A NOTE ON THE MINIMAL ESSENTIAL SET OF COINCIDENT POINTS FOR SET-VALUED MAPPINGS

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Motivated by the ideas of Kinoshita, we introduce the concept of minimal essential set of the coincident points for set-valued mappings, and we prove that there exists at least one minimal essential set and one essential component of the coincident points for set-valued mappings (satisfying some conditions).

1. Introduction

Kinoshita [3] introduced the notion of essential component to the set of fixed points and proved that for any continuous mapping of the Hilbert cube into itself, there exists at least one essential component of the set of its fixed points. The natural extension of fixed point theory is the study of coincident points. Tan et al. [5] introduced the concept of essential coincident points for multivalued mappings, they also discussed the generic stability of coincident points for multivalued mappings. However, as can be seen in Example 2.9, there exist no essential coincident points.

In this paper, motivated by the ideas of Kinoshita, we introduce the concept of minimal essential set of the coincident points for set-valued mappings, and we prove that there exists at least one minimal essential set of the coincident points for set-valued mappings (satisfying some conditions), and hence there exists at least one essential component of the coincident points.

2. Preliminaries

Let K be a subset of a metric space (E,d); for any $\delta > 0$, we denote by $O(K,\delta) = \{x \in E : d(x,K) < \delta\}$ the open neighborhood of K with radius δ in E.

Let X be a nonempty compact convex subset of a Banach space V. Let

$$S = \{f : X \longrightarrow 2^X \text{ upper semicontinuous and nonempty closed convex values}\}, \quad (2.1)$$

where 2^X denotes the family of all nonempty subsets of X.

For any $f, f' \in S$, define

$$\rho_1(f,f') = \sup_{x\in X} H(f(x), f'(x)), \quad (2.2)$$

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where H is the Hausdorff metric defined on X. Clearly, (S,ρ_1) is a complete metric space. Let

$$Y=\left\{(f,g)\in S\times S: f,g\in S,\ \text{for any } x\in \mathrm{Bd}X,\ (f(x)-g(x))\cap\left(\bigcup_{\lambda>0}\lambda(X-x)\right)\neq\varnothing\right\}, \tag{2.3}$$

where $\mathrm{Bd}X$ denotes the boundary of X, then (Y,ρ) is a complete metric space, where $\rho((f,g),(f',g'))=\rho_1(f,f')+\rho_1(g,g')$.

Theorem 2.1. *$Y\subset S\times S$ is a closed subset.*

Proof. Let $y_\alpha=(f_\alpha,g_\alpha)\in Y$ with $y_\alpha\to y=(f,g)\in S\times S$. Since $(f_\alpha,g_\alpha)\in Y$, for any $x\in \mathrm{Bd}X$, one has

$$(f_\alpha(x)-g_\alpha(x))\cap\left(\bigcup_{\lambda>0}\lambda(X-x)\right)\neq\varnothing. \tag{2.4}$$

Then there exist $u_\alpha\in f_\alpha(x)$ and $v_\alpha\in g_\alpha(x)$ such that

$$u_\alpha-v_\alpha\in\bigcup_{\lambda>0}\lambda(X-x). \tag{2.5}$$

Note that $f_\alpha\to f$, $g_\alpha\to g$, X is compact, $\{u_\alpha\}$ has a cluster point $u_0\in f(x)$, and $\{v_\alpha\}$ has a cluster point $v_0\in g(x)$. Without loss of generality, we may assume that $u_\alpha\to u_0\in f(x)$, $v_\alpha\to v_0\in g(x)$.

(1) If there exists infinite α such that $u_\alpha=v_\alpha$, then $u_0=v_0$, and hence $u_0-v_0\in\bigcup_{\lambda>0}\lambda(X-x)$.

