MULTIOBJECTIVE DUALITY WITH $\rho-(\eta,\theta)$-INVEXITY

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Under $\rho-(\eta,\theta)$-invexity assumptions on the functions involved, weak, strong, and converse duality theorems are proved to relate properly efficient solutions of the primal and dual problems for a multiobjective programming problem.

1. Introduction

The notion of η-invexity was originally introduced by Hanson [6] who showed that, for a nonlinear programming problem whose objective and constrained functions are η-invex (all with respect to the same η), the Karush-Kuhn-Tucker necessary optimality conditions are also sufficient. The term invex (for invariant convex) was coined by Craven [2] to signify the fact that the invexity property of a function is invariant under certain types of coordinate transformations. Evidently, convex functions, in general, do not possess this property.

Various properties, extensions, and applications of η-invex functions are discussed in [1, 2, 7] among others. Later the concept of $\rho-(\eta,\theta)$-invexity has been introduced by Zalmai [11], which generalizes the notion of invexity.

Recently programs with several conflicting objectives have been extensively studied in the literature. Introducing the concept of proper efficiency of solutions, Geoffrion [5] proved an equivalence between multiobjective program with convex functions and a related parametric (scalar) objective program. Using this equivalence, Weir [9] formulated a dual program for a multiobjective program having differentiable convex functions. Subsequently, Egudo [4] and Weir [9] proved duality results for a differentiable multiobjective program with pseudoconvex/quasiconvex functions. Das and Nanda [3] have studied the duality theorems of Mond-Weir type for a multiobjective programming problem with semilocally invex functions. Xu [10] has studied mixed-type duality in multiobjective programming problems.

In the present paper, duality results (weak, strong, and converse duality theorems) are proved for multiobjective programming problem under $\rho-(\eta,\theta)$-invexity assumptions on the functions involved.

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2. Preliminaries

In [5], Geoffrion considered the following multiobjective programming problem:
(VP)

$$\underset{x\in X}{\text{Minimize}}\, f(x), \tag{2.1}$$

where $f:X\to\mathbb{R}^p$ and X is an open subset of $\mathbb{R}^n$, and minimization means obtaining efficient solutions in the following sense.

A point $\bar{x}\in X$ is an efficient solution for $f=(f_1,f_2,\ldots,f_p)$ if there is no $x\in X$ such that $f(x)\leq f(\bar{x})$ and $f(x)\neq f(\bar{x})$.

An efficient solution $\bar{x}\in X$, for which there exits a scalar $M>0$ such that for each $i=1,2,\ldots,p$, we have

$$\frac{f_i(x)-f_i(\bar{x})}{f_j(\bar{x})-f_j(x)}\leq M \tag{2.2}$$

for some j such that $f_j(x)>f_j(\bar{x})$ and $f_i(x)<f_i(\bar{x})$ for $x\in X$, is called a properly efficient solution of (VP) (see Geoffrion [5]). Geoffrion [5] proved the following results.

Lemma 2.1. *If for fixed $0<\lambda\in\mathbb{R}^p$, $\bar{x}$ is an optimal solution of the parametric programming problem*
(P_λ)

$$\underset{x\in X}{\text{Minimize}}\, \lambda^T f(x), \tag{2.3}$$

where $0<\lambda\in\mathbb{R}^p$ is a vector, then $\bar{x}$ is a properly efficient solution of the multiobjective problem (VP).

Lemma 2.2. *If X is convex and $f_i, i=1,2,\ldots,p$, are all convex functions, then $\bar{x}$ is a properly efficient solution for (VP) if and only if $\bar{x}$ is an optimal solution of the parametric programming problem (P_λ) for some $\lambda\in\mathbb{R}^p$ with strictly positive components.*

Recently Hanson and Mond [7] have generalized Lemma 2.2 to invex functions. They have shown that if f_i, $i=1,2,\ldots,p$, are differentiable invex functions with respect to the same $\eta(x,u)$ (n-dimensional) for $x\in X$, $u\in X$, then $\bar{x}$ is properly efficient solution in the multiobjective programming problem (VP) if and only if $\bar{x}$ is an optimal solution of the parametric programming problem (P_λ) for some $\lambda\in\mathbb{R}^p$ with strictly positive components.

A differentiable function $f(x)$ is said to be invex (see [1, 6]) at a point $u\in X$ over X if there exists $\eta(x,u)\in\mathbb{R}^n$ such that

$$f(x)-f(u)\geq\eta(x,u)^T\nabla f(u)\quad \forall x\in X. \tag{2.4}$$

Here ∇f denotes the gradient of f and the subscript "T" stands for the transpose of a vector. Later the concept of $\rho-(\eta,\theta)$-invexity has been studied by Zalmai (see [11]), which generalizes the notion of invexity function.

