EXACT AND APPROXIMATE SOLUTIONS OF DELAY DIFFERENTIAL EQUATIONS WITH NONLOCAL HISTORY CONDITIONS

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We study the exact and approximate solutions of a delay differential equation with various types of nonlocal history conditions. We establish the existence and uniqueness of mild, strong, and classical solutions for a class of such problems using the method of semidiscretization in time. We also establish a result concerning the global existence of solutions. Finally, we consider some examples and discuss their exact and approximate solutions.

1. Introduction

We are concerned here with exact and approximate solutions of the following delay differential equation:

$$\begin{gathered}\frac{\partial w}{\partial t}(x,t)-\frac{\partial^2 w}{\partial x^2}(x,t)=f\big(x,t,w(x,t),w(x,t-\tau)\big),\quad 0<t\leq T<\infty,\ x\in(a,b),\\ w(a,t)=w(b,t)=0,\quad t\geq 0,\\ g\big(w_{[-\tau,0]}\big)=\phi,\end{gathered}\tag{1.1}$$

where the sought-for real-valued function w is defined on $(a,b)\times[-\tau,T]$, $\tau>0$, $a<b$, f is a smooth real-valued function defined on $(a,b)\times[0,T]\times\mathbb{R}^2$, g is a map from $\mathscr{C}_0:=C([-\tau,0];L^2(a,b))$ into $L^2(a,b)$, $w_{[-\tau,0]}$ is the restriction of w on $(a,b)\times[-\tau,0]$, and $\phi\in L^2(a,b)$.

Some of the cases of the nonlocal history function g in which we will be interested are the following.

(I) $g(\psi)(x)=\int_{-\tau}^{0}k(s)\psi(s)(x)ds$ for $x\in(a,b)$ and $\psi\in\mathscr{C}_0$, where $k\in L^1(-\tau,0)$ with $\kappa:=\int_{-\tau}^{0}k(s)ds\neq 0$.

(II) $g(\psi)(x)=\sum_{i=1}^{n}c_i\psi(\theta_i)(x)$ for $x\in(a,b)$ and $\psi\in\mathscr{C}_0$, where $-\tau\leq\theta_1<\theta_2<\cdots<\theta_n\leq 0$ and $C:=\sum_{i=1}^{n}c_i\neq 0$.

(III) $g(\psi)(x)=\sum_{i=1}^{n}(c_i/\epsilon_i)\int_{\theta_i-\epsilon_i}^{\theta_i}\psi(s)(x)ds$ for $x\in(a,b)$ and $\psi\in\mathscr{C}_0$, where θ_i and c_i are as in (II) and $\epsilon_i>0$ for $i=1,2,\ldots,n$.

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Nonlocal abstract differential and functional differential equations have been extensively studied in the literature. We refer to the works of Byszewski [6], Byszewski and Lakshmikantham [8], Balachandran and Chandrasekaran [5], and Lin and Liu [11]. Most of them used semigroup theory and fixed point theorem to establish the unique existence and regularity of solution. In [7], Byszewski and Akca applied Schauder's fixed point principle to prove the theorems for existence of mild and classical solutions of nonlocal Cauchy problem of the form

$$u'(t)+Au(t)=f(t,u(t),u(b_1(t)),\dots,u(b_m(t))),\quad t\in(0,T],$$
$$u(0)+g(u)=u_0, \tag{1.2}$$

where $-A$ is the infinitesimal generator of a compact C_0 semigroup in a Banach space.

In our recent work [1, 2], we studied the functional differential equation (1.2) with the nonlocal history condition $h(u_{[-\tau,0]})=\phi$, where h is a Volterra-type operator from $\mathscr{C}_0$ into itself and $\phi\in\mathscr{C}_0$. We made use of method of semidiscretization in time to derive the existence and uniqueness of a strong solution. Many authors have used and developed the method of semidiscretization for nonlinear evolution and nonlinear functional evolution equations, see, for instance, the papers of Kartsatos and Parrott [9], Kartsatos and Zigler [10], Bahuguna and Raghavendra [4], and the references listed therein.

Our purpose here is to study the exact and approximate solutions of the delay differential equation (1.1) with a nonlocal condition. In doing so, we first use the method of semidiscretization to derive the existence of a unique strong solution, then we prove that strong solution is a classical solution if additional conditions are assumed on the operator. The global existence of a solution for (1.1), a nonconsidered problem in [1, 2], is also established with an additional assumption (see Theorem 4.1). The result of the paper consists, among other things, in that we obtain a solution of problem of much stronger regularity than in [1, 2].

2. Existence and uniqueness of solutions

The existence and uniqueness results have been established for the more general case of (2.2) in Bahuguna [3]. For the sake of completeness, we briefly mention the ideas and the main result of the existence and uniqueness.