(2) If there exists infinite α such that $u_\alpha\neq v_\alpha$, then there exists $k>0$ such that

$$u_\alpha-v_\alpha\in\bigcup_{0<\lambda<k}\lambda(X-x). \tag{2.6}$$

Hence there exists λ_α with $0<\lambda_\alpha<k$ such that $u_\alpha-v_\alpha\in\lambda_\alpha(X-x)$. So there exists $z_\alpha\in X$ such that $u_\alpha-v_\alpha=\lambda_\alpha(z_\alpha-x)$. Note that X is compact, $\{z_\alpha\}$ has a cluster point $z_0\in X$, we may assume that $z_\alpha\to z_0$. And since $0<\lambda_\alpha<k$, we may assume that $\lambda_\alpha\to\lambda_0(\geq 0)$, hence

$$u_0-v_0=\lambda_0(z_0-x). \tag{2.7}$$

If $\lambda_0=0$, then $u_0-v_0=0\in\bigcup_{\lambda>0}\lambda(X-x)$. If $\lambda_0\neq 0$, then $u_0-v_0=\lambda_0(z_0-x)\in\lambda_0(X-x)\subset\bigcup_{\lambda>0}\lambda(X-x)$. Hence, for any $x\in\mathrm{Bd}X$,

$$(f(x)-g(x))\cap\left(\bigcup_{\lambda>0}\lambda(X-x)\right)\neq\varnothing. \tag{2.8}$$

Therefore $Y\subset S\times S$ is a closed subset. □

For any $y=(f,g)\in Y$, we denote by $CC(y)=\{x\in X: f(x)\cap g(x)\neq\varnothing\}$ the set of coincident points of the set-valued mappings f and g, by [2, Theorem 10], $CC(y)\neq\varnothing$, thus $y\to CC(y)$ indeed defines a set-valued mapping of coincident points from Y to X and we have the following theorem.

THEOREM 2.2. *The mapping $CC : Y \to 2^X$ is upper semicontinuous with nonempty compact values.*

Proof. For any $y = (f,g) \in Y$, we need to prove that $CC(y) \subset X$ is compact. Let a sequence $\{x_\alpha\} \subset CC(y)$ and $x_\alpha \to x_0 \in X$. Since $x_\alpha \in CC(y)$, we have $f(x_\alpha) \cap g(x_\alpha) \neq \emptyset$.

Suppose that $f(x_0) \cap g(x_0) = \emptyset$, then there exists $\delta > 0$ such that

$$O(f(x_0),\delta) \cap O(g(x_0),\delta) = \emptyset. \tag{2.9}$$

By upper semicontinuities of f and g, and since $x_\alpha \to x_0$, there exists α_0 such that for any $\alpha > \alpha_0$, $f(x_\alpha) \subset O(f(x_0),\delta)$ and $g(x_\alpha) \subset O(f(x_0),\delta)$, then $f(x_\alpha) \cap g(x_\alpha) = \emptyset$, which contradicts the fact that $f(x_\alpha) \cap g(x_\alpha) \neq \emptyset$, hence $x_0 \in CC(y)$ and hence $CC(y)$ is compact.

Since X is compact, we want to prove that the mapping CC is upper semicontinuous, we only need to prove that the Graph CC of CC is closed:

$$\operatorname{Graph} CC = \{(y,x) \in Y \times X : x \in CC(y),\ y \in Y\}. \tag{2.10}$$

Let a sequence $\{(y_\alpha,x_\alpha)\} \subset \operatorname{Graph} CC$ and $(y_\alpha,x_\alpha) \to (y_0,x_0) \in Y \times X$. Denote $y_\alpha = (f_\alpha,g_\alpha)$, $y_0 = (f_0,g_0)$, then $x_\alpha \in CC(y_\alpha)$ and $f_\alpha(x_\alpha) \cap g_\alpha(x_\alpha) \neq \emptyset$.

