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COMMON FIXED POINT THEOREMS FOR A PAIR OF COUNTABLY CONDENSING MAPPINGS IN ORDERED BANACH SPACES

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In this paper some common fixed point theorems for a pair of multivalued weakly isotone mappings on an ordered Banach space are proved.
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1 Introduction

Let X be a Banach space with norm $\|\,.\,\|$. A nonempty closed subset K of X is called a cone if

(i) $K + K \subseteq K$,

(ii) $\lambda K \subseteq K$ for all $\lambda > 0$ and

(iii) $(-K) \cap K = \{0\}$ where 0 is the zero element of X.

We define an order relation $\leq$ in X with the help of the cone K as follows: for $x, y \in X$ then $x \leq y$ iff $y - x \in K$. By an ordered Banach space X we mean the

Banach space X equipped with a partial ordering $\leq$ induced by K. Let $z, w \in X$ be such that $z \leq w$. Then by an order interval $[z, w]$ we mean a set in X defined by

$$[z, w] = \{x \in X : \ z \leq x \leq w\}.$$

A cone K is normal in X if every order interval in X is bounded in norm (see [1, 4]).

Definition 1.1: A map $T : X \to X$ is said to be isotone increasing if for $x, \ y \in X$ and $x \leq y$ we have $T\,x \leq T\,y$.

The measure of noncompactness [2] of $A \subseteq X$ is defined by

$$\alpha(A) = \inf\{r > 0 : \ A \subseteq \cup_{i=1}^{n} A_i \ \text{ and } \ diam\,(A_i) \leq r \ \text{ for } \ i \in \{1, ..., n\}\}.$$

Definition 1.2: Let $Q \subseteq X$. A map $T : Q \to X$ is said to be countably condensing if $T(Q)$ is bounded and if for any countably bounded set A of Q with $\alpha(A) > 0$ we have $\alpha(T(A)) < \alpha(A)$.

In Section 2 we prove new common fixed point theorems for a pair of single valued maps and in Section 3 we prove the multivalued analogue of these theorems. The results in this paper complement and extend results in the literature; see [3] and the references therein.

2 Pairs of Single-Valued Mappings

In this section we prove some common fixed point theorems for a pair of mappings defined on a closed subset of an ordered Banach space X.

Condition D_Q: Let $Q \subseteq X$. Two maps $S, T : Q \to Q$ are said to satisfy condition D_Q if for any countable set A of Q and for any fixed $a \in Q$ the condition

$$A \subseteq \{a\} \cup S(A) \cup T(A)$$

implies $\overline{A}$ is compact.

Definition 2.1: Two maps S and T on an ordered Banach space X into itself are said to be weakly isotone increasing if $S\,x \leq T\,S\,x$ and $T\,x \leq S\,T\,x$ for all $x \in X$. Similarly S and T are said to be weakly isotone decreasing if $S\,x \geq T\,S\,x$ and $T\,x \geq S\,T\,x$ for all $x \in X$. Also two mappings S and T are called weakly isotone if they are either weakly isotone increasing or weakly isotone decreasing.

Now we are ready to prove our main result.

Theorem 2.1: *Let B be a closed subset of an ordered Banach space X and let $S, T : B \to B$ be two continuous and weakly isotone mappings satisfying condition D_B. Then S and T have a common fixed point.*

Proof: Let $x \in B$ be arbitrary. Suppose S and T are weakly isotone increasing. Define a sequence $\{x_n\} \subseteq B$ as follows:

$$x_0 = x, \ x_{2\,n+1} = S\,x_{2\,n}, \ x_{2\,n+2} = T\,x_{2\,n+1} \ \text{ for } \ n \geq 0. \tag{2.1}$$

Note $x_1 = S\,x_0 \leq T\,S\,x_0 = T\,x_1 = x_2$ and so we have

$$x_1 \leq x_2 \leq x_3 \leq ....$$

Let $A = \{x_0, x_1, ...\}$. Now A is countable and

$$A = \{x_0\} \cup \{x_1, x_3, ...\} \cup \{x_2, x_4, ...\} \subseteq \{x_0\} \cup S(A) \cup T(A).$$

Now S and T satisfy condition D_B so $\overline{A}$ is compact. Thus $\{x_n\}$ has a convergent subsequence which converges to say $x^\star \in B$. However $\{x_n\}$ is nondecreasing from above, so the original sequence $\{x_n\}$ converges to $x^\star \in B$. Also the continuity of T and S imply

$$x^\star = \lim_{n\to\infty} x_{2\,n+1} = \lim_{n\to\infty} S\,x_{2\,n} = S(\lim_{n\to\infty} x_{2\,n}) = S(x^\star)$$

and

$$x^\star = \lim_{n\to\infty} x_{2\,n+2} = \lim_{n\to\infty} T\,x_{2\,n+1} = T(\lim_{n\to\infty} x_{2\,n+1}) = T(x^\star).$$

Thus S and T have a common fixed point. The case when S and T are weakly isotone decreasing is similar.

