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ON AN EVEN ORDER NEUTRAL DIFFERENTIAL INEQUALITY

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In this paper, we prove new results related to the nonexistence criteria for eventually positive solutions of certain even order neutral differential inequality with distributed deviating arguments.
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1 Introduction

In order to make this paper self-contained, we introduce the following definition.

Definition 1: The function $f(t)$ is said to be eventually zero if there exists a sufficiently large t_μ such that $f(t) \equiv 0$ holds for $t \geq t_\mu$.

This paper is concerned with nonexistence conditions of eventually positive solutions of the even order neutral differential inequality with distributed deviating arguments

$$[x(t) + c(t)x(t-\tau)]^{(n)} + \int_a^b p(t,\xi) f(x[g(t,\xi)]) d\sigma(\xi) \leq 0, \qquad t \geq t_0, \tag{1}$$

in which $\tau > 0$ is a constant, n is an even positive integer; $c(t) \in C(I,R)$, $0 \leq c(t) \leq 1$, and $p(t,\xi) \in C(I \times J, R_+)$ is not eventually zero on any $I_\mu \times J$, $I = [t_0, \infty)$, $J = [a,b]$,

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$I_\mu = [t_\mu, \infty)$, $t_\mu \geq t_0$, $R_+ = [0, \infty)$. Furthermore, we assume that $g(t, \xi) \in C(I \times J, R)$ is nondecreasing with respect to t and ξ, respectively, $\frac{d}{dt} g(t, a)$ exists, $g(t, \xi) \leq t$ for $\xi \in J$, and $\liminf_{t \to \infty, \xi \in J} \{g(t, \xi)\} = \infty$; $f(x) \in C(R, R)$, and $x f(x) > 0$ $(x \neq 0)$; $\sigma(\xi) \in (J, R)$ is nondecreasing; integral of inequality (1) is in Lebesgue-Stieltjes sense.

Recently, Li and Cui [1] have obtained some results dealing with a class of even order neutral differential inequalities with applications. On the other hand, Liu and Fu [2] have studied nonlinear differential inequality with distributed deviating arguments and their applications. These authors provided some results on nonexistence conditions of eventually positive solutions of inequality (1). For example,

Theorem A:(See [1]) *If* $0 \leq c(t) \leq 1$, *and*

$$\int_{t_0}^{t} \int_a^b p(s, \xi)\{1 - c[g(s, \xi)]\} d\sigma(\xi) ds = \infty,$$

then inequality (1) *has no eventually positive solutions.*

Theorem B:(See ([2]) *Assume that* $f(-x) = -f(x)$, $x \in (0, \infty)$, *and*

$$\frac{f(x)}{x} \geq \lambda, \quad x \in (0, \infty), \quad \text{for some constant} \quad \lambda > 0. \tag{2}$$

If there exists a monotonically increasing function $\varphi(t) \in C'(I, (0, \infty))$ *such that*

$$\int_{t_0}^{t} [\lambda \varphi(s) \int_a^b p(s, \xi)\{1 - c[g(s, \xi)]\} d\sigma(\xi) - A\varphi'(s)] ds = \infty$$

for any number $A > 0$, *then inequality (1) has no eventually positive solutions.*

The purpose of this paper is to obtain two new results related to the nonexistence criteria for eventually positive solutions of inequality (1). In the established nonexistence criteria, there is a general class of function $H(t, s)$ as the parameter function. By choosing a different function $H(t, s)$, we are able to derive some useful corollaries.

Definition 2: The solution $x(t) \in C^{(n)}(I, R)$ of inequality (1) is said to be eventually positive if there exists a sufficiently large positive number $T \geq t_0$ such that the inequality $x(t) > 0$ holds for $t \geq T$.

