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EXISTENCE OF SOLUTIONS OF SOBOLEV-TYPE SEMILINEAR MIXED INTEGRODIFFERENTIAL INCLUSIONS IN BANACH SPACES

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The existence of mild solutions of Sobolev-type semilinear mixed integrodifferential inclusions in Banach spaces is proved using a fixed point theorem for multivalued maps on locally convex topological spaces.
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1 Introduction

The problem of proving the existence of mild solutions for differential and integrodifferential equations in abstract spaces has been studied by several authors [2, 4, 11, 12, 13]. Balachandran and Uchiyama [3] established the existence of solutions of nonlinear integrodifferential equations of Sobolev type with nonlocal conditions in Banach spaces. Benchohra [6] studied the existence of mild solutions on infinite intervals for a class of differential inclusions in Banach spaces. For the existence results of differential inclusions on compact intervals, one can refer to the papers of Avgerinos and Papageorgiou [1], and Papageorgiou [14, 15]. Benchohra and Ntouyas [7] discussed the existence results for first order integrodifferential inclusions of the form

$$\frac{dy}{dt} - Ay \in F(t, \int_0^t k(t,s,y)ds) \quad t \in I = [0,\infty),$$
$$y(0) = y_0.$$

In this paper, we consider the Sobolev-type semilinear mixed integrodifferential inclusion of the type

$$(Eu(t))' + Au \in G\left(t, u, \int_0^t k(t,s,u)ds, \int_0^a b(t,s,u)ds\right) \quad t \in I = [0,\infty), \tag{1.1}$$

$$u(0) = u_0,$$

where $G : I \times X \times X \times X \to 2^Y$ is a bounded, closed, convex, multivalued map $k : \Delta \times X \to X$, $b : \Delta \times X \to X$, where $\Delta = \{(t,s) \in I \times I; t \geq s\}, u_0 \in X$, a is a real constant, X, Y are real Banach spaces with norms $\|.\|$ and $|.|$, respectively. Our method is to reduce the problem (1.1) to a fixed point problem of a suitable multivalued map in the Frechet space $C(I,X)$ and we make use of a fixed point theorem due to Ma [10] for multivalued maps in locally convex topological spaces.

2 Preliminaries

In this section we introduce the notations, definitions and preliminary facts from multivalued analysis which are used in this paper. I_m is the compact interval $[0,m] (m \in N)$. $C(I,X)$ is the linear metric Frechet space of continuous functions from I into X with the metric

$$d(u,z) = \sum_{m=0}^{\infty} \frac{2^{-m}\|u-z\|_m}{1+\|u-z\|_m} \text{ for each } u,z \in C(I,X),$$

where $\|u\|_m = \sup\{\|u(t)\| : t \in I_m\}$. $B(X)$ denotes the Banach space of bounded linear operators from X into X. A measurable function $u : I \to X$ is Bochner integrable if and only if $|u|$ is Lebesgue integrable. Let $L^1(I,X)$ denote the Banach space of continuous functions $u : I \to X$ which are Bochner integrable normed by

$$\|u\|_{L^1} = \int_0^{\infty} \|u(t)\| dt,$$

and U_r is a neighbourhood of 0 in $C(I,X)$ defined by

$$U_r = \{u \in C(I,X) : \|u\|_m \leq r\}$$

for each $m \in N$. The convergence in $C(I,X)$ is the uniform convergence on compact intervals, that is, $u_j \to u$ in $C(I,X)$ if and only if for each $m \in N, \|u_j - u\|_m \to 0$ in $C(I_m,X)$ as $j \to \infty$. $BCC(X)$ denotes the set of all nonempty bounded, closed, and convex subsets of X.

A multivalued map $G : X \to 2^X$ is convex(closed) valued if $G(x)$ is convex(closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \bigcup_{x \in B} G(x)$ is bounded in X for any bounded set B of X (that is, $\sup_{x \in B}\{\sup\{\|u\| : u \in G(x)\}\} < \infty$). G is called upper semi continuous on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open subset B of X containing $G(x_0)$, there exists an open neighbourhood A of x_0 such that $G(A) \subseteq B$. G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$. If the multivalued map G is completely continuous with nonempty compact values, then G is upper semicontinuous if and only if G has a closed graph (that is, $x_n \to x_0, u_n \to u_0, u_n \in Gx_n$ imply $u_0 \in Gx_0$).