Definition 2.3. A differentiable function $h : X \to \mathbb{R}$ is called $\rho - (\eta, \theta)$-invex with respect to vector-valued functions η and θ if there exists some real number ρ such that for all $x, u \in X$,

$$h(x) - h(u) \geq \eta^T(x,u)\nabla h(u) + \rho\|\theta(x,u)\|^2. \tag{2.5}$$

If $\rho > 0$, then $f(x)$ is called strongly $\rho - (\eta, \theta)$-invex, if $\rho = 0$, we obviously get the usual notion of invexity, and if $\rho < 0$, then $f(x)$ is called weakly $\rho - (\eta, \theta)$-invex. It is clear that

$$\text{strongly } \rho - (\eta,\theta)\text{-invex} \Longrightarrow \text{invex} \Longrightarrow \text{weakly } \rho - (\eta,\theta)\text{-invex}. \tag{2.6}$$

Definition 2.4. h is said to be $\rho - (\eta, \theta)$-pseudoinvex with respect to vector-valued functions η and θ, if there exists some real number ρ such that for all $x, u \in X$

$$\eta^T(x,u)\nabla h(u) \geq -\rho\|\theta(x,u)\|^2 \Longrightarrow h(x) \geq h(u). \tag{2.7}$$

Definition 2.5. h is said to be $\rho - (\eta, \theta)$-quasi-invex with respect to vector-valued functions η and θ if there exists some real number ρ such that for all $x, u \in X$,

$$h(x) \leq h(u) \Longrightarrow \eta^T(x,u)\nabla h(u) \leq -\rho\|\theta(x,u)\|^2. \tag{2.8}$$

3. Duality

Consider the following multiobjective programming problems:

(PVP) Minimize$_{x\in X}$ $f(x)$ subject to $g(x) \leq 0$,

(DVP) Maximize$_{x\in X,\lambda,y}$, $f(u) + y^T g(u)e$ subject to $\nabla\lambda^T f(u) + \nabla y^T g(u) = 0$, $y \geq 0$, $\lambda \geq 0$, $\lambda^T e = 1$,

where $f : X \to \mathbb{R}^p$, $g : X \to \mathbb{R}^m$, $y \in \mathbb{R}^m$, $\lambda \in \mathbb{R}^p$, and e is p-tuple of 1's. Thus parametric (scalar) programming problems corresponding (PVP) and (DVP) are

(PC_λ) Minimize$_{x\in X}$ $\lambda^T f(x)$ subject to $g(x) \leq 0$,

(DC_λ) Maximize$_{x\in X,y}$ $\lambda^T f(u) + y^T g(u)$ subject to $\nabla\lambda^T f(u) + \nabla y^T g(u) = 0$, $y \geq 0$,

respectively. In programming problems (PC_λ) and (DC_λ), the vector $0 < \lambda \in \mathbb{R}^p$ is predetermined. In [4], Egudo and Hanson proved weak and strong duality theorems between (PVP) and (DVP) for invex functions. We prove the following duality theorems.

Theorem 3.1 (weak duality). *Let S be the feasible region for the primal problem (PVP), that is, $S = \{x \in X, g(x) \leq 0\}$. Let (u, λ, y) be a feasible point in the dual problem (DVP) such that $\lambda^T f$ is $\rho - (\eta, \theta)$-invex at $u \in S$ and $y^T g$ is $\rho_1 - (\eta, \theta)$-invex at $u \in S$ with $\rho + \rho_1 \geq 0$. Then*

$$\lambda^T f(x) \geq \lambda^T f(u) + y^T g(u) \quad \forall x \in S. \tag{3.1}$$

Proof. Since $\lambda^T f$ is $\rho - (\eta, \theta)$-invex at u over S and $y^T g$ is $\rho_1 - (\eta, \theta)$-invex at u over S, we have

$$\begin{aligned} \lambda^T f(x) - \lambda^T f(u) &\geq \eta^T(x,u)\nabla(\lambda^T f(u)) + \rho\|\theta(x,u)\|^2, \\ y^T g(x) - y^T g(u) &\geq \eta^T(x,u)\nabla(y^T g(u)) + \rho_1\|\theta(x,u)\|^2, \end{aligned} \tag{3.2}$$

that is,

$$\begin{aligned}\{\lambda^T f(x)+y^T g(x)-\lambda^T f(u)-y^T g(u)\\ \geq \eta^T(x,u)[\nabla(\lambda^T f(u))+\nabla(y^T g(u))]+(\rho+\rho_1)\|\theta(x,u)\|^2\}.\end{aligned} \tag{3.3}$$