To develop the nonexistence criteria of eventually positive solutions of inequality (1), we first need the following Lemmas:

Lemma 1: (See [1]) *Assume that* $x(t)$ *is an eventually positive solution of inequality* (1). *Let*

$$y(t) = x(t) + c(t)x(t - \tau). \tag{3}$$

Then there exists a $t_1 \geq t_0$ *such that*

$$y(t) > 0, \;\; y'(t) > 0, \;\; y^{(n-1)}(t) > 0 \;\; \textit{and} \;\; y^{(n)}(t) \leq 0, \qquad t \geq t_1.$$

Lemma 2: (See [3]) *Let* $x^{(n)}(t) \in C(I, R_+)$. *If* $x^{(n)}(t)$ *is eventually of one sign for all large* t, *and* $x^{(n)}(t) \times x^{(n-1)}(t) \leq 0$ *for* $t_1 > t_0$, *then there exists a constant* $\theta \in (0, 1)$ *such that for sufficiently large* t, *there exists a constant* $M_\theta > 0$ *satisfying*

$$|u'(t/2)| \geq M_\theta t^{n-2} |u^{(n-1)}(t)|.$$

2 Main Results

The following theorems provide sufficient conditions leading to nonexistence of eventually positive solutions for inequality (1).

Theorem 1: *Assume that the condition of Theorem B holds, and there exist functions* $H(t,s) \in C'(D;R)$, $h(t,s) \in C(D;R)$, *with* $D = \{(t,s) | t \geq s \geq t_0\}$ *satisfying*

(H_1) $H(t,t) = 0$, $t \geq t_0$; $H(t,s) > 0$, $t > s \geq t_0$;

(H_2) $H_t{}'(t,s) \geq 0$, $H_s{}'(t,s) \leq 0$, *and* $-H_s{}'(t,s) = h(t,s)\sqrt{H(t,s)}$, $(t,s) \in D$.

If

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s) \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\} d\sigma(\xi) ds = \infty, \tag{4}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{h^2(t,s)}{g^{n-2}(s,a)g'(s,a)} ds < \infty, \tag{5}$$

then inequality (1) *has no eventually positive solutions.*

Proof: Assume to the contrary that $x(t)$ is an eventually positive solution of inequality (1). Then from $\lim_{t\to\infty,} \inf_{\xi\in J}\{g(t,\xi)\} = \infty$, there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(t-\tau) > 0$ and $x[g(t,\xi)] > 0$ for $t \geq t_1$ and $\xi \in J$. From (2) and (3), inequality (1) can be written as

$$\begin{aligned} 0 &\geq y^{(n)}(t) + \int_a^b p(t,\xi) f(x[g(t,\xi)]) d\sigma(\xi) \\ &\geq y^{(n)}(t) + \lambda \int_a^b p(t,\xi)\{y[g(t,\xi)] - c[g(t,\xi)]x[g(t,\xi)-\tau]\} d\sigma(\xi). \end{aligned} \tag{6}$$

From Lemma 1, $y'(t) > 0$ and $y(t) \geq x(t)$, $t \geq t_1$, hence $y[g(t,\xi)] \geq y[g(t,\xi)-\tau] \geq x[g(t,\xi)-\tau]$. Thus

$$y^{(n)}(t) + \lambda \int_a^b p(t,\xi)\{1 - c[g(t,\xi)]\} y[g(t,\xi)] d\sigma(\xi) \leq 0, \qquad t \geq t_1. \tag{7}$$

Furthermore, in view of $g(t,\xi)$ being nondecreasing with respect to ξ, we have

$$y^{(n)}(t) + \lambda y[g(t,a)] \int_a^b p(t,\xi)\{1 - c[g(t,\xi)]\} d\sigma(\xi) \leq 0, \qquad t \geq t_2. \tag{8}$$

Let

$$z(t) = \frac{y^{(n-1)}(t)}{y[\frac{g(t,a)}{2}]}. \tag{9}$$

Then $z(t) \geq 0$. Since $\frac{d}{dt}g(t,a)$ exists, we obtain $y'[g(t,a)] = \frac{dy}{dg}\frac{d}{dt}g(t,a)$. Furthermore, from Lemma 1, $y^{(n)}(t) \leq 0$, and in view of $g(t,\xi)$ being nondecreasing with respect to ξ, $g(t,\xi) \leq t$ for $\xi \in J$, we obtain $y^{(n-1)}(t) \leq y^{(n-1)}[g(t,a)] \leq y^{(n-1)}[\frac{g(t,a)}{2}]$. Thus, from