Suppose that $f_0(x_0) \cap g_0(x_0) = \emptyset$, then there exists $\delta^* > 0$ such that

$$O(f_0(x_0),\delta^*) \cap O(g_0(x_0),\delta^*) = \emptyset. \tag{2.11}$$

Since $f_\alpha \to f_0$, $g_\alpha \to g_0$, $x_\alpha \to x_0$, and f_0, g_0 are upper semicontinuous, there exists α^* such that

$$\begin{aligned} f_\alpha(x_\alpha) &\subset O\left(f_0(x_\alpha),\frac{\delta^*}{2}\right) \subset O(f_0(x_0),\delta^*), \quad \forall \alpha > \alpha^*, \\ g_\alpha(x_\alpha) &\subset O\left(g_0(x_\alpha),\frac{\delta^*}{2}\right) \subset O(g_0(x_0),\delta^*), \quad \forall \alpha > \alpha^*. \end{aligned} \tag{2.12}$$

Hence $f_\alpha(x_\alpha) \cap g_\alpha(x_\alpha) = \emptyset$, which contradicts the fact that $f_\alpha(x_\alpha) \cap g_\alpha(x_\alpha) \neq \emptyset$. So the mapping CC is upper semicontinuous with nonempty compact values. □

For each $y \in Y$, the component of a point $x \in CC(y)$ is the union of all connected subsets of $CC(y)$ which contain the point x, see [1, page 356], components are connected closed subsets of $CC(y)$ and are also connected compact. It is easy to see that the components of two distinct points of $CC(y)$ either coincide or are disjoint, so that all components constitute a decomposition of $CC(y)$ into connected pairwise disjoint compact subsets, that is,

$$CC(y) = \bigcup_{\alpha \in \Lambda} C_\alpha(y), \tag{2.13}$$

where Λ is an index set, for any $\alpha \in \Lambda$, $C_\alpha(y)$ is a nonempty connected compact subset and for any $\alpha, \beta \in \Lambda$ $(\alpha \neq \beta)$, $C_\alpha(y) \cap C_\beta(y) = \emptyset$.

Definition 2.3. For $y \in Y$, $CC(y) = \bigcup_{\alpha\in\Lambda} C_\alpha(y)$, $C_\alpha(y)$ is called an essential component if for each open set O containing $C_\alpha(y)$, there exists $\delta > 0$ such that for any $y' \in Y$ with $\rho(y,y') < \delta$, $CC(y') \cap O \neq \varnothing$.

Definition 2.4. For $y \in Y$, $e(y) \subset CC(y)$ is a nonempty closed set, $e(y)$ is called an essential set of $CC(y)$ (with respect to Y) if for any open set U with $U \supset e(y)$, there is $\delta > 0$ such that for any $y' \in Y$ with $\rho(y,y') < \delta$, $CC(y') \cap U \neq \varnothing$.

Definition 2.5. For $y \in Y$, $m(y) \subset CC(y)$ is an essential set, $m(y)$ is called a minimal essential set of $CC(y)$ (with respect to Y) if $m(y)$ is a minimal element of the family of essential sets of $CC(y)$ ordered by set inclusion.

Remark 2.6. If $e_1(y) \subset CC(y)$ is an essential set of $CC(y)$ (with respect to Y), $e_2(y) \subset CC(y)$ is closed, and $e_1(y) \subset e_2(y)$, then $e_2(y)$ is also an essential set of $CC(y)$.

Remark 2.7. If $x \in CC(y)$ is an essential coincident point (see [5]) of $CC(y)$, then $\{x\}$ is an essential set of $CC(y)$; $e(y) \subset CC(y)$ is an essential set and $e(y) = \{x\}$, then $x \in CC(y)$ is an essential coincident point of $CC(y)$.

Remark 2.8. If $A \subset CC(y)$ is closed, $x \in A \subset CC(y)$, and x is an essential coincident point of $CC(y)$, then A is an essential set and $\{x\}$ is a minimal essential set of $CC(y)$.