If we take $H:=L^2(a,b)$, the real Hilbert space of all real-valued square-integrable functions on the interval (a,b), and the linear operator A defined by

$$D(A):=\{u\in H: u''\in H,\ u(a)=u(b)=0\},\quad Au=-u'', \tag{2.1}$$

then it is well known that $-A$ generates an analytic semigroup e^{tA}, $t\geq 0$, in H. If we define $u:[-\tau,T]\to H$ given by $u(t)(x)=w(x,t)$, then (1.1) may be rewritten as the following evolution equation:

$$u'(t)+Au(t)=F(t,u(t),u(t-\tau)),\quad 0<t\leq T,$$
$$h(u_{[-\tau,0]})=\Phi, \tag{2.2}$$

for a suitably defined function $F : [0,T] \times H^2 \to H$, $0 < T < \infty$, $\Phi \in \mathcal{C}_0 := C([-\tau,0];H)$, the linear operator A, defined from the domain $D(A) \subset H$ into H, is such that $-A$ is the infinitesimal generator of a C_0 semigroup $S(t)$, $t \geq 0$, of contractions in H, the map h is defined from $\mathcal{C}_0$ into $\mathcal{C}_0$. Here $\mathcal{C}_t := C([-\tau,t];H)$ for $t \in [0,T]$ is the space of all continuous functions from $[-\tau,t]$ into H endowed with supremum norm

$$\|\psi\|_t = \sup_{-\tau \leq \eta \leq t} \|\psi(\eta)\|, \quad \psi \in \mathcal{C}_t. \tag{2.3}$$

Suppose that there exists a $\chi \in \mathcal{C}_0$ such that $h(\chi) = \Phi$. Let $\widetilde{T}$ be any number such that $0 < \widetilde{T} \leq T$. A function $u \in \mathcal{C}_{\widetilde{T}}$ such that

$$u(t) = \begin{cases} \chi(t), & t \in [-\tau,0], \\ S(t)\chi(0) + \int_0^t S(t-s)F(s,u(s),u(s-\tau))\,ds, & t \in [0,\widetilde{T}], \end{cases} \tag{2.4}$$

is called a mild solution of (2.2) on $[-\tau,\widetilde{T}]$. By a strong solution u of (2.2) on $[-\tau,\widetilde{T}]$, we mean a function $u \in \mathcal{C}_{\widetilde{T}}$ such that $u(t) \in D(A)$ for a.e. $t \in [0,\widetilde{T}]$, u is differentiable a.e. on $[0,\widetilde{T}]$ and

$$u'(t) + Au(t) = F(t,u(t),u(t-\tau)), \quad \text{a.e. } t \in [0,\widetilde{T}]. \tag{2.5}$$

A mild solution u of (1.1) on $[-\tau,\widetilde{T}]$ is called a classical solution of (1.1) if $u(t) \in D(A)$ for all $t \in (0,\widetilde{T}]$ and $u \in C^1((0,\widetilde{T});H)$, and

$$u'(t) + Au(t) = F(t,u(t),u(t-\tau)), \quad t \in (0,\widetilde{T}]. \tag{2.6}$$

We have the following existence and uniqueness result for (2.2).

Theorem 2.1. *Suppose that there exists a Lipschitz continuous $\chi \in \mathcal{C}_0$ such that $h(\chi) = \Phi$ and F satisfies the condition*

$$\|F(t_1,u_1,v_1) - F(t_2,u_2,v_2)\| \leq L_F(r)[\,|t_1 - t_2| + \|u_1 - u_2\| + \|v_1 - v_2\|], \tag{2.7}$$

for all $t_i \in [0,T]$, $u_i, v_i \in B_r(H,\chi(0))$, $i = 1,2$, where $B_r(Z,z_0)$ denotes the closed ball of radius $r > 0$ centered at z_0 in the Banach space Z. Then there exists a strong solution u of (2.2) either on the whole interval $[-\tau,T]$ or on a maximal interval $[-\tau,t_{\max})$, $0 < t_{\max} \leq T$, such that u is a strong solution of (2.2) on $[-\tau,\widetilde{T}]$ for every $0 < \widetilde{T} < t_{\max}$, and in the latter case,

$$\lim_{t \to t_{\max}-} \|u(t)\| = \infty. \tag{2.8}$$

If, in addition, $S(t)$ is an analytic semigroup in H, then u is a classical Lipschitz continuous solution on every compact subinterval of the interval of existence. Furthermore, u is unique in $\{\psi \in \mathcal{C}_{\widetilde{T}} : \psi = \chi \text{ on } [-\tau,0]\}$ for every compact subinterval $[-\tau,\widetilde{T}]$ of the interval of existence.

3. Approximations

In this section, we consider the application of the method of semidiscretization in time and the convergence of the approximate solutions. We first establish the existence and uniqueness of a strong solution of (2.2) for any given $\chi \in \mathscr{C}_0$ and $\chi(0) \in D(A)$. Fix $R > 0$ and let $R_0 := R + \sup_{t\in[-\tau,0]} \|\chi(t) - \chi(0)\|$. We choose t_0 such that

$$0 < t_0 \le T, \qquad t_0 M_0 \le R, \tag{3.1}$$

where, $M_0 := \|A\chi(0)\| + L_f(R_0)(T + 5R_0) + \|f(0,\chi(0),\chi(0))\|$.

