A-MONOTONICITY AND APPLICATIONS TO NONLINEAR VARIATIONAL INCLUSION PROBLEMS

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A new notion of the A-monotonicity is introduced, which generalizes the H-monotonicity. Since the A-monotonicity originates from hemivariational inequalities, and hemivariational inequalities are connected with nonconvex energy functions, it turns out to be a useful tool proving the existence of solutions of nonconvex constrained problems as well.

Recently, Fang and Huang [1] introduced a new class of mappings—*h-monotone* mappings—in the context of solving a system of variational inclusions involving a combination of *h-monotone* and strongly monotone mappings based on the resolvent operator technique. The notion of the *h-monotonicity* has revitalized the theory of maximal monotone mappings in several directions, especially in the domain of applications. Here we announce the notion of the *A-monotone* mappings and its applications to the solvability of systems of nonlinear variational inclusions. The class of the *A-monotone* mappings generalizes the *h-monotonicity*. On the top of that, *A-monotonicity* originates from hemivariational inequalities, and emerges as a major contributor to the solvability of nonlinear variational problems on nonconvex settings. As a matter of fact, some nice examples on *A-monotone* (or generalized maximal monotone) mappings can be found in Naniewicz and Panagiotopoulos [2] and Verma [4]. Hemivariational inequalities—initiated and developed by Panagiotopoulos [3]—are connected with nonconvex energy functions and turned out to be useful tools proving the existence of solutions of nonconvex constrained problems. We note that the *A-monotonicity* is defined in terms of relaxed monotone mappings—a more general notion than the monotonicity/strong monotonicity—which gives a significant edge over the *h-monotonicity*.

Definition 1 [1]. Let $h : H \to H$ and $M : H \to 2^H$ be any two mappings on H. The map M is said to be *h-monotone* if M is monotone and $(h + \rho M)(H) = H$ holds for $\rho > 0$. This is equivalent to stating that M is h-monotone if M is monotone and $(h + \rho M)$ is maximal monotone.

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Let X denote a reflexive Banach space and X^* its dual. Inspired by [2, 4], we introduce the notion of the A-*monotonicity* as follows.

Definition 2. Let $A : X \to X^*$ and $M : X \to 2^{X^*}$ be any mappings on X. The map M is said to be A-*monotone* if M is m-*relaxed* monotone and $(A + \rho M)$ is maximal monotone for $\rho > 0$.

Lemma 3. *Let $A : H \to H$ be r-strongly monotone and $M : H \to 2^H$ be A-monotone. Then the resolvent operator $J^{\rho}_{A,M} : H \to H$ is $(1/(r - \rho m))$-Lipschitz continuous for $0 < \rho < r/m$.*

Example 4 [2, Lemma 7.11]. Let $A : X \to X^*$ be (m)-*strongly* monotone and $f : X \to R$ be locally Lipschitz such that ∂f is (α)-*relaxed* monotone. Then ∂f is A-*monotone*, that is, $A + \partial f$ is maximal monotone for $m - \alpha > 0$, where $m, \alpha > 0$.

Example 5 [4, Theorem 4.1]. Let $A : X \to X^*$ be (m)-*strongly* monotone and let $B : X \to X^*$ be (c)-*strongly* Lipschitz continuous. Let $f : X \to R$ be locally Lipschitz such that ∂f is (α)-*relaxed* monotone. Then ∂f is $(A - B)$-*monotone*.

Let H_1 and H_2 be two real Hilbert spaces and K_1 and K_2, respectively, be nonempty closed convex subsets of H_1 and H_2. Let $A : H_1 \to H_1$, $B : H_2 \to H_2$, $M : H_1 \to 2^{H_1}$, and $N : H_2 \to 2^{H_2}$ be nonlinear mappings. Let $S : H_1 \times H_2 \to H_1$ and $T : H_1 \times H_2 \to H_2$ be any two multivalued mappings. Then the problem of finding $(a, b) \in H_1 \times H_2$ such that

$$0 \in S(a,b) + M(a), \qquad 0 \in T(a,b) + N(b) \tag{1}$$

is called the system of nonlinear variational inclusion (SNVI) problems.

When $M(x) = \partial_{K_1}(x)$ and $N(y) = \partial_{K_2}(y)$ for all $x \in K_1$ and $y \in K_2$, where K_1 and K_2, respectively, are nonempty closed convex subsets of H_1 and H_2, and ∂_{K_1} and ∂_{K_2} denote indicator functions of K_1 and K_2, respectively, the SNVI (1) reduces to determine an element $(a, b) \in K_1 \times K_2$ such that

$$\langle S(a,b), x - a\rangle \geq 0 \quad \forall x \in K_1, \tag{2}$$

$$\langle T(a,b), y - b\rangle \geq 0 \quad \forall y \in K_2. \tag{3}$$

Lemma 6. *Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_1$ and $B : H_2 \to H_2$ be strictly monotone, let $M : H_1 \to 2^{H_1}$ be A-monotone, and let $N : H_2 \to 2^{H_2}$ be B-monotone. Let $S : H_1 \times H_2 \to H_1$ and $T : H_1 \times H_2 \to H_2$ be any two multivalued mappings. Then a given element $(a, b) \in H_1 \times H_2$ is a solution to the SNVI (1) problem if and only if (a, b) satisfies*

$$a = J^{\rho}_{A,M}(A(a) - \rho S(a,b)), \qquad b = J^{\eta}_{B,N}(B(b) - \eta T(a,b)). \tag{4}$$

Theorem 7. *Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_1$ be (r_1)-strongly monotone and (α_1)-Lipschitz continuous, and let $B : H_2 \to H_2$ be (r_2)-strongly monotone and (α_2)-Lipschitz continuous. Let $M : H_1 \to 2^{H_1}$ be A-monotone and let $N : H_2 \to 2^{H_2}$ be B-monotone. Let $S : H_1 \times H_2 \to H_1$ be such that $S(\cdot, y)$ is (γ, r)-relaxed cocoercive and (μ)-Lipschitz continuous in the first variable and $S(x, \cdot)$ is (ν)-Lipschitz continuous in the second variable for all $(x, y) \in H_1 \times H_2$. Let $T : H_1 \times H_2 \to H_2$ be such that $T(u, \cdot)$ is (λ, s)-relaxed* *cocoercive and* (β)*-Lipschitz continuous in the second variable and* $T(\cdot,v)$ *is* (τ)*-Lipschitz continuous in the first variable for all* $(u,v)\in H_1\times H_2$*. If, in addition, there exist positive constants* ρ *and* η *such that*

$$\begin{aligned} &\sqrt{\alpha_1-2\rho r+2\rho\gamma\mu^2+\rho^2\mu^2}+\eta\tau<r_1,\\ &\sqrt{\alpha_2-2\eta s+2\eta\lambda\beta^2+\eta^2\beta^2}+\rho\nu<r_2, \end{aligned} \tag{5}$$

then the SNVI (1) problem has a unique solution.

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