Lemma 2, we have

$$\begin{aligned} z'(t) &= \frac{y^{(n)}(t)}{y[\frac{g(t,a)}{2}]} - \frac{1}{2}\frac{y^{(n-1)}(t)y'[\frac{g(t,a)}{2}]g'(t,a)}{y^2[\frac{g(t,a)}{2}]} \\ &\leq \frac{y^{(n)}(t)}{y[\frac{g(t,a)}{2}]} - \frac{M_\theta}{2}g^{n-2}(t,a)g'(t,a)z^2(t), \end{aligned} \tag{10}$$

Furthermore, from $y'(t) > 0$ and (8), for $t \geq t_2$, we obtain

$$z'(t) \leq -\lambda \int_a^b p(t,\xi)\{1 - c[g(t,\xi)]\}d\sigma(\xi) - \frac{M_\theta}{2}g^{n-2}(t,a)g'(t,a)z^2(t). \tag{11}$$

Integrating by parts for any $t > T \geq t_1$, and using the properties (H_1) and (H_2), we have

$$\begin{aligned} &\lambda \int_T^t H(t,s)\int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\}d\sigma(\xi)ds \\ &\leq -\int_T^t H(t,s)z'(s)ds - \frac{M_\theta}{2}\int_T^t H(t,s)g^{n-2}(s,a)g'(s,a)z^2(s)ds \\ &= -\int_T^t H(t,s)dz(s) - \frac{M_\theta}{2}\int_T^t H(t,s)g^{n-2}(s,a)g'(s,a)z^2(s)ds \\ &= H(t,T)z(T) - \int_T^t h(t,s)\sqrt{H(t,s)}z(s)ds \\ &-\frac{M_\theta}{2}\int_T^t H(t,s)g^{n-2}(s,a)g'(s,a)z^2(s)ds \\ &= H(t,T)z(T) - \frac{1}{2}\int_T^t \Big[\sqrt{M_\theta H(t,s)g^{n-2}(s,a)g'(s,a)}z(s) \\ &+ \frac{h(t,s)}{\sqrt{M_\theta g^{n-2}(s,a)g'(s,a)}}\Big]^2 ds + \int_T^t \frac{h^2(t,s)}{2M_\theta g^{n-2}(s,a)g'(s,a)}ds, \end{aligned}$$

which implies that

$$\begin{aligned} &\int_T^t \left[\lambda H(t,s)\int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\}d\sigma(\xi) - \frac{h^2(t,s)}{2M_\theta g^{n-2}(s,a)g'(s,a)}\right]ds \\ &\leq H(t,T)z(T) - \frac{1}{2}\int_T^t \Big[\sqrt{M_\theta H(t,s)g^{n-2}(s,a)g'(s,a)}z(s) \\ &\quad + \frac{h(t,s)}{\sqrt{M_\theta g^{n-2}(s,a)g'(s,a)}}\Big]^2 ds. \end{aligned} \tag{12}$$

Furthermore, in view of (H_2), for $t_1 \geq t_0$, we have $H(t,t_1) \leq H(t,t_0)$. Thus

$$\begin{aligned} &\int_{t_1}^t \left[\lambda H(t,s)\int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\}d\sigma(\xi) - \frac{h^2(t,s)}{2M_\theta g^{n-2}(s,a)g'(s,a)}\right]ds \\ &\leq H(t,t_1)z(t_1) \leq H(t,t_0)z(t_1) \end{aligned}$$

$$\begin{aligned}
&\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[\lambda H(t,s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)-\frac{h^2(t,s)}{2M_\theta g^{n-2}(s,a)g'(s,a)}\right]ds\\
=\ &\frac{1}{H(t,t_0)}\left[\int_{t_0}^{t_1}+\int_{t_1}^{t}\right]\left[\lambda H(t,s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)\right.\\
&\left.-\frac{h^2(t,s)}{2M_\theta g^{n-2}(s,a)g'(s,a)}\right]ds\\
\le\ & z(t_1)+\int_{t_0}^{t_1}\frac{H(t,s)}{H(t,t_0)}\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds\\
\le\ & z(t_1)+\int_{t_0}^{t_1}\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds. \qquad (13)
\end{aligned}$$