We assume the following conditions:

(i) The operator $A : D(A) \subset X \to Y$ and $E : D(E) \subset X \to Y$ satisfy the following conditions

$[C_1]$ A and E are closed linear operators.

$[C_2]$ $D(E) \subset D(A)$ and E is bijective.

$[C_3]$ $E^{-1} : Y \to D(E)$ is continuous.

$[C_4]$ The resolvent $R(\lambda, -AE^{-1})$ is a compact operator for some $\lambda \in \rho(-AE^{-1})$ and resolvent set of $-AE^{-1}$.

Conditions $[C_1], [C_2]$, and the closed graph theorem imply the boundedness of the linear operator $AE^{-1} : Y \to Y$.

(ii) $G : I \times X \times X \times X \to BCC(Y)$ is measurable with respect to t for each $u \in X$, upper semi continuous with respect to u for each $t \in I$, and for each $u \in C(I, X)$ the set

$$S_{G,u} = \{g \in L^1(I; R) : g(t) \in G(t, u, \int_0^t k(t, s, u)ds, \int_0^a b(t, s, u)ds)$$

for a.e $t \in I\}$ is nonempty.

(iii) There exist functions $p(t), q(t) \in C(I; R)$ such that

$$|\int_0^t k(t, s, u)ds| \leq p(t)\|u\| \text{ and } |\int_0^a b(t, s, u)ds| \leq q(t)\|u\| \text{for a.e } t, s \in I, u \in X.$$

(iv) There exists a function $\alpha(t) \in L^1(I; R^+)$ such that

$$\|G(t, u, v, w)\| \leq \alpha(t)\Omega(\|u\| + \|v\| + \|w\|)$$

for a.e $t \in I, u \in X$, where $\Omega : R_+ \to (0, \infty)$ is continuous increasing function satisfying $\Omega(p(t)x + q(t)y) \leq p(t)\Omega(x) + q(t)\Omega(y)$ and

$$M \int_0^m \alpha(s)(1 + p(s) + q(s))ds < \int_c^\infty \frac{du}{\Omega(u)}$$

for each $m \in N$, where $c = \|E^{-1}\|M|Eu_0|$ and $M = \max\{\|T(t)\|; t \in I\}$.

(v) For each neighbourhood U_r of $0, u \in U_r$ and $t \in I$, the set

$$\{E^{-1}T(t)Eu_0 + \int_0^t E^{-1}T(t - s)g(s)ds, \ \ g \in S_{G,u}\}$$

is relatively compact.

Definition 2.1: A continuous function $u(t)$ of the integral inclusion

$$u(t) \in E^{-1}T(t)Eu_0 + \int_0^t E^{-1}T(t - s)G\left(s, u, \int_0^s k(s, \tau, u(\tau))d\tau, \int_0^a b(s, \tau, u(\tau))d\tau\right) ds$$

is called a mild solution of (1.1) on I.

Lemma 2.1: [9]. *Let I be a compact real interval and let X be a Banach space. Let G be a multivalued map satisfying (i) and let Γ be a linear continuous mapping from $L^1(I, X)$ to $C(I, X)$. Then the operator*

$$\Gamma \circ S_G : C(I, X) \to X, \ \ (\Gamma \circ S_G)(y) = \Gamma(S_{G,y})$$

is a closed graph operator in $C(I,X) \times C(I,X)$.

Lemma 2.2: [10]. *Let* X *be a locally convex space. Let* $N : X \to X$ *be a compact, convex valued, upper semicontinuous, multivalued map such that there exists a closed neighbourhood* U_r *of* 0 *for which* $N(U_r)$ *is a relatively compact set for each* $r \in N$. *If the set* $\zeta = \{y \in X : \lambda y \in N(y)\}$ *for some* $\lambda > 1$ *is bounded, then* N *has a fixed point.*

Remark: [9]. If $dim X < \infty$ and I is a compact real interval, then for each $u \in C(I,X)$, $S_{G,u}$ is nonempty.