Example 2.9. Let $X = [0,1]$, for any $x \in X$, $f(x) = [0,x]$, $g(x) = [x,1]$, then $y = (f,g) \in Y$ and $CC(y) = \{x \in [0,1] : f(x) \cap g(x) \neq \varnothing\} = [0,1]$. But x_0 is not an essential coincident point for any $x_0 \in CC(y)$. If $x_0 \in (0,1)$, for all $\varepsilon > 0$, take $\delta > 0$ ($\delta < \varepsilon/2$) such that $O(x_0,\delta) = (x_0-\delta, x_0+\delta) \subset [0,1]$.

Define the set-valued mappings $f^\varepsilon, g^\varepsilon : X \to 2^X$ by

$$g^\varepsilon(x) = g(x),$$

$$f^\varepsilon(x) = \begin{cases} [0,x], & x \in [0, x_0-\delta], \\ \left[0, \left(1-\dfrac{\varepsilon}{2\delta}\right)x + \dfrac{\varepsilon}{2\delta}(x_0-\delta)\right], & x \in [x_0-\delta, x_0], \\ \left[0, \left(1+\dfrac{\varepsilon}{2\delta}\right)x - \dfrac{\varepsilon}{2\delta}(x_0+\delta)\right], & x \in (x_0, x_0+\delta], \\ [0,x], & x \in (x_0+\delta, 1], \end{cases} \tag{2.14}$$

then $y^\varepsilon = (f^\varepsilon, g^\varepsilon) \in Y$ and $\rho(y,y^\varepsilon) < \varepsilon$, but $CC(y^\varepsilon) \cap O(x_0,\delta) = \varnothing$, hence $x_0 \in (0,1)$ is not an essential coincident point.

Similarly, if $x_0 = 1$, for all $\varepsilon : 0 < \varepsilon < 1/2$, take $\delta > 0$ ($\delta < \varepsilon/2$) such that $(1-\delta, 1] \subset (0,1]$.

Define the set-valued mappings $f^\varepsilon, g^\varepsilon : X \to 2^X$ by

$$g^\varepsilon(x) = g(x),$$

$$f^\varepsilon(x) = \begin{cases} [0,x] & \text{if } x \in [0, 1-\delta], \\ \left[0, \left(1-\dfrac{\varepsilon}{2\delta}\right)x + \dfrac{\varepsilon}{2\delta}(1-\delta)\right] & \text{if } x \in (1-\delta, 1]. \end{cases} \tag{2.15}$$

If $x_0 = 0$, for all $\varepsilon > 0$ ($< 1/2$), take $\delta > 0$ ($\delta < \varepsilon/2$) such that $[0,\delta) \subset [0,1)$.

Define the set-valued mappings $f^\varepsilon, g^\varepsilon : X \to 2^X$ by

$$f^\varepsilon(x) = f(x),$$
$$g^\varepsilon(x) = \begin{cases} \left[\left(1 - \dfrac{\varepsilon}{2\delta}\right)x + \dfrac{\varepsilon}{2}, 1\right] & \text{if } x \in [0,\delta], \\ [x,1] & \text{if } x \in (\delta, 1]. \end{cases} \tag{2.16}$$

Hence, for any $x_0 \in CC(y) = [0,1]$, x_0 is not an essential coincident point.

3. The minimal essential set of coincident points

By Zorn lemma, we obtain the following theorem.

Theorem 3.1. *For any $y \in Y$, there exists at least one minimal essential set of $CC(y)$.*

Proof. By Theorem 2.2, the map $CC : Y \to 2^X$ is upper semicontinuous and $CC(y)$ is compact for any $y \in Y$, then $CC(y)$ is an essential set.