It follows from (13) that

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\lambda H(t,s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)\\
\le\ &\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[\lambda H(t,s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)\right.\\
&\left.-\frac{h^2(t,s)}{2M_\theta g^{n-2}(s,a)g'(s,a)}\right]ds\\
&+\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{h^2(t,s)}{g^{n-2}(s,a)g'(s,a)}ds\\
\le\ & z(t_1)+\int_{t_0}^{t_1}\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds\\
&+\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{h^2(t,s)}{g^{n-2}(s,a)g'(s,a)}ds\\
<\ &\infty,
\end{aligned}$$

which contradicts (4). Therefore, the proof of Theorem 1 is complete.

Remark 1: From Theorem 1, we can establish various sufficient conditions by means of the choices of parameter function $H(t,s)$. For example, choosing $H(t,s)=(t-s)^{m-1}$, $t\ge s\ge t_0$, in which $m>2$ is an integer, we obtain $h(t,s)=(m-1)(t-s)^{\frac{m-3}{2}}$, $t\ge s\ge t_0$. From Theorem 1, we have

Corollary 1: *If there exists an integer $m>2$ such that*

$$\limsup_{t\to\infty}\frac{1}{t^{m-1}}\int_{t_0}^{t}(t-s)^{m-1}\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds=\infty, \qquad (14)$$

$$\limsup_{t\to\infty}\frac{1}{t^{m-1}}\int_{t_0}^{t}\frac{(m-1)^2(t-s)^{m-3}}{g^{n-2}(s,a)g'(s,a)}ds<\infty, \qquad (15)$$

then inequality (1) has no eventually positive solutions.

If

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds<\infty, \qquad (16)$$

we have the following result:

Theorem 2: *Assume that the conditions of Theorem 1 and (16) hold. If $H'_t(t,s)$ is nondecreasing, and there exists a function $\varphi(t) \in C(I,R)$ satisfying*

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_u^t \left[\lambda H(t,s) \int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi) - \frac{h^2(t,s)}{2M_\theta g^{n-2}(s,a)g'(s,a)} \right] ds \geq \varphi(u), u \geq t_0, \tag{17}$$

$$\lim_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t H(t,s)g^{n-2}(s,a)g'(s,a)\varphi_+^2(s)ds = \infty, \quad \varphi_+(s) = \max_{s\geq t_0}\{\varphi(s),0\}, \tag{18}$$

then inequality (1) has no eventually positive solutions.

Proof: Assume to the contrary that $x(t)$ is an eventually positive solution of inequality (1). Then from the proof of Theorem 1, there exists a $t_1 \geq t_0$ such that

$$z'(t) \leq -\lambda \int_a^b p(t,\xi)\{1-c[g(t,\xi)]\}d\sigma(\xi) - \frac{M_\theta}{2} g^{n-2}(t,a)g'(t,a)z^2(t). \tag{19}$$

Thus

$$\begin{aligned} &\lambda \int_{t_1}^t H(t,s) \int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds \\ \leq\ & -\int_{t_1}^t H(t,s)z'(s)ds - \frac{M_\theta}{2}\int_{t_1}^t H(t,s)g^{n-2}(s,a)g'(s,a)z^2(s)ds \\ =\ & H(t,t_1)z(t_1) - \int_{t_1}^t \sqrt{H(t,s)}h(t,s)z(s)ds \\ & -\frac{M_\theta}{2}\int_{t_1}^t H(t,s)g^{n-2}(s,a)g'(s,a)z^2(s)ds \end{aligned} \tag{20}$$

and

$$\begin{aligned} &\lambda \int_{t_1}^t H(t,s) \int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds \\ \leq\ & H(t,t_1)z(t_1) - \frac{1}{2}\int_{t_1}^t \left\{ \sqrt{M_\theta g^{n-2}(s,a)g'(s,a)H(t,s)}z(s) + \frac{h(t,s)}{\sqrt{M_\theta g^{n-2}(s,a)g'(s,a)}} \right\}^2 ds \\ & + \int_{t_1}^t \frac{h^2(t,s)}{2M_\theta g^{n-2}(s,a)g'(s,a)}ds \\ \leq\ & H(t,t_1)z(t_1) + \int_{t_1}^t \frac{h^2(t,s)}{2M_\theta g^{n-2}(s,a)g'(s,a)}ds. \end{aligned} \tag{21}$$

