdt. \tag{4.23}$$

Substituting (2.3) in (4.23) (with $\theta = z$) and integrating by parts (with respect to x), by taking into account (4.22), we obtain

$$\int_Q\int \frac{\partial u}{\partial t}((l-x)\mathfrak{T}_x z - 2\mathfrak{T}_x^2 z)dxdt = \epsilon\int_Q\int a(x,t)\frac{\partial u}{\partial x}(l-x)\frac{\partial^2\left(\rho_\epsilon^{-1}\right)^* z}{\partial x\partial t}\,dxdt$$
$$-\int_Q\int a(x,t)\frac{\partial u}{\partial x}(l-x)\frac{\partial\left(\rho_\epsilon^{-1}\right)^* z}{\partial x}dxdt. \tag{4.24}$$

Putting

$$u = \mathfrak{T}_t\left(e^{c_7\tau}\left(\rho_\epsilon^{-1}\right)^* z\right) = \int_0^t e^{c_7\tau}\left(\rho_\epsilon^{-1}\right)^* z d\tau \tag{4.25}$$

in relation (4.24), where c_7 is a constant such that $c_7c_0 - c_2 - c_2^2/2c_0 \geq 0$, and integrating by parts with respect to t on each term of the right-hand side of the obtained equality, we obtain, by taking into account (2.4) and due to $u \in D_0(L)$ that

$$\epsilon\int_Q\int (l-x)a(x,t)\,\frac{\partial u}{\partial x}\cdot\frac{\partial^2\left(\rho_\epsilon^{-1}\right)^*}{\partial x\partial t}dxdt$$
$$= -\epsilon\int_Q\int (l-x)e^{c_7 t}a(x,t)\left(\frac{\partial\left(\rho_\epsilon^{-1}\right)^* z}{\partial x}\right)^2 dxdt \tag{4.26}$$
$$-\epsilon\int_Q\int (l-x)\frac{\partial a(x,t)}{\partial t}\,\frac{\partial u}{\partial x}\cdot\frac{\partial\left(\rho_\epsilon^{-1}\right)^* z}{\partial x}dxdt,$$

$$\int_Q\int (l-x)a(x,t)\frac{\partial u}{\partial x}\,\frac{\partial\left(\rho_\epsilon^{-1}\right)^* z}{\partial x}dxdt$$
$$= -\int_Q\int (l-x)e^{-c_7 t}a(x,t)\frac{\partial u}{\partial x}\,\frac{\partial^2 u}{\partial x\partial t}dxdt \tag{4.27}$$
$$= -\frac{1}{2}\int_0^l (l-x)e^{-c_7 T}a(x,T)\left(\frac{\partial u(x,T)}{\partial x}\right)^2 dx$$

$$-\frac{1}{2}\int\int_Q (l-x)e^{-c_7t}\left(c_7 a(x,t)-\frac{\partial a(x,t)}{\partial t}\right)\left(\frac{\partial u}{\partial x}\right)^2 dxdt.$$

Elementary calculations, starting from (4.26) and (4.27), yield the inequalities

$$\epsilon\int\int_Q (l-x)a(x,t)\frac{\partial u}{\partial x}\cdot\frac{\partial^2\left(\rho_\epsilon^{-1}\right)^* z}{\partial x\partial t}dxdt$$

$$\leq\frac{\epsilon c_3^2}{4c_0}\int\int_Q (l-x)e^{-c_7t}\left(\frac{\partial u}{\partial x}\right)^2 dxdt, \tag{4.28}$$

and

$$-\int\int_Q (l-x)a(x,t)\frac{\partial u}{\partial x}\cdot\frac{\partial\left(\rho_\epsilon^{-1}\right)^* z}{\partial x\partial t}dxdt$$

$$\leq -\frac{1}{2}\left(c_7c_0-c_2-\frac{\epsilon c_2^2}{2c_0}\right)\int\int_Q (l-x)e^{-c_7t}\left(\frac{\partial u}{\partial x}\right)^2 dxdt, \tag{4.29}$$

Substituting (4.28) and (4.29) into (4.24), we get

$$\int\int_Q e^{c_7t}\left(\rho_\epsilon^{-1}\right)^* z((l-x)\Im_x z-2\Im_x^2 z)dxdt$$

$$\leq -\frac{1}{2}\left(c_7c_0-c_2-\frac{\epsilon c_2^2}{2c_0}\right)\int\int_Q (l-x)e^{-c_7t}\left(\frac{\partial u}{\partial x}\right)^2 dxdt.$$

Hence, for sufficiently small $\epsilon\leq 1$, we have

$$\int\int_Q e^{c_7t}\left(\rho_\epsilon^{-1}\right)^* z((l-x)\Im_x z-2\Im_x^2 z)dxdt\leq 0. \tag{4.30}$$

Passing to the limit in the above inequality and integrating by parts with respect to x, we obtain, by Lemma 4, that

$$\int\int_Q e^{c_7t}(\Im_x z)^2 dxdt\leq 0$$

and thus $z=0$. Hence, $\hbar=0$, which completes the proof. □

Now, we return to the proof of Theorem 2. Since F is a Hilbert space, we have that $R(L)=F$ is equivalent to the orthogonality of vector $(\hbar,\hbar_0)\in F$ to the set $R(L)$, i.e., if and only if, the relation

$$(\mathfrak{L}u,\hbar)_{0,Q}+\left(\frac{\partial\ell u}{\partial x},\frac{\partial\hbar_0}{\partial x}\right)_{L^2_\sigma(0,l)}=0, \tag{4.31}$$

where u runs over B and $(\hbar, \hbar_0) \in F$, implies that $\hbar = 0$ and $\hbar_0 = 0$.