Let $E(y)$ denote the family of all essential sets of $CC(y)$ ordered by set inclusion. Let $\{e_\alpha(y)\}_{\alpha\in\Gamma}$ be a decreasing chain of $E(y)$, then $\lim e_\alpha(y) = \bigcap_{\alpha\in\Gamma} e_\alpha(y) \neq \emptyset$ and is compact. Denoting $e(y) = \lim e_\alpha(y)$, we need to prove that $e(y)$ is the lower bound of the chain $\{e_\alpha\}_{\alpha\in\Gamma}$, that is, $e(y) \in E(y)$. Since $e_\alpha(y)$ is compact, by [4, page 43], $H(e_\alpha(y), e(y)) \to 0$, where H is the Hausdorff metric defined on X, hence for any open set O with $O \supset e(y)$, there is $\alpha_1 \in \Gamma$ such that $e_\alpha(y) \subset O$ for any $\alpha > \alpha_1$. Since $e_\alpha(y)$ is an essential set of $CC(y)$, there exists $\delta > 0$ such that $CC(y') \cap O \neq \emptyset$ for any $y' \in Y$ with $\rho(y,y') < \delta$, then $e(y)$ is an essential set of $CC(y)$, $e(y)$ is the lower bound of the chain $\{e_\alpha\}_{\alpha\in\Gamma}$. Therefore, by Zorn lemma, $E(y)$ has a minimal element and this minimal element is a minimal essential set of $CC(y)$. □

Theorem 3.2. *For any $y \in Y$, the minimal essential set of $CC(y)$ is connected.*

Proof. Let $m(y)$ be a minimal essential set of $CC(y)$. Suppose that $m(y)$ was not connected, then there exist two nonempty closed sets $C_1(y)$, $C_2(y)$ and two open U_1, U_2 such that $C_1(y) \subset U_1$, $C_2(y) \subset U_2$ and $m(y) = C_1(y) \cup C_2(y)$, $U_1 \cap U_2 = \emptyset$. Because $m(y)$ is a minimal essential set of $CC(y)$, $C_1(y)$ and $C_2(y)$ are not essential sets. Since $C_1(y)$ and $C_2(y)$ are compact, there exist two open sets is V_1 and V_2 which satisfy

$$C_1(y) \subset V_1 \subset \bar{V}_1 \subset U_1, \qquad C_2(y) \subset V_2 \subset \bar{V}_2 \subset U_2, \tag{3.1}$$

where $\bar{V}_i$ denotes the closure of V_i, $i = 1,2$.

For any $\delta > 0$, there exist $y_1 = (f_1, g_1), y_2 = (f_2, g_2) \in Y$ with $\rho(y, y_1) < \delta$, $\rho(y, y_2) < \delta$ such that

$$CC(y_1) \cap V_1 = \emptyset, \qquad CC(y_2) \cap V_2 = \emptyset. \tag{3.2}$$

Define two set-valued maps $f^*: X \to 2^X$ and $g^*: X \to 2^X$ as follows:

$$f^*(x) = \begin{cases} f_1(x) & \text{if } x \in \bar{V}_1, \\ f_2(x) & \text{if } x \in \bar{V}_2, \\ \xi(x) f_1(x) + \eta(x) f_2(x) & \text{if } x \in X \setminus \bar{V}_1 \cup \bar{V}_2, \end{cases}$$
$$g^*(x) = \begin{cases} g_1(x) & \text{if } x \in \bar{V}_1, \\ g_2(x) & \text{if } x \in \bar{V}_2, \\ \xi(x) g_1(x) + \eta(x) g_2(x) & \text{if } x \in X \setminus \bar{V}_1 \cup \bar{V}_2, \end{cases} \tag{3.3}$$

where

$$\xi(x) = \frac{d(x, \bar{V}_2)}{d(x, \bar{V}_2) + d(x, \bar{V}_1)}, \qquad \eta(x) = \frac{d(x, \bar{V}_1)}{d(x, \bar{V}_2) + d(x, \bar{V}_1)}. \tag{3.4}$$

It is easy to see that $y^* = (f^*, g^*) \in Y$, then $CC(y^*) \neq \emptyset$ and $CC(y^*) \cap (V_1 \cup V_2) = \emptyset$.