Corollary 2.1: *Let B be a closed subset of an ordered Banach space X and let $S, T : B \to B$ be two continuous, countably condensing and weakly isotone mappings. Then S and T have a common fixed point.*

Proof: The result follows from Theorem 2.1 once we show S and T satisfy condition D_B. To see this let A be a countable subset of B, $a \in B$ fixed, and $A \subseteq \{a\} \cup S(A) \cup T(A)$. Now A is bounded since S and T are condensing (so in particular $S(B)$ and $T(B)$ are bounded). Now if $\alpha(A) \neq 0$ then

$$\alpha(A) \leq \max\{\alpha(\{a\}), \alpha(S(A)), \alpha(T(A))\} < \alpha(A),$$

which is a contradiction. Thus $\alpha(A) = 0$, so $\overline{A}$ is compact.

In our next result let X be a Banach space, K a cone in X and let $\overline{x}, \overline{y} \in X$ be such that $\overline{x} \leq \overline{y}$. Also $[\overline{x}, \overline{y}]$ denotes an order interval in X as described in Section 1.

Condition R: Two weakly isotone maps $S, T : [\overline{x}, \overline{y}] \to [\overline{x}, \overline{y}]$ are said to satisfy condition R if for any countable set A of $[\overline{x}, \overline{y}]$,

(i) the condition
$$A \subseteq \{\overline{x}\} \cup S(A) \cup T(A)$$
implies $\overline{A}$ is compact if S and T are weakly isotone increasing,

(ii) whereas the condition
$$A \subseteq \{\overline{y}\} \cup S(A) \cup T(A)$$
implies $\overline{A}$ is compact if S and T are weakly isotone decreasing.

Corollary 2.2: *Suppose $S, T : [\overline{x}, \overline{y}] \to [\overline{x}, \overline{y}]$ are two continuous and weakly isotone mappings satisfying condition R. Then S and T have a common fixed point.*

Proof: If S and T are weakly isotone increasing we define a sequence $\{x_n\} \subseteq [\overline{x}, \overline{y}]$ as in (2.1) with $x_0 = \overline{x}$, whereas if S and T are weakly isotone decreasing we define a sequence $\{x_n\} \subseteq [\overline{x}, \overline{y}]$ as in (2.1) with $x_0 = \overline{y}$. □

Corollary 2.3: *Suppose $S, T : [\overline{x}, \overline{y}] \to [\overline{x}, \overline{y}]$ are two continuous, countably condensing and weakly isotone mappings. Then S and T have a common fixed point $x^\star \in [\overline{x}, \overline{y}]$.*

Remark 2.1: If in Corollary 2.3 the cone K is normal in X then the condition that $S([\overline{x}, \overline{y}])$ and $T([\overline{x}, \overline{y}])$ are bounded in the definition of countably condensing is automatically satisfied since $[\overline{x}, \overline{y}]$ is bounded in norm.

Remark 2.2: Assume $S, T : [\overline{x}, \overline{y}] \to [\overline{x}, \overline{y}]$ are two continuous weakly isotone mappings satisfying condition R and in addition suppose S and T are isotone increasing.

(i) Suppose S and T are weakly isotone increasing.

We know from Corollary 2.2 that the sequence $\{x_n\}$ defined by

$$x_0 = \overline{x},\ x_{2n+1} = S\,x_{2n},\ x_{2n+2} = T\,x_{2n+1} \text{ for } n \geq 0$$

converges to a common fixed point $x^\star \in [\overline{x}, \overline{y}]$ of S and T.

Now let $z \in [\overline{x}, \overline{y}]$ be any common fixed point of S and T. We claim that $x^\star \leq z$. To see this notice since S and T are isotone increasing that

$$x_1 = S\,\overline{x} \leq S\,z = z,\ x_2 = T\,x_1 \leq T\,z = z, \ldots\ldots .$$

Thus $x_n \leq z$ for all $n \in \{1, 2, \ldots.\}$. Since $\{x_n\}$ converges to $x^\star$ we have $x^\star \leq z$.

(ii) Suppose S and T are weakly isotone decreasing.

We know from Corollary 2.2 that the sequence $\{x_n\}$ defined by

$$x_0 = \overline{y},\ x_{2n+1} = S\,x_{2n},\ x_{2n+2} = T\,x_{2n+1} \text{ for } n \geq 0$$

converges to a common fixed point $y^\star \in [\overline{x}, \overline{y}]$ of S and T.

Now let $z \in [\overline{x}, \overline{y}]$ be any common fixed point of S and T. Similar reasoning as in (i) guarantees that $z \leq y^\star$.

3 Pairs of Multivalued Mappings.

Let X be an ordered Banach space and let 2^X (respectively $C(X)$) denote the family of all nonempty (respectively, nonempty closed) subsets of X. Let $A, B \in 2^X$. Then $A \leq B$ means $a \leq b$ for all $a \in A$ and $b \in B$.