For $n \in \mathbb{N}$, let $h_n = t_0/n$. We set $u_0^n = \chi(0)$ for all $n \in \mathbb{N}$ and define each of $\{u_j^n\}_{j=1}^n$ as the unique solution of the equation

$$\frac{u - u_{j-1}^n}{h_n} + Au = F\big(t_j^n, u_{j-1}^n, \tilde{u}_{j-1}^n(t_j^n - \tau)\big), \tag{3.2}$$

where $\tilde{u}_0^n(t) = \chi(t)$ for $t \in [-\tau,0]$, $\tilde{u}_0^n(t) = \chi(0)$ for $t \in [0,t_1^n]$, and for $2 \le j \le n$,

$$\tilde{u}_{j-1}^n(t) = \begin{cases} \chi(t), & t \in [-\tau,0], \\ u_{i-1}^n + \dfrac{1}{h_n}\big(t - t_{i-1}^n\big)\big(u_i^n - u_{i-1}^n\big), & t \in \big[t_{i-1}^n, t_i^n\big],\ i = 1,2,\dots,j-1, \\ u_{j-1}^n, & t \in \big[t_{j-1}^n, t_j^n\big]. \end{cases} \tag{3.3}$$

The existence of a unique $u_j^n \in D(A)$ satisfying (3.2) is a consequence of the m-monotonicity of A. We define the sequence $\{U^n\} \subset \mathscr{C}_{t_0}$ of polygonal functions

$$U^n(t) = \begin{cases} \chi(t), & t \in [-\tau,0], \\ u_{j-1}^n + \dfrac{1}{h_n}\big(t - t_{j-1}^n\big)\big(u_j^n - u_{j-1}^n\big), & t \in \big(t_{j-1}^n, t_j^n\big], \end{cases} \tag{3.4}$$

and prove the convergence of $\{U^n\}$ to a unique strong solution u of (2.2) as $n \to \infty$. Before proving the convergence, we state and prove some lemmas which will be used to establish the main result.

Lemma 3.1. *For $n \in \mathbb{N}$, $j = 1,2,\dots,n$,*

$$\big\|u_j^n - \chi(0)\big\| \le R. \tag{3.5}$$

Proof. From (3.2) for $j = 1$, we have

$$\big\|u_1^n - \chi(0)\big\| \le h_n M_0 \le R. \tag{3.6}$$

Assume that $\|u_i^n - \chi(0)\| \leq R$ for $i = 1,2,\ldots,j-1$. Now, for $2 \leq j \leq n$,

$$||u_j^n - \chi(0)|| \leq ||u_{j-1}^n - \chi(0)|| + h_n M_0. \tag{3.7}$$

Repeating the above inequality, we obtain

$$||u_j^n - \chi(0)|| \leq j h_n M_0 \leq R, \tag{3.8}$$

as $jh_n \leq t_0$ for $0 \leq j \leq n$. This completes the proof of the lemma. □

Lemma 3.2. *There exists a positive constant K independent of the discretization parameters n, j, and h_n such that*

$$\left|\left|\frac{u_j^n - u_{j-1}^n}{h_n}\right|\right| \leq K, \quad j = 1,2,\ldots,n, \; n = 1,2,\ldots. \tag{3.9}$$

Proof. In this proof and subsequently, K will represent a generic constant independent of j, h_n, and n. From (3.2), for $j = 1$ and monotonicity of A, we have

$$\left|\left|\frac{u_1^n - u_0^n}{h_n}\right|\right| \leq M_0 \leq K. \tag{3.10}$$

Now, for $2 \leq j \leq n$, using monotonicity of A and local Lipschitz-like condition (2.7) of F, we get

$$\max_{\{1 \leq k \leq j\}} \left|\left|\frac{u_k^n - u_{k-1}^n}{h_n}\right|\right| \leq (1 + Ch_n) \max_{\{1 \leq k \leq j-1\}} \left|\left|\frac{u_k^n - u_{k-1}^n}{h_n}\right|\right| + Ch_n, \tag{3.11}$$

where C is a positive constant independent of j, h_n, and n. Repeating the above inequality, we obtain

$$\left|\left|\frac{u_j^n - u_{j-1}^n}{h_n}\right|\right| \leq K. \tag{3.12}$$

This completes the proof of the lemma. □

We introduce another sequence $\{X^n\}$ of step functions from $[-h_n, t_0]$ into H by

$$X^n(t) = \begin{cases} \chi(0), & t \in [-h_n, 0], \\ u_j^n, & t \in (t_{j-1}^n, t_j^n]. \end{cases} \tag{3.13}$$

For notational convenience, let

$$f^n(t) = f\left(t_j^n, u_{j-1}^n, \tilde{u}_{j-1}^n(t_j^n - \tau)\right), \quad t \in (t_{j-1}^n, t_j^n], \; 1 \leq j \leq n. \tag{3.14}$$

Then (3.2) may be rewritten as

$$\frac{d^-}{dt}U^n(t)+AX^n(t)=f^n(t), \quad t\in(0,t_0], \tag{3.15}$$

where d^-/dt denotes the left derivative in $(0,t_0]$. Also, for $t\in(0,t_0]$, we have

$$\int_0^t AX^n(s)ds=\chi(0)-U^n(t)+\int_0^t f^n(s)ds. \tag{3.16}$$

Next, we prove the convergence of U^n to u in $\mathscr{C}_{t_0}$.