Lemma 2.3: [16]. *Let* $S(t)$ *be a uniformly continuous semigroup and let* A *be its infinitesimal generator. If the resolvent set* $R(\lambda : A)$ *of* A *is compact for every* $\lambda \in \rho(A)$, *then* $S(t)$ *is a compact semigroup.*

From the above fact, $-AE^{-1}$ generates a compact semigroup $T(t)$ in Y. Thus, $\max_{t \in I} |T(t)|$ is finite and so denote $M = \max_{t \in I} |T(t)|$.

3 Main Result

Theorem 3.1: *If the assumptions* (i)–(v) *are satisfied, then the initial value problem* (1.1) *has at least one mild solution on* I.

Proof: A solution to (1.1) is a fixed point for the multivalued map $N : C(I,X) \to 2^{C(I,X)}$ defined by

$$N(u) = \{h \in C(I,X) : h(t) = E^{-1}T(t)Eu_0 + \int_0^t E^{-1}T(t-s)g(s)ds, \ \ g \in S_{G,u}\},$$

where

$$S_{G,u} = \{g \in L^1(I,X) : g(t) \in G(t,u,\int_0^t k(t,s,u(s))ds, \int_0^a b(t,s,u(s))ds)$$

for a.e $t \in I\}$.

First we shall prove $N(u)$ is convex for each $u \in C(I,X)$. Let $h_1, h_2 \in N(u)$, then there exist $g_1, g_2 \in S_{G,u}$ such that

$$h_i(t) = E^{-1}T(t)Eu_0 + \int_0^t E^{-1}T(t-s)g_i(s)ds, i = 1,2, t \in I$$

Let $0 \leq k_1 \leq 1$, then for each $t \in I$ we have

$$(k_1h_1 + (1-k_1)h_2)t = E^{-1}T(t)Eu_0 + \int_0^t E^{-1}T(t-s)(k_1g_1(s) + (1-k_1)g_2(s))ds.$$

Since $S_{G,u}$ is convex, thus $kh_1 + (1-k)h_2 \in N(u)$. Hence, $N(u)$ is convex for each $u \in C(I,X)$.

Let $U_r = \{u \in C(I,X); \|u\| \leq r\}$ be a neighbourhood of 0 in $C(I,X)$ and $u \in U_r$. Then for each $h \in N(u)$ there exists $g \in S_{G,u}$ such that for $t \in I$, we have

$$\begin{aligned}
\|h(t)\| &\leq \|E^{-1}\|\|T(t)\||Eu_0| + \int_0^t \|E^{-1}\|\|T(t-s)\|\|g(s)\|ds \\
&\leq \|E^{-1}\|M|Eu_0| + \|E^{-1}\|M\int_0^t \alpha(s)\Omega(\|u\| + p(t)\|u\| + q(t)\|u\|)ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \|E^{-1}\|M|Eu_0| + \|E^{-1}\|M\int_0^t \alpha(s)(\Omega(\|u\|) + p(t)\Omega(\|u\|) + q(t)\Omega(\|u\|))ds \\
&\leq \|E^{-1}\|M|Eu_0| + \|E^{-1}\|M\int_0^t \alpha(s)(1 + p(s) + q(s))\Omega(\|u\|)ds \\
&\leq \|E^{-1}\|M|Eu_0| + \|E^{-1}\|M\|\alpha\|_{L^1(I_m)}\|(1 + p(s) + q(s))\| \sup_{u\in U_r} \Omega(\|u\|)
\end{aligned}
$$

Hence, $N(U_r)$ is bounded in $C(I, X)$ for each $r \in N$. Next we shall prove $N(U_r)$ is an equicontinuous set in $C(I, X)$ for each $r \in N$. Let $t_1, t_2 \in I_m$ with $t_1 < t_2$. Then for all $h \in N(u)$ with $u \in U_r$, we have

$$
\begin{aligned}
\|h(t_1) - h(t_2)\| \leq\ & \|E^{-1}\|\|(T(t_2) - T(t_1))Eu_0\| \\
&+ \|E^{-1}\|\|\int_0^{t_2} (T(t_2 - s) - T(t_1 - s))g(u)ds\| \\
&+ \|E^{-1}\|\|\int_{t_1}^{t_2} T(t_1 - s)g(u)ds\| \\
\leq\ & \|E^{-1}\|\|(T(t_2) - T(t_1))Eu_0\| \\
&+ \|E^{-1}\|\|\int_0^{t_2} (T(t_2 - s) - T(t_1 - s))g(u)ds\| \\
&+ M(t_2 - t_1)\|E^{-1}\| \int_0^m \|g(s)\|ds.
\end{aligned}
$$