Since $\rho(y, y^*) = \rho_1(f, f^*) + \rho_1(g, g^*)$, by [6, Lemma 3.1], we have $\rho(y, y^*) < \delta$, but $m(y) \subset C_1(y) \cup C_2(y) \subset V_1 \cup V_2$, by Definition 2.4, $m(y)$ is not an essential set of $CC(y)$, which contradicts the fact that $m(y)$ is a minimal essential set, hence $m(y)$ is connected and the proof is complete. □

By Theorems 3.1 and 3.2, we have the following corollaries.

Corollary 3.3. *For any $y \in Y$, there exists at least one connected minimal essential set of $CC(y)$.*

Corollary 3.4. *For any $y \in Y$, there exists at least one essential component of $CC(y)$.*

Proof. For any $y \in Y$, by Corollary 3.3, there exists at least one connected minimal essential set $m(y)$ of $CC(y)$, since $m(y)$ is connected, there exists a component $M(y)$ of $CC(y)$ such that $m(y) \subset M(y)$, by Definition 2.3, $M(y)$ is an essential component of $CC(y)$. □

Remark 3.5. If $g(x) = x$ for any $x \in X$, then for any $f \in S$ and $x \in \mathrm{Bd}\, X$,

$$(f(x) - g(x)) \cap \left(\bigcup_{\lambda > 0} \lambda(X - x) \right) = (f(x) - x) \cap \left(\bigcup_{\lambda > 0} \lambda(X - x) \right) \neq \emptyset. \tag{3.5}$$

Therefore $y = (f, g) \in Y$ and $CC(y) = F(f)$, where $F(f)$ denotes the set of fixed points of f.

By Corollary 3.4, we have the following corollary.

Corollary 3.6. *For any $f \in S$, there is at least one essential component of $F(f)$.*

Remark 3.7. Corollary 3.6 is a generalization of [3, Theorem 3].

Example 3.8. Let $X = [-1,1]$,

$$f(x) = \begin{cases} \{0\}, & -1 \le x < 0, \\ [0,x], & 0 \le x \le 1, \end{cases}$$
$$g(x) = \begin{cases} [x,-1], & -1 \le x < 0, \\ [0,x-1], & 0 \le x \le 1. \end{cases} \tag{3.6}$$

Then $y = (f,g) \in Y$ and $CC(y) = \{x \in [0,1] : f(x) \cap g(x) \neq \emptyset\} = [0,1] \subset [-1,1]$.

Suppose that $[0,1]$ is not an essential set of $CC(y)$, then there exists an open set U with $U \supset [0,1]$ (Let $U = (-\varepsilon,1]$, $0 < \varepsilon < 1$), for all $\delta > 0$, there exists $y^\delta \in Y$ with $H(y,y^\delta) < \delta$ such that $CC(y^\delta) \cap U = \emptyset$, that is, $CC(y^\delta) \subset [-1,-\varepsilon]$.

Take $\delta = \varepsilon/4$, for any $y^0 = (f^0,g^0) \in Y$ with $\rho_1(f,f^0) < \delta/2$ and $\rho_1(g,g^0) < \delta/2$, one has $\rho(y,y^0) < \delta$, and for all $x \in [-1,-\varepsilon]$, $H(f(x),f^0(x)) < \rho_1(f,f^0) < \delta/2$, $H(g(x),g^0(x)) < \rho_1(g,g^0) < \delta/2$, then $f^0(x) \subset (-\delta/2,\delta/2)$, $g^0(x) \subset [-\varepsilon+\delta/2,-1] = [(-7/2)\delta,-1]$, and $[-\delta/2,\delta/2] \cap [(-7/2)\delta,-1] = \emptyset$, hence $f^0(x) \cap g^0(x) = \emptyset$ for any $x \in [-1,-\varepsilon]$, $CC(y^0) \subset (-\varepsilon,1]$ which contradicts the fact that $CC(y^0) \subset [-1,-\varepsilon]$. Therefore, $[0,1]$ is an essential set and hence $[0,1]$ is a minimal essential set.

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