LEMMA 3.3. *There exists $u\in\mathscr{C}_{t_0}$ such that $U^n\to u$ in $\mathscr{C}_{t_0}$ as $n\to\infty$. Moreover, u is Lipschitz continuous on* $[0,t_0]$.

Proof. It can be easily proved using monotonicity of A and condition (2.7) of F in (3.15) (cf. [1, 2]). □

Proof of Theorem 2.1. By proceeding as in Agarwal and Bahuguna [2] we can show the existence and uniqueness of the strong solution on $[-\tau,t_0]$ as well as the continuation of the solution on $[-\tau,T]$. Thus we have that there exists a strong solution of (2.2) either on the whole interval $[-\tau,T]$ or on the maximal interval of existence $[-\tau,t_{\max})$, $0<t_{\max}\le T$. In the latter case, if $\lim_{t\to t_{\max}-}\|u(t)\|<\infty$, we have that $\lim_{t\to t_{\max}-}u(t)$ is in the closure of $D(A)$ in H, and if it is in $D(A)$, then, following the same steps as before, $u(t)$ can be extended beyond $t_{\max}$, which contradicts the definition of the maximal interval of existence.

To prove the remaining part of Theorem 2.1, we assume the interval of existence $[-\tau, T]$. The proof may be modified for the interval $[-\tau,t_{\max})$. Also $-A$ is the infinitesimal generator of C_0 semigroup. The function $\bar{F}:[0,T]\to H:=L^2(a,b)$ given by

$$\bar{F}(t)=F(t,u(t),u(t-\tau)) \tag{3.17}$$

is Lipschitz continuous and therefore continuous on $[0,T]$ and $\bar{F}\in L^1((0,T);H)$. Now it is easy to see that if u is the strong solution of (2.2), then u is given by

$$u(t)=\begin{cases}\chi(t), & t\in[-\tau,0],\\ S(t)\chi(0)+\int_0^t S(t-s)\bar{F}(s)ds, & t\in[0,T],\end{cases} \tag{3.18}$$

and therefore is a mild solution of (2.2). If $S(t)$ is an analytic semigroup in H, then by the use of Corollary 3.3 in Pazy [12, page 113], we obtain that u is a classical solution of (2.2). Clearly, if $\chi\in\mathscr{C}_0$ satisfying that $h(\chi)=\Phi$ is unique on $[-\tau,0]$, u is unique since for two $\chi,\tilde{\chi}$ in $\mathscr{C}_0$ satisfying $h(\chi)=h(\tilde{\chi})=\Phi$ with $\chi\ne\tilde{\chi}$, the corresponding solutions u_χ and $u_{\tilde{\chi}}$ belonging to $\{\psi\in\mathscr{C}_{\tilde{T}}:\psi=\chi \text{ on } [-\tau,0]\}$ and $\{\psi\in\mathscr{C}_{\tilde{T}}:\psi=\tilde{\chi} \text{ on } [-\tau,0]\}$, respectively, are different. □

4. Global existence

We turn now to global existence. Here further assumptions are made, under the consideration of which, the existence of a global solution is established.

THEOREM 4.1. *Let $-A$ be the infinitesimal generator of a compact C_0 semigroup $S(t)$, $t \geq 0$, on H. Let $F : [0,\infty) \times H \to H$ be continuous and map bounded sets in $[0,\infty) \times H$ into bounded sets in H. Also there exist two locally integrable functions $k_1(s)$ and $k_2(s)$ such that*

$$||F(s,u,v)|| \leq k_1(s)(\|u\| + \|v\|) + k_2(s), \quad \text{for } 0 \leq s < \infty,\ u,v \in H. \tag{4.1}$$

Then, for every $\chi \in \mathscr{C}_0$ satisfying $h(\chi) = \Phi$, problem (2.2) has a global solution $u \in C([-\tau,\infty),H)$.

Proof. We know that the corresponding solution u exists on the interval $[-\tau,T)$ and is given by

$$u(t) = \begin{cases} \chi(t), & t \in [-\tau,0], \\ S(t)\chi(0) + \int_0^t S(t-s)F(s,u(s),u(s-\tau))\,ds, & t \in [0,T). \end{cases} \tag{4.2}$$

We also know that $\|S(t)\| \leq Me^{\omega t}$ for some $M \geq 1$ and $\omega \geq 0$. Let

$$\xi(t) = (M+1)\|\chi\|_0 + \int_0^t Me^{-\omega s}k_2(s)ds. \tag{4.3}$$

The function ξ thus defined is obviously continuous on $[0,\infty)$.