Now since (u,λ,y) is feasible in (DVP), we have $\eta^T(x,u)[\nabla(\lambda^T f(u))+\nabla(y^T g(u))]=0$ and $y^T g(x)\leq 0$ for all $x\in S$, therefore inequality (3.3) reduces to

$$\lambda^T f(x)\geq \lambda^T f(u)+y^T g(u)+(\rho+\rho_1)\|\theta(x,u)\|^2. \tag{3.4}$$

Again $\rho+\rho_1\geq 0$, so

$$\lambda^T f(x)\geq \lambda^T f(u)+y^T g(u) \quad \forall x\in S. \tag{3.5}$$

□

Theorem 3.2 (strong duality). *Let $\bar{x}$ be a properly efficient solution of the multiobjective programming problem (PVP) at which a constraint qualification is satisfied. Then there exists $(\bar{\lambda},\bar{y})$ such that $(\bar{x},\bar{\lambda},\bar{y})$ is a feasible solution in the programming problem (DVP) and $y^T g(\bar{x})=0$. If also for each feasible $(u,\bar{\lambda},y)$ in the dual programming problem (DVP), $\bar{\lambda}^T f$ is $\rho-(\eta,\theta)$-invex and $y^T g$ is $\rho_1-(\eta,\theta)$-invex at u over the primal feasible region $S=\{x\mid x\in X: g(x)\leq 0\}$ with $\rho+\rho_1\geq 0$, then $(\bar{x},\bar{\lambda},\bar{y})$ is a properly efficient solution of the dual programming problem (DVP) and the objective values are equal.*

Proof. Since a constraint qualification [8] (also see Ben-Israel and Mond [1]) is satisfied at $\bar{x}$, then, from Kuhn-Tucker necessary conditions [5], there exists $(\bar{\lambda},\bar{y})$ such that $(\bar{x},\bar{\lambda},\bar{y})$ is a feasible solution in the programming (DVP) and $y^T g(\bar{x})=0$. Hence, objective function values are equal. Also since for each feasible $(u,\bar{\lambda},y)$ in the dual programming problem (DVP), $\bar{\lambda}^T f$ is $\rho-(\eta,\theta)$-invex and $y^T g$ is $\rho_1-(\eta,\theta)$-invex at u over S, then

$$\begin{aligned}\bar{\lambda}^T f(x)-\bar{\lambda}^T f(u)\geq \eta^T(x,u)\nabla(\bar{\lambda}^T f(u))+\rho\|\theta(x,u)\|^2,\\ y^T g(x)-y^T g(u)\geq \eta^T(x,u)\nabla(y^T g(u))+\rho_1\|\theta(x,u)\|^2,\end{aligned} \tag{3.6}$$

that is,

$$\begin{aligned}\{\bar{\lambda}^T f(x)+y^T g(x)\geq \bar{\lambda}^T f(u)+y^T g(u)\\ +\eta^T(x,u)[\nabla(\bar{\lambda}^T f(u))+\nabla y^T g(u)]+(\rho+\rho_1)\|\theta(x,u)\|^2\}.\end{aligned} \tag{3.7}$$

But $[\nabla(\bar{\lambda}^T f(u))+\nabla(y^T g(u))]=0$, therefore

$$\bar{\lambda}^T f(x)+y^T g(x)\geq \bar{\lambda}^T f(u)+y^T g(u)+(\rho+\rho_1)\|\theta(x,u)\|^2. \tag{3.8}$$

Since $y\geq 0$ and $g(x)\leq 0$, for all $x\in X$, we have $y^T g(x)\leq 0$, for all $x\in S$. Hence, for all feasible $(u,\bar{\lambda},y)$ in the dual programming problem (DVP), we have

$$\bar{\lambda}^T f(x)\geq \bar{\lambda}^T f(u)+y^T g(u)+(\rho+\rho_1)\|\theta(x,u)\|^2 \quad \forall x\in S. \tag{3.9}$$

As $\rho + \rho_1 \geq 0$, so

$$\bar{\lambda}^T f(x) \geq \bar{\lambda}^T f(u) + y^T g(u) \quad \forall x \in S. \tag{3.10}$$

By assumption, $\bar{x}$ is feasible in the primal (PVP) and we have shown that $(\bar{x}, \bar{\lambda}, \bar{y})$ in the dual (DVP) we have

$$\bar{\lambda}^T f(u) + y^T g(u) \leq \bar{\lambda}^T f(\bar{x}). \tag{3.11}$$

Now (3.11) implies that for $\bar{\lambda}$, $(\bar{x}, \bar{y})$ solves the parametric problem (DC_λ). Since $\bar{\lambda} > 0$, from Geoffrion's [5] sufficient conditions, we conclude that $(\bar{x}, \bar{\lambda}, \bar{y})$ is properly efficient for the problem (DVP). □

4. Converse duality

In this section, we study the converse duality theorem.