Putting $u \in D_0(L)$ in (4.31), we obtain

$$(\mathcal{L}u, \hbar)_{0,Q} = 0.$$

Hence Proposition 2 implies that $\hbar = 0$. Thus, (4.31) takes the form

$$\left(\frac{\partial \ell u}{\partial x}, \frac{\partial \hbar_0}{\partial x}\right)_{L^2_\sigma(0,l)} = 0, \quad u \in D(L).$$

Since the range of the trace operator ℓ is dense in the Hilbert space with the norm $\left\|\frac{\partial \hbar_0}{\partial x}\right\|_{L^2_\sigma(0,l)}$, from the last equality, it follows that $\hbar_0 = 0$ (we recall that $\hbar_0$ satisfies the compatibility conditions (1.11)). Hence, $R(L)$ is dense in F. □

References

[1] Adams, R.A., *Sobolev Spaces*, Academic Press, New York 1975.

[2] Batten, Jr., G.W., Second order correct boundary condition for the numerical solution of the mixed boundary problem for parabolic equations, *Math. Comput.* **17** (1963), 405-413.

[3] Benouar, N.E. and Yurchuk, N.I., Mixed problem with an integral condition for parabolic equations with the Bessel operator, *Differentsial'nye Uravneniya* **27**:12 (1991), 2094-2098.

[4] Bouziani, A., Mixed problem for certain nonclassical equations with a small parameter, *Bulletin de la Classe des Sciences*, Académie Royale de Belgique **5** (1994), 389-400.

[5] Bouziani, A., Problème aux limites pour certaines équations de type non classicque du troisième ordre, *Bulletin de la Classe des Sciences*, Académie Royale de Belgique **6** (1995), 215-222.

[6] Bouziani, A., Mixed problem with integral conditions for a certain parabolic equation, *J. of Appl. Math. and Stoch. Anal.* **9**:3 (1996), 323-330.

[7] Bouziani, A., Mixed problem with nonlocal condition for certain pluriparabolic equations, *Hiroshima Math. J.* **27**:3 (1997), 373-390.

[8] Bouziani, A. and Benouar, N.E., Problème mixte avec conditions intégrales pour une classe d'équations parabliques, *Comptes Rendus de l'Académie des Sciences*, Paris **321**:I (1995), 1177-1182.

[9] Bouziani, A. and Benouar, N.E., Problème mixte avec conditions intégrales pour une classe d'équations hyperboliques, *Bulletin de la Société Math. de Belgique*, **31** (1996), 125-133.

[10] Bouziani, A. and Benouar, N.E., Problème aux limites pour une classe d'équations de type non classique pour une structure pluri-dimensionnelle, *Bull. of the Polish Acad. of Sciences- Mathematics* **43**:4 (1995), 317-328.

[11] Cahlon, B. and Kulkarni, D.M. and Shi, P., Stewise stability for the heat equation with a nonlocal constraint, *SIAM J. Numer. Anal.* **32**:2 (1995), 571-593.

[12] Cannon, J.R., The solution of the heat equation subject to the specification of energy, *Quart. Appl. Math.* **21**:2 (1963), 155-160.

[13] Cannon, J.R. and Van der Hoek, J., The existence and the continuous dependence for the solution of the heat equation subject to the specification of energy, *Boll. Uni. Math. Ital. Suppl.* **1** (1981), 253-282.

[14] Cannon, J.R. and Van der Hoek, J., An implicit finite difference scheme for the diffusion of mass in a portion of the domain, *Numer. Solutions of PDEs* (ed. by J. Noye), North-Holland, Amsterdam (1982), 527-539.

[15] Cannon, J.R., Esteva, S.P. and Van der Hoek, J., A Galerkin procedure for the diffusion equation subject to the specification of mass, *SIAM J. Numer. Anal.* **24** (1987), 499-515.

[16] Gårding, L., *Cauchy's Problem for Hyperbolic Equations*, Univ. of Chicago Lecture Notes 1957.

[17] Ionkin, N.I., Solution of boundary value problem in heat conduction theory with nonlocal boundary conditions, *Differentsial'nye Uravneniya* **13**:2 (1977), 294-304.

[18] Jones, S., Jumarhon, B., McKee, S. and Scott, J.A., A mathematical model of biosensor, *J. Eng. Math.* **30** (1996), 312-337.

[19] Jumarhon, B. and McKee, S., On the heat equation with nonlinear and nonlocal boundary conditions, *J. Math. Anal. Appl.* **190** (1995), 806-820.

[20] Jumarhon, B. and McKee, S., Product integration methods for solving a system of nonlinear Volterra integral equations, *J. Comput. Appl. Math.* **69** (1996), 285-301.

[21] Kamynin, N.I., A boundary value problem in the theory of the heat conduction with non-classical boundary condition, *Th. Vychisl. Mat. Mat. Fiz.* **43**:6 (1964), 1006-1024.

[22] Rektorys, K., *Variational Methods in Mathematics, Sciences and Engineering*, 2nd ed., Dordrecht, Boston, Reidel 1979.

[23] Shi, P., Weak solution to an evolution problem with a nonlocal constraint, *SIAM. J. Math. Anal.* **24**:1 (1993), 46-58.

[24] Yurchuk, N.I., Mixed problem with an integral condition for certain parabolic equations, *Differentsial'nye Uravneniya* **22**:19 (1986), 2117-2126.