For $t \in [-\tau,0]$,

$$||u(t)||e^{-\omega t} = ||\chi(t)||e^{-\omega t} \leq \|\chi\|_0 \leq M\|\chi\|_0, \tag{4.4}$$

and for $t \in [0,T)$,

$$\begin{aligned} ||u(t)||e^{-\omega t} &\leq e^{-\omega t}||S(t)\chi(0)|| + e^{-\omega t}\int_0^t ||S(t-s)F(s,u(s),u(s-\tau))||ds \\ &\leq M\|\chi\|_0 + M\int_0^t e^{-\omega s}k_1(s)(||u(s)|| + ||u(s-\tau)||)ds + M\int_0^t e^{-\omega s}k_2(s)ds \\ &\leq M\|\chi\|_0 + M\int_0^t e^{-\omega s}k_2(s)ds + 2M\int_0^t e^{-\omega s}k_1(s)\sup_{\eta\in[-\tau,s]}||u(\eta)||ds. \end{aligned} \tag{4.5}$$

The above inequality implies that

$$e^{-\omega t}\sup_{\eta\in[-\tau,t]}||u(\eta)|| \leq \xi(t) + 2M\int_0^t e^{-\omega s}k_1(s)\sup_{\eta\in[-\tau,s]}||u(\eta)||ds. \tag{4.6}$$

By the application of Gronwall's inequality, we have

$$e^{-\omega t}\sup_{\eta\in[-\tau,t]}||u(\eta)|| \leq \xi(t) + 2M\int_0^t k_1(s)\xi(s)\exp\left\{2M\int_s^t k_1(r)dr\right\}ds, \tag{4.7}$$

which implies the boundedness of $\|u(t)\|$ by a continuous function. Consequently, there exists a global solution u of (2.2) (see Theorem 2.2 on page 193 in Pazy [12]). □

5. Examples

In this section, to illustrate the applicability of our work, we discuss the exact and approximate solutions of some initial boundary value problems.

As a first example, we consider the equation

$$\frac{\partial u}{\partial t}(t,x)-\frac{\partial^2 u}{\partial x^2}(t,x)=u(t-\tau,x)-e^{-2t}(1+e^{2\tau})\sin x,\quad t>0,\ x\in[0,\pi],\tag{5.1}$$

with the boundary condition

$$u(t,0)=u(t,\pi)=0,\quad t>0,\tag{5.2}$$

and a nonlocal history condition

$$\frac{1}{\tau}\int_{-\tau}^{0}e^{2s}u(s,x)ds=\sin x,\quad x\in[0,\pi],\tag{5.3}$$

where $\tau>1$ is arbitrary. Let $H=L^2([0,\pi])$. The operator A with domain $D(A)=\{v\in H:v''\in H, v(0)=v(\pi)=0\}$ is given by

$$Av=-\frac{d^2v}{dx^2}\quad\text{for } v\in D(A).\tag{5.4}$$

Then $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$, $t\geq 0$, in H.

An exact solution of (5.1) is

$$u(t,x)=e^{-2t}\sin x,\quad t\geq-\tau,\ x\in[0,\pi].\tag{5.5}$$

In this case $\chi_1\in\mathscr{C}_0:=C([-\tau,0];L^2([0,\pi]))$ is given by

$$\chi_1(t)(x)=e^{-2t}\sin x,\tag{5.6}$$

so that the history condition is satisfied.

Divide the interval $I=[0,1]$ into ten subintervals $I_1,I_2,\dots,I_{10}$ ($I_j=[t_{j-1},t_j], j=1,2,\dots,10$) of length $h=0.1$. For $t_0=0$, set $u_0(x)=\sin x$ and find, subsequently, for t_j, the approximate solutions u_j, $j=1,2,\dots,10$, so that

$$\begin{gathered}\frac{u_j(x)-u_{j-1}(x)}{h}-u_j''(x)=\chi_1(t_j-\tau)(x)-e^{-2t_j}(1+e^{2\tau})\sin x,\\ u_j(0)=u_j(\pi)=0,\end{gathered}\tag{5.7}$$

that is,

$$\begin{gathered}\frac{u_j(x)-u_{j-1}(x)}{h}-u_j''(x)=e^{-2t_j}\sin x,\\ u_j(0)=u_j(\pi)=0,\end{gathered}\tag{5.8}$$

is satisfied for $j=1,2,\dots,10$.

Table 5.1. Approximate and exact solutions for the first example ($\chi = e^{-2t}\sin x$).

Approximate solution	Exact solution
$u_1(x) = 0.834661\sin x$	$u(x,t_1) = 0.818731\sin x$
$u_2(x) = 0.697844\sin x$	$u(x,t_2) = 0.670320\sin x$
$u_3(x) = 0.584512\sin x$	$u(x,t_3) = 0.548812\sin x$
$u_4(x) = 0.490526\sin x$	$u(x,t_4) = 0.449329\sin x$
$u_5(x) = 0.412490\sin x$	$u(x,t_5) = 0.367879\sin x$
$u_6(x) = 0.347609\sin x$	$u(x,t_6) = 0.301194\sin x$
$u_7(x) = 0.293590\sin x$	$u(x,t_7) = 0.246597\sin x$
$u_8(x) = 0.248546\sin x$	$u(x,t_8) = 0.201896\sin x$
$u_9(x) = 0.210924\sin x$	$u(x,t_9) = 0.165299\sin x$
$u_{10}(x) = 0.179446\sin x$	$u(x,t_{10}) = 0.135335\sin x$

For $j = 1$, (5.8) becomes

$$u_1''(x) - \frac{1}{h}u_1(x) = \left(-\frac{1}{h} + e^{-2t_1}\right)\sin x, \qquad u_1(0) = u_1(\pi) = 0. \tag{5.9}$$