Theorem 4.1 (converse duality). *Let $(\bar{x}, \bar{\lambda}, \bar{y})$ be a feasible solution for the dual problem (DVP) such that $\bar{\lambda} f$ is $\rho - (\eta, \theta)$*-invex *at $\bar{u}$ on S and $\bar{y}^T g$ is $\rho_1 - (\eta, \theta)$*-invex *at $\bar{u}$ over the primal feasible region $S = \{x \mid x \in X : g(x) \leq 0\}$ with $\rho + \rho_1 \geq 0$. Suppose there exists $\bar{x} \in S$ such that $\bar{\lambda}^T f(\bar{x}) = \bar{\lambda}^T f(\bar{u}) + \bar{y}^T g(\bar{u})$. Then $\bar{x}$ is properly efficient solution of (PVP). If also for each feasible $(u, \bar{\lambda}, y)$ in the dual programming problem (DVP), $\bar{\lambda}^T f$ is $\rho - (\eta, \theta)$*-invex *at u over S and $y^T g$ is $\rho_1 - (\eta, \theta)$*-invex *at u over the primal feasible region $S = \{x \mid x \in X : g(x) \leq 0\}$ with $\rho + \rho_1 \geq 0$, then $(\bar{x}, \bar{\lambda}, \bar{y})$ is also properly efficient of the dual multiobjective programming problem (DVP).*

Proof. By Theorem 3.1 we have

$$\bar{\lambda}^T f(x) \geq \bar{\lambda}^T f(\bar{u}) + \bar{y}^T g(\bar{u}) \quad \forall x \in S. \tag{4.1}$$

Now since there exists $\bar{x} \in S$ such that

$$\bar{\lambda}^T f(\bar{x}) = \bar{\lambda}^T f(\bar{u}) + \bar{y}^T g(\bar{u}), \tag{4.2}$$

so

$$\bar{\lambda}^T f(x) \geq \bar{\lambda}^T f(\bar{x}) \quad \forall x \in S. \tag{4.3}$$

Since $x \in S$, (4.3) implies that for $\bar{\lambda}$, $\bar{x}$ is an optimal solution of the parametric programming (PC_λ). As $\bar{\lambda} > 0$, from Geoffrion's sufficient conditions, $\bar{x}$ is properly efficient solution of the primal multiobjective programming problem (PVP).

Again because $\bar{\lambda}^T f$ is $\rho - (\eta, \theta)$-invex and $y^T g$ is $\rho_1 - (\eta, \theta)$-invex at u over S with $\rho + \rho_1 \geq 0$, we have, for each feasible $(u, \bar{\lambda}, y)$ in the dual programming problem (DVP),

$$\bar{\lambda}^T f(x) \geq \bar{\lambda}^T f(u) + y^T g(u), \quad \forall x \in S, \tag{4.4}$$

and because $\bar{x} \in S$ and $\bar{\lambda}^T f(\bar{x}) = \bar{\lambda}^T f(\bar{u}) + \bar{y}^T g(\bar{u})$, it follows that

$$\bar{\lambda}^T f(u) + y^T f(u) \leq \bar{\lambda}^T f(\bar{x}) = \bar{\lambda}^T f(\bar{u}) + \bar{y}^T g(\bar{u}), \tag{4.5}$$

and (4.5) holds for all feasible $(u,\bar{\lambda},y)$ in the dual programming problem (DVP). This implies that for $\bar{\lambda}$, $(\bar{u},\bar{y})$ is an optimal solution of the parametric programming problem (DC_λ). Since $\bar{\lambda}>0$, it now follows from Geoffrion's [5] sufficient condition that $(\bar{x},\bar{\lambda},\bar{y})$ is a properly efficient solution of the dual programming problem (DVP). □

Concluding remark. As $\rho-(\eta,\theta)$-invexity/pseudoinvexity is a generalization of invexity, multiobjective variational problem and multiobjective control problem under $\rho-(\eta,\theta)$-invexity will orient future research of the authors.

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