Furthermore, for $t > u \geq t_0$, we have

$$\frac{1}{H(t,t_0)}\int_u^t \left[\lambda H(t,s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi) - \frac{h^2(t,s)}{2M_\theta g^{n-2}(s,a)g'(s,a)}\right]ds \leq \frac{H(t,u)}{H(t,t_0)}z(u). \tag{22}$$

From (17) and (H_2), we conclude that

$$\begin{aligned}\varphi(u) &\leq \frac{1}{H(t,t_0)}\int_u^t \left[\lambda H(t,s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)\right. \\ &\quad \left. - \frac{h^2(t,s)}{2M_\theta g^{n-2}(s,a)g'(s,a)}\right]ds \\ &\leq \frac{H(t,u)}{H(t,t_0)}z(u) \leq z(u),\end{aligned} \tag{23}$$

which implies that

$$\varphi_+^2(u) \leq z^2(u). \tag{24}$$

Let

$$\begin{aligned} v(t) &= \frac{1}{H(t,t_0)}\int_{t_1}^t \sqrt{H(t,s)}h(t,s)z(s)ds \\ w(t) &= \frac{1}{H(t,t_0)}\int_{t_1}^t \frac{M_\theta}{2}H(t,s)g^{n-2}(s,a)g'(s,a)z^2(s)ds. \end{aligned}$$

Then, from (20), we find

$$v(t)+w(t) \leq \frac{H(t,t_1)}{H(t,t_0)}z(t_1) - \frac{1}{H(t,t_0)}\int_{t_1}^t \lambda H(t,s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds. \tag{25}$$

It follows from (17) that

$$\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_u^t \lambda H(t,s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds \geq \varphi(u).$$

Furthermore, we obtain

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_1}^t \lambda H(t,s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds \\ &-\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_1}^t \frac{h^2(t,s)}{2M_\theta g^{n-2}(s,a)g'(s,a)}ds \geq \varphi(t_1).\end{aligned} \tag{26}$$

It turns out from (26) and (16)

$$\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_1}^t \frac{h^2(t,s)}{2M_\theta g^{n-2}(s,a)g'(s,a)}ds < \infty.$$

Thus, there exists a sequence $\{t_n\}_1^\infty$ in $[t_1,\infty)$ such that $\lim_{n\to\infty} t_n=\infty$ that satisfies

$$\lim_{n\to\infty}\frac{1}{H(t_n,t_0)}\int_{t_1}^{t_n}\frac{h^2(t_n,s)}{2M_\theta g^{n-2}(s,a)g'(s,a)}ds<\infty. \tag{27}$$

Result (27) implies that

$$\begin{aligned}\limsup_{t\to\infty}&\{v(t)+w(t)\}\\ &\le z(t_1)-\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_1}^{t}\lambda H(t,s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds\\ &\le z(t_1)-\varphi(t_1)\triangleq M. \end{aligned}\tag{28}$$

Then, for any sufficiently large n, we have

$$v(t_n)+w(t_n)<M_1, \tag{29}$$

where $M_1>M$, M and M_1 are constant. According to the definition of $w(t)$, we have

$$w'(t)=\int_{t_1}^{t}\frac{M_\theta\left(H_t'(t,s)H(t,t_0)-H_t'(t,t_0)H(t,s)\right)}{2H^2(t,t_0)}g^{n-2}(s,a)g'(s,a)z^2(s)ds.$$