Consequently, we solve a second-order ordinary differential equation. In this case, the solution is

$$u_1(x) = \frac{1}{1+h}\left(1 - he^{-2h}\right)\sin x. \tag{5.10}$$

Similarly, for $j = 2$, (5.8) yields

$$u_2''(x) - \frac{1}{h}u_2(x) = \left(-\frac{1}{h(1+h)}\left(1 - he^{-2t_1}\right) + e^{-2t_2}\right)\sin x, \qquad u_2(0) = u_2(\pi) = 0. \tag{5.11}$$

On solving this equation in the same way as before, we get

$$u_2(x) = \frac{1}{(1+h)^2}\left[1 - he^{-2h}\left(1 + (1+h)e^{-2h}\right)\right]\sin x. \tag{5.12}$$

Similar results are easily obtained for $j = 3,4,\ldots,10$. Thus we have

$$u_j(x) = \frac{1}{(1+h)^j}\left[1 - he^{-2h}\left(1 + (1+h)e^{-2h} + (1+h)^2e^{-4h} + \cdots + (1+h)^{j-1}e^{2(j-1)h}\right)\right]\sin x \tag{5.13}$$

or

$$u_j(x) = \frac{1}{(1+h)^j}\left[1 - he^{-2h}\left(\frac{1-(1+h)^je^{-2jh}}{1-(1+h)e^{-2h}}\right)\right]\sin x, \quad j = 1,2,\ldots,10. \tag{5.14}$$

Putting here $h = 0.1$ and rounding off to six decimals, we finally obtain the approximate solutions $u_j(x)$ at t_j, $j = 1,2,\ldots,10$ (see Table 5.1).

We also calculate the exact solution of (5.1) for $t = t_1 = 0.1,\dots,t = t_{10} = 1$ (see Table 5.1).

In the next step we choose another function

$$\chi_2(t)(x) = \frac{2\tau}{1-e^{-2\tau}}\sin x \tag{5.15}$$

in $\mathscr{C}_0$ which differs from χ_1 and satisfies the history condition (5.3).

Divide the interval $I = [0,1]$ into the same number of subintervals with step length $h = 0.1$. For $t_0 = 0$, set $u_0(x) = (2\tau/1-e^{-2\tau})\sin x$ and find the approximations u_j so that

$$\frac{u_j(x)-u_{j-1}(x)}{h} - u_j''(x) = \chi_2(t_j-\tau)(x) - e^{-2t_j}(1+e^{2\tau})\sin x, \quad u_j(0) = u_j(\pi) = 0, \tag{5.16}$$

that is,

$$\frac{u_j(x)-u_{j-1}(x)}{h} - u_j''(x) = \left[\frac{-2\tau}{1-e^{-2\tau}} + e^{-2t_j}(1+e^{2\tau})\right]\sin x, \quad u_j(0) = u_j(\pi) = 0, \tag{5.17}$$

is fulfilled for $j = 1,2,\dots,10$.

Following the calculations similar to the previous case, we obtain the approximate solutions u_j, $j = 1,2,\dots,10$, as follows:

$$\begin{aligned} u_1(x) &= \left[\frac{\tau}{\sinh 2\tau} - \frac{h}{(1+h)}e^{-2h}\right](1+e^{2\tau})\sin x, \\ u_2(x) &= \left[\frac{\tau}{\sinh 2\tau} - \frac{he^{-2h}}{(1+h)^2}(1+(1+h)e^{-2h})\right](1+e^{2\tau})\sin x, \end{aligned} \tag{5.18}$$

and

$$u_j(x) = \left[\frac{\tau}{\sinh 2\tau} - \frac{he^{-2h}}{(1+h)^j}\left(\frac{1-(1+h)^j e^{-2jh}}{1-(1+h)e^{-2h}}\right)\right](1+e^{2\tau})\sin x, \quad j = 1,2,\dots,10. \tag{5.19}$$

Putting here $h = 0.1$ and rounding off to six decimals, we get approximate solutions which are tabulated in Table 5.2.

In this case the exact solution is obtained by solving the partial differential equation

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) - \frac{\partial^2 u}{\partial x^2}(t,x) &= \frac{2\tau}{1-e^{-2\tau}}\sin x - e^{-2t}(1+e^{2\tau})\sin x, \quad t > 0,\ x \in [0,\pi], \\ u(t,0) = u(t,\pi) &= 0, \quad t > 0, \\ u(x,0) &= \frac{2\tau}{1-e^{-2\tau}}\sin x, \quad x \in [0,\pi]. \end{aligned} \tag{5.20}$$

We take the solution of the form

$$u(t,x) = T(t)\sin x. \tag{5.21}$$

Table 5.2. Approximate and exact solutions for the first example ($\chi = (2\tau/(1-e^{-2\tau}))\sin x$).