Hence, by the Ascoli-Arzela Theorem, we conclude that $N : C(I, X) \to 2^{C(I,X)}$ is a completely continuous multivalued map. Next we shall prove that N has a closed graph. Let $u_n \to u_*, h_n \in N(u_n)$ and $h_n \to h_0$, then we shall prove that $h_0 \in N(u_*)$. Here, $h_n \in N(u_n)$ means that there exists $g_n \in S_{G,u_n}$ such that

$$h_n(t) = E^{-1}T(t)Eu_0 + \int_0^t E^{-1}T(t - s)g_n(s)ds, \quad t \in I.$$

We must also prove that there exists $g_0 \in S_{G,u}$ such that

$$h_0(t) = E^{-1}T(t)Eu_0 + \int_0^t E^{-1}T(t - s)g_0(s)ds, \quad t \in J. \tag{3.1}$$

To prove the above, we use the fact that $h_n \to h_0$; and $h_n - E^{-1}T(t)Eu_0 \in \Gamma(S_{G,u})$, where

$$(\Gamma g)(t) = \int_0^t E^{-1}T(t - s)g(s)ds, \quad t \in I.$$

Consider the functions $u_n, h_n - E^{-1}T(t)Eu_0$ and g_n defined on the interval $[k, k+1]$ for any $k \in N \cup \{0\}$. Then using Lemma 2.1, we can conclude (3.1) is true on the compact interval $[k, k + 1]$. That is,

$$[h_0(t)]_{[k,k+1]} = E^{-1}T(t)Eu_0 + \int_0^t E^{-1}T(t - s)g_0^k(s)ds$$

for a suitable L^1-selection g_0^k of $G(t, u, \int_0^t k(t, s, u)ds, \int_0^T b(t, s, u)ds)$ on the interval $[k, k + 1]$. Let $g_0(t) = g_0^k(t)$ for $t \in [k, k + 1)$. Then g_0 is an L^1-selection and (3.1) will satisfied. Clearly we have $\|(h_n - E^{-1}T(t)Eu_0) - (h_0 - E^{-1}T(t)Eu_0)\|_\infty \to 0$ as $n \to \infty$. Consider for all $k \in N \cup \{0\}$, the mapping

$$S_G^k : C([k,k+1],X) \to L^1([k,k+1],X),$$

$$y \to S_{G,y}^k = \{g \in L^1([k,k+1],X) : g(t) \in G(t,u,\int_0^t k(t,s,u)ds, \int_0^a b(t,s,u)ds)$$

for a.e $t \in [k,k+1]\}$.

Now we consider the linear continuous operators

$$\Gamma_k : L^1([k,k+1],X) \to C([k,k+1],X),$$

$$g \to \Gamma_k(g)(t) = \int_0^t E^{-1}T(t-s)g(s)ds.$$

From Lemma 2.1 it follows that $\Gamma_k \circ S_G^k$ is a closed graph operator for all $k \in N \cup \{0\}$. Moreover, we have

$$(h_n(t) - E^{-1}T(t)Eu_0)|_{[k,k+1]} \in \Gamma_k(S_{G,u_n}^k)$$

and $u_n \to u_*$. From Lemma 2.1, we have $(h_0(t) - E^{-1}T(t)Eu_0)|_{[k,k+1]} \in \Gamma_k(S_{G,u_*}^k)$,

$$(h_0(t) - E^{-1}T(t)Eu_0)|_{[k,k+1]} = \int_0^t E^{-1}T(t-s)g_0^k(s)ds \text{for some} g_0^k \in S_{G,u_*}^k.$$

Hence, the function g_0 defined on I by $g_0(t) = g_0^k(t)$ for $t \in [k,k+1]$ is in S_{G,u_*}. Therefore, $N(U_r)$ is relatively compact for each $r \in N$ where N is upper semicontinuous with convex closed values. Finally we prove the set $\zeta = \{u \in C(I,X); \lambda u \in Nu\}$, for some $\lambda > 1$, is bounded.