Since $H_t'(t,s)$ is nondecreasing and (H_2) holds, we have $w'(t)\ge 0$, thus, $w(t)$ is increasing, and $\lim\limits_{t\to\infty} w(t)=l$ exists, where l is finite or infinite. In the case of $\lim\limits_{n\to\infty} w(t_n)=\infty$. Consequently, it follows from (29) that

$$\lim_{n\to\infty} v(t_n)=-\infty, \tag{30}$$

and

$$\frac{v(t_n)}{w(t_n)}+1<\frac{M_1}{w(t_n)}.$$

Thus, for any $0<\varepsilon<1$ and sufficiently large n, we have

$$\frac{v(t_n)}{w(t_n)}<\varepsilon-1<0. \tag{31}$$

On the other hand, by using the Schwartz inequality, for $t\ge t_1$, we obtain

$$\begin{aligned} 0 &\le v^2(t_n)=\frac{1}{H^2(t_n,t_0)}\left\{\int_{t_1}^{t_n}\sqrt{H(t_n,s)}h(t_n,s)z(s)ds\right\}^2\\ &\le \left\{\frac{1}{H(t_n,t_0)}\int_{t_1}^{t_n}\frac{M_\theta}{2}H(t_n,s)g^{n-2}(s,a)g'(s,a)z^2(s)ds\right\}\\ &\qquad\times\left\{\frac{1}{H(t_n,t_0)}\int_{t_1}^{t_n}\frac{2h^2(t_n,s)}{M_\theta g^{n-2}(s,a)g'(s,a)}ds\right\}\\ &= w(t_n)\frac{1}{H(t_n,t_0)}\int_{t_1}^{t_n}\frac{2h^2(t_n,s)}{M_\theta g^{n-2}(s,a)g'(s,a)}ds. \end{aligned}$$

Then,

$$0\le\frac{v^2(t_n)}{w(t_n)}\le\frac{1}{H(t_n,t_0)}\int_{t_1}^{t_n}\frac{2h^2(t_n,s)}{M_\theta g^{n-2}(s,a)g'(s,a)}ds. \tag{32}$$

It follows from (27) that

$$0 \le \lim_{n\to\infty} \frac{v^2(t_n)}{w(t_n)} < \infty. \tag{33}$$

In view of (31), we obtain

$$\lim_{n\to\infty} \frac{v(t_n)}{w(t_n)} = \lim_{n\to\infty} \frac{v'(t_n)}{w'(t_n)} \le \varepsilon - 1 < 0,$$

and then,

$$\lim_{n\to\infty} \frac{v^2(t_n)}{w(t_n)} = \lim_{n\to\infty} \frac{2v(t_n)v'(t_n)}{w'(t_n)} = 2\lim_{n\to\infty} v(t_n) \lim_{n\to\infty} \frac{v'(t_n)}{w'(t_n)} = \infty,$$

which contradicts (33). Thus, we have $\lim_{t\to\infty} w(t) = l < \infty$. Furthermore, according to (24), we conclude that

$$\lim_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_1}^{t} H(t,s) g^{n-2}(s,a) g'(s,a) \varphi_+^2(s) ds$$

$$\le \lim_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_1}^{t} H(t,s) g^{n-2}(s,a) g'(s,a) z^2(s) ds = \frac{2}{M_\theta} \lim_{t\to\infty} w(t) < \infty, \tag{34}$$

which implies that

$$\begin{aligned} \lim_{t\to\infty} & \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s) g^{n-2}(s,a) g'(s,a) \varphi_+^2(s) ds \\ = & \lim_{t\to\infty} \frac{1}{H(t,t_0)} \left[\int_{t_0}^{t_1} + \int_{t_1}^{t}\right] H(t,s) g^{n-2}(s,a) g'(s,a) \varphi_+^2(s) ds \\ \le & \int_{t_0}^{t_1} H(t,s) g^{n-2}(s,a) g'(s,a) \varphi_+^2(s) ds + \frac{2}{M_\theta} \lim_{t\to\infty} w(t) < \infty. \end{aligned}$$

The latter contradicts (18). Therefore, the proof of Theorem 2 is complete.

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