Approximate solution	Exact solution
$u_1(x) = \left(\frac{q}{\sinh 2q} - 0.074430\right)(1+e^{2q})\sin x$	$u(x,t_1) = \left(\frac{q}{\sinh 2q} - 0.086106\right)(1+e^{2q})\sin x$
$u_2(x) = \left(\frac{q}{\sinh 2q} - 0.128602\right)(1+e^{2q})\sin x$	$u(x,t_2) = \left(\frac{q}{\sinh 2q} - 0.148411\right)(1+e^{2q})\sin x$
$u_3(x) = \left(\frac{q}{\sinh 2q} - 0.166803\right)(1+e^{2q})\sin x$	$u(x,t_3) = \left(\frac{q}{\sinh 2q} - 0.192007\right)(1+e^{2q})\sin x$
$u_4(x) = \left(\frac{q}{\sinh 2q} - 0.192487\right)(1+e^{2q})\sin x$	$u(x,t_4) = \left(\frac{q}{\sinh 2q} - 0.220991\right)(1+e^{2q})\sin x$
$u_5(x) = \left(\frac{q}{\sinh 2q} - 0.208432\right)(1+e^{2q})\sin x$	$u(x,t_5) = \left(\frac{q}{\sinh 2q} - 0.238651\right)(1+e^{2q})\sin x$
$u_6(x) = \left(\frac{q}{\sinh 2q} - 0.216865\right)(1+e^{2q})\sin x$	$u(x,t_6) = \left(\frac{q}{\sinh 2q} - 0.247617\right)(1+e^{2q})\sin x$
$u_7(x) = \left(\frac{q}{\sinh 2q} - 0.219568\right)(1+e^{2q})\sin x$	$u(x,t_7) = \left(\frac{q}{\sinh 2q} - 0.249988\right)(1+e^{2q})\sin x$
$u_8(x) = \left(\frac{q}{\sinh 2q} - 0.217961\right)(1+e^{2q})\sin x$	$u(x,t_8) = \left(\frac{q}{\sinh 2q} - 0.247432\right)(1+e^{2q})\sin x$
$u_9(x) = \left(\frac{q}{\sinh 2q} - 0.213174\right)(1+e^{2q})\sin x$	$u(x,t_9) = \left(\frac{q}{\sinh 2q} - 0.241271\right)(1+e^{2q})\sin x$
$u_{10}(x) = \left(\frac{q}{\sinh 2q} - 0.206097\right)(1+e^{2q})\sin x$	$u(x,t_{10}) = \left(\frac{q}{\sinh 2q} - 0.232544\right)(1+e^{2q})\sin x$

Putting this into (5.20), we get a first-order linear differential equation in $T(t)$ which can be solved by calculating the integrating factor. Thus we have

$$T(t) = \frac{2\tau}{1-e^{-2\tau}} - (1+e^{2\tau})(e^{-t} - e^{-2t}). \tag{5.22}$$

Therefore, the exact solution is

$$u(t,x) = \left[\frac{\tau}{\sinh 2\tau} - (e^{-t} - e^{-2t})\right](1+e^{2\tau})\sin x. \tag{5.23}$$

Exact solutions for $t = t_1 = 0.1, \dots, t = t_{10} = 1$ are tabulated in Table 5.2.

On comparison of approximate solutions with exact solution of problem (5.1) at discrete values of variable t in both cases, it is observed that they are very much similar to each other. It is also seen that for $\chi_1 \neq \chi_2$ in $\mathcal{C}_0$, the corresponding solutions are different, which implies the existence of unique solution of (5.1).

As a second example we consider the same partial differential equation with a different nonlocal history condition:

$$\begin{gathered}
\frac{\partial u}{\partial t}(t,x) - \frac{\partial^2 u}{\partial x^2}(t,x) = u(t-\tau,x) - e^{-2t}(1+e^{2\tau})\sin x, \quad t>0,\ x\in[0,\pi], \\
u(t,0) = u(t,\pi) = 0, \quad t>0, \\
\frac{1}{2e^{2\tau}}u(-\tau,x) + \frac{1}{2}u(0,x) = \sin x, \quad x\in[0,\pi].
\end{gathered} \tag{5.24}$$

Table 5.3. Approximate and exact solutions for the second example ($\chi = e^{-2t}\sin x$).

Approximate solution	Exact solution
$u_1(x) = 0.834661\sin x$	$u(x,t_1) = 0.818731\sin x$
$u_2(x) = 0.697844\sin x$	$u(x,t_2) = 0.670320\sin x$
$u_3(x) = 0.584512\sin x$	$u(x,t_3) = 0.548812\sin x$
$u_4(x) = 0.490526\sin x$	$u(x,t_4) = 0.449329\sin x$
$u_5(x) = 0.412490\sin x$	$u(x,t_5) = 0.367879\sin x$
$u_6(x) = 0.347609\sin x$	$u(x,t_6) = 0.301194\sin x$
$u_7(x) = 0.293590\sin x$	$u(x,t_7) = 0.246597\sin x$
$u_8(x) = 0.248546\sin x$	$u(x,t_8) = 0.201896\sin x$
$u_9(x) = 0.210924\sin x$	$u(x,t_9) = 0.165299\sin x$
$u_{10}(x) = 0.179446\sin x$	$u(x,t_{10}) = 0.135335\sin x$