Let $\lambda u = Nu$ for some $\lambda > 1$. Then there exists $g \in S_{G,u}$ such that

$$u(t) = \lambda^{-1}E^{-1}T(t)Eu_0 + \lambda^{-1}\int_0^t E^{-1}T(t-s)g(s)ds,\ t \in I,$$

$$\|u(t)\| \le \|E^{-1}\|M|Eu_0| + \|E^{-1}\|M\int_0^t \alpha(s)(1+p(s)+q(s))\Omega(\|u\|)ds.$$

Let $v(t) = \|E^{-1}\|M|Eu_0| + \|E^{-1}\|M\int_0^t \alpha(s)(1+p(s)+q(s))\Omega(\|u\|)ds$. Then we have $v(0) = \|E^{-1}\|M\|Eu_0\| = c$ and $\|u(t)\| \le v(t), t \in I_m$. Using the increasing character of Ω we get

$$v'(t) \quad \le \quad \|E^{-1}\|M\alpha(t)(1+p(t)+q(t))\Omega(v(t)),\ \ t \in I_m.$$

The above proves that for each $t \in I_m$,

$$\int_{v(0)}^{v(t)} \frac{du}{\Omega(u)} \le \|E^{-1}\|M\int_0^m \alpha(s)(1+p(s)+q(s))ds < \int_0^\infty \frac{du}{\Omega(u)}.$$

The above inequality implies that there exists a constant M_0 such that $v(t) \le M_0, t \in I_m$, and hence that $\|u\|_\infty \le M_0$ where M_0 depends on m and on the functions α, p, Ω. Hence, ζ is bounded. Thus by Lemma 2.2, N has a fixed point that is a mild solution of (1.1).

4 Nonlocal Initial Conditions

Several authors have studied the nonlocal Cauchy problem in abstract spaces [2, 3, 4, 11, 12, 13]. The importance of nonlocal conditions is discussed in [4, 5]. In this section we consider a first order Sobolev-type, semilinear, mixed, integrodifferential inclusion (1.1) with the nonlocal initial condition

$$u(0) + f(u) = u_0 \tag{4.1}$$

In addition to the five assumptions in Section 2, we also assume the following.

(vi) $f : C(I, X) \to X$ is a continuous function, and there exists a constant $L > 0$ such that $\|f(u)\| \leq L$ for each $u \in X$.

(vii) $\|E^{-1}\|M \int_0^m \alpha(s)(1 + p(s) + q(s))ds < \int_{c_1}^{\infty} \frac{du}{\Omega(u)}$ where $c_1 = \|E^{-1}\|M|Eu_0| + L\|E^{-1}\|M|Eu_0|$.

(viii) For each neighbourhood U_r of $0, u \in U_r$ and $t \in I$, the set $\{E^{-1}T(t)Eu_0 - E^{-1}T(t)Ef(u) + \int_0^t E^{-1}T(t-s)g(s)ds,\ g \in S_{G,u}\}$ is relatively compact.

Definition 4.1: A continuous function $u(t)$ of the integral inclusion

$$u(t) \in E^{-1}T(t)Eu_0 - E^{-1}T(t)Ef(u) + \int_0^t E^{-1}T(t-s)G\left(s, u, \int_0^s k(s,\tau,u(\tau))d\tau, \int_0^a b(s,\tau,u(\tau))d\tau\right)ds$$

is called a mild solution of (1.1)-(4.1) on I.

Theorem 4.1: *If the assumptions* (i)–(iii)*,* (vi)–(viii) *are satisfied, then the nonlocal initial value problem* (1.1)–(4.1) *has at least one mild solution on* I*.*

The proof of Theorem 4.1 is similar to Theorem 3.1 and hence, is omitted.

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