Definition 3.1: A map $T : X \to 2^X$ is said to be isotone increasing if for x, $y \in X$ and $x \leq y$, $x \neq y$ we have $T\,x \leq T\,y$.

Definition 3.2: Two maps $S, T : X \to 2^X$ are said to be weakly isotone increasing if for any $x \in X$ we have $S\,x \leq T\,y$ for all $y \in S\,x$ and $T\,x \leq S\,y$ for all $y \in T\,x$. S and T are called weakly isotone decreasing if for any $x \in X$ we have $S\,x \geq T\,y$ for all $y \in S\,x$ and $T\,x \geq S\,y$ for all $y \in T\,x$. Also two mappings S and T are called weakly isotone if they are either weakly isotone increasing or weakly isotone decreasing.

Condition D_Q: Let $Q \subseteq X$. Two maps $S, T : Q \to 2^Q$ are said to satisfy condition D_Q if for any countable set A of Q and for any fixed $a \in Q$ the condition

$$A \subseteq \{a\} \cup S(A) \cup T(A)$$

implies $\overline{A}$ is compact; here $T(A) = \cup_{x \in A} T\,x$.

Theorem 3.1: *Let B be a closed subset of an ordered Banach space X and let $S, T : B \to C(B)$ be two closed (i.e. have closed graph) weakly isotone mappings satisfying condition D_B. Then S and T have a common fixed point.*

Proof: Let $x \in B$ be arbitrary. Suppose S and T are weakly isotone increasing. Define a sequence $\{x_n\} \subseteq B$ as follows:

$$x_0 = x,\ x_{2n+1} \in S\,x_{2n},\ x_{2n+2} \in T\,x_{2n+1} \text{ for } n \geq 0. \tag{3.1}$$

Note $x_1 \in S\,x_0$ and since $S\,x_0 \leq T\,y$ for all $y \in S\,x_0$ we have $S\,x_0 \leq T\,x_1$. In particular $S\,x_0 \leq x_2$, and so $x_1 \leq x_2$. As a result we have

$$x_1 \leq x_2 \leq x_3 \leq \ldots\ldots .$$

Let $A = \{x_0, x_1, \ldots..\}$. Now A is countable and

$$A = \{x_0\} \cup \{x_1, x_3, \ldots.\} \cup \{x_2, x_4, \ldots.\} \subseteq \{x_0\} \cup S(A) \cup T(A).$$

Now S and T satisfy condition D_B so $\overline{A}$ is compact. Thus $\{x_n\}$ has a convergent subsequence which converges to say $x^\star \in B$. However $\{x_n\}$ is nondecreasing from above, so the original sequence $\{x_n\}$ converges to $x^\star \in B$. Now

$$x_{2\,n} \to x^\star,\ x_{2\,n+1} \to x^\star,\ x_{2\,n+1} \in S\,x_{2\,n},$$

together with the fact that S has closed graph implies $x^\star \in S\,x^\star$. Similarly $x^\star \in T\,x^\star$. The case when S and T are weakly isotone decreasing is similar.

Definition 3.3: Let $Q \subseteq X$. A map $T : Q \to 2^X$ is said to be countably condensing if $T(Q)$ is bounded and if for any countably bounded set A of Q with $\alpha(A) > 0$ we have $\alpha(T(A)) < \alpha(A)$.

Essentially the same reasoning as in Corollary 2.1 establishes the following result.

Corollary 3.1: *Let B be a closed subset of an ordered Banach space X and let $S, T : B \to C(B)$ be two closed, countably condensing, weakly isotone mappings. Then S and T have a common fixed point.*

In our next result let X be a Banach space, K a cone in X and let $\overline{x}, \overline{y} \in X$ be such that $\overline{x} \leq \overline{y}$. Also $[\overline{x}, \overline{y}]$ denotes an order interval in X as described in Section 1.

Condition R**:** Two weakly isotone maps $S, T : [\overline{x}, \overline{y}] \to 2^{[\overline{x}, \overline{y}]}$ are said to satisfy condition R if for any countable set A of $[\overline{x}, \overline{y}]$,

(i) the condition
$$A \subseteq \{\overline{x}\} \cup S(A) \cup T(A)$$
implies $\overline{A}$ is compact if S and T are weakly isotone increasing,

(ii) whereas the condition
$$A \subseteq \{\overline{y}\} \cup S(A) \cup T(A)$$
implies $\overline{A}$ is compact if S and T are weakly isotone decreasing.

Corollary 3.2: *Suppose $S, T : [\overline{x}, \overline{y}] \to C([\overline{x}, \overline{y}])$ are two closed, weakly isotone mappings satisfying condition R. Then S and T have a common fixed point.*

Proof: If S and T are weakly isotone increasing we define a sequence $\{x_n\} \subseteq [\overline{x}, \overline{y}]$ as in (3.1) with $x_0 = \overline{x}$, whereas if S and T are weakly isotone decreasing we define a sequence $\{x_n\} \subseteq [\overline{x}, \overline{y}]$ as in (3.1) with $x_0 = \overline{y}$.

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