An exact solution of (5.24) is

$$u(t,x) = e^{-2t}\sin x, \quad t \ge -\tau,\ x \in [0,\pi]. \tag{5.25}$$

In a similar manner as before, for $\chi_1 \in \mathscr{C}_0$ given by $\chi_1(t)(x) = e^{-2t}\sin x$, approximations u_j at discrete values t_j, $j = 1,2,\ldots,10$, of t are

$$u_j(x) = \frac{1}{(1+h)^j}\left[1 - he^{-2h}\left(\frac{1-(1+h)^j e^{-2jh}}{1-(1+h)e^{-2h}}\right)\right]\sin x, \quad j = 1,2,\ldots,10. \tag{5.26}$$

Next, we choose $\chi_2 \in \mathscr{C}_0$, such that $\chi_2 \neq \chi_1$ satisfying the nonlocal history condition of (5.24), and given by

$$\chi_2(t)(x) = \frac{2e^{2\tau}}{1+e^{2\tau}}\sin x. \tag{5.27}$$

Following the similar steps of the previous example, here we get the approximate solutions

$$u_j(x) = \left[\frac{2e^{2\tau}}{(1+e^{2\tau})^2} - \frac{he^{-2h}}{(1+h)^j}\left(\frac{1-(1+h)^j e^{-2jh}}{1-(1+h)e^{-2h}}\right)\right](1+e^{2\tau})\sin x, \quad j = 1,2,\ldots,10, \tag{5.28}$$

and the exact solution

$$u(t,x) = \left[\frac{2e^{2\tau}}{(1+e^{2\tau})^2} - (e^{-t} - e^{-2t})\right](1+e^{2\tau})\sin x, \quad j = 1,2,\ldots,10. \tag{5.29}$$

Putting $h = 0.1$ in both cases, approximate as well as exact solutions are obtained. These approximate solutions u_j, $j = 1,2,\ldots,10$, corresponding to χ_1 and χ_2 along with their respective exact solutions are shown in Tables 5.3 and 5.4, respectively.

From these observations we arrive at a conclusion similar to the one of the previous example.

Table 5.4. Approximate and exact solutions for the second example ($\chi = (2e^{2\tau}/(1+e^{2\tau}))\sin x$).

Approximate solution	Exact solution
$u_1(x) = \left(\frac{2e^{2q}}{(1+e^{2q})^2} - 0.074430\right)(1+e^{2q})\sin x$	$u(x,t_1) = \left(\frac{2e^{2q}}{(1+e^{2q})^2} - 0.086106\right)(1+e^{2q})\sin x$
$u_2(x) = \left(\frac{2e^{2q}}{(1+e^{2q})^2} - 0.128602\right)(1+e^{2q})\sin x$	$u(x,t_2) = \left(\frac{2e^{2q}}{(1+e^{2q})^2} - 0.148411\right)(1+e^{2q})\sin x$
$u_3(x) = \left(\frac{2e^{2q}}{(1+e^{2q})^2} - 0.166803\right)(1+e^{2q})\sin x$	$u(x,t_3) = \left(\frac{2e^{2q}}{(1+e^{2q})^2} - 0.192007\right)(1+e^{2q})\sin x$
$u_4(x) = \left(\frac{2e^{2q}}{(1+e^{2q})^2} - 0.192487\right)(1+e^{2q})\sin x$	$u(x,t_4) = \left(\frac{2e^{2q}}{(1+e^{2q})^2} - 0.220991\right)(1+e^{2q})\sin x$
$u_5(x) = \left(\frac{2e^{2q}}{(1+e^{2q})^2} - 0.208432\right)(1+e^{2q})\sin x$	$u(x,t_5) = \left(\frac{2e^{2q}}{(1+e^{2q})^2} - 0.238651\right)(1+e^{2q})\sin x$
$u_6(x) = \left(\frac{2e^{2q}}{(1+e^{2q})^2} - 0.216865\right)(1+e^{2q})\sin x$	$u(x,t_6) = \left(\frac{2e^{2q}}{(1+e^{2q})^2} - 0.247617\right)(1+e^{2q})\sin x$
$u_7(x) = \left(\frac{2e^{2q}}{(1+e^{2q})^2} - 0.219568\right)(1+e^{2q})\sin x$	$u(x,t_7) = \left(\frac{2e^{2q}}{(1+e^{2q})^2} - 0.249988\right)(1+e^{2q})\sin x$
$u_8(x) = \left(\frac{2e^{2q}}{(1+e^{2q})^2} - 0.217961\right)(1+e^{2q})\sin x$	$u(x,t_8) = \left(\frac{2e^{2q}}{(1+e^{2q})^2} - 0.247432\right)(1+e^{2q})\sin x$
$u_9(x) = \left(\frac{2e^{2q}}{(1+e^{2q})^2} - 0.213174\right)(1+e^{2q})\sin x$	$u(x,t_9) = \left(\frac{2e^{2q}}{(1+e^{2q})^2} - 0.241271\right)(1+e^{2q})\sin x$
$u_{10}(x) = \left(\frac{2e^{2q}}{(1+e^{2q})^2} - 0.206097\right)(1+e^{2q})\sin x$	$u(x,t_{10}) = \left(\frac{2e^{2q}}{(1+e^{2q})^2} - 0.232544\right)(1+e^{2q})\sin x$

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