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Research Article

On Zeros of Self-Reciprocal Random Algebraic Polynomials

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This paper provides an asymptotic estimate for the expected number of level crossings of a trigonometric polynomial $T_N(\theta) = \sum_{j=0}^{N-1}\{\alpha_{N-j}\cos(j+1/2)\theta + \beta_{N-j}\sin(j+1/2)\theta\}$, where α_j and β_j, $j = 0,1,2,\dots,N-1$, are sequences of independent identically distributed normal standard random variables. This type of random polynomial is produced in the study of random algebraic polynomials with complex variables and complex random coefficients, with a self-reciprocal property. We establish the relation between this type of random algebraic polynomials and the above random trigonometric polynomials, and we show that the required level crossings have the functionality form of $\cos(N+\theta/2)$. We also discuss the relationship which exists and can be explored further between our random polynomials and random matrix theory.

1. Introduction

Let $\{\alpha_j\}_{j=1}^{n-1}$ and $\{\beta_j\}_{j=1}^{n-1}$ be sequences of independently normally distributed random variables with means zero and variances σ^2. For a sequence of complex numbers $\eta_j = \alpha_j + i\beta_j$, $j = 1,2,\dots,n-1$, with $\eta_n \equiv \eta_0 \equiv 1$, we define a (complex) random algebraic polynomial as

$$P_n(z) = \sum_{j=0}^{n}\eta_j z^j. \tag{1.1}$$

Although there have been many results concerning real and complex roots of $P_n(z)$, most of them assume identical distributions for α_j's and β_j's, and therefore η_j's. These results, for the real case, are initiated by fundamental works of Kac [3] and Rice [1, 2], and recently they are re-examined in an interesting work by Wilkins [4]. The study of the mathematical behavior of $P_n(z)$, later generalized to the complex case first described by Ibragimov and Zeitouni [5] and then by Farahmand and Jahangiri [6], is reviewed in [7]. For physical applications and developments, we refer interested readers to [8] and the references therein. Further for the case of identical coefficients, Farahmand and Grigorash [9] study a case of nonzero means.

However, in the study of random matrix theory, it turns out that a special form of $P_n(z)$, known as self-reciprocal random algebraic polynomial, is of interest in which the polynomial required, for all n and z, satisfies the relation $P_n(z) = z^n P_n(1/z)$. This yields a polynomial where $\eta_n \equiv \eta_0 \equiv 1$, and η_{n-j} is the complex conjugate of η_j, $j = 1,2,\dots,n-1$. The assumption of $\eta_n \equiv \eta_0 \equiv 1$ is motivated by the requirement that in the random matrix theory we are interested in polynomials whose (complex) zeros are located in the unit circle. Our above form of $P_n(z)$ satisfies this requirement when $\eta_{n-1}z^{n-1} + \eta_{n-2}z^{n-2} + \cdots + \eta_1 z \equiv 0$. The properties of zeros of reciprocal polynomials with deterministic coefficients are also discussed by Lakatos and Losonczi [10].

2. Random trigonometric polynomials

With simple transformation, for $z = r\exp(i\theta)$, we can rewrite $P_n(z)$ in (1.1) as

$$\begin{aligned} e^{in\theta/2}p_n(e^{i\theta}) = {} & 2\cos\left(\frac{n\theta}{2}\right) + 2\alpha_1\cos\left(\frac{n-2}{2}\right)\theta + 2\beta_1\sin\left(\frac{n-2}{2}\right)\theta \\ & + 2\alpha_2\cos\left(\frac{n-4}{2}\right)\theta + 2\beta_2\sin\left(\frac{n-4}{2}\right)\theta \\ & \vdots \\ & + f_n(\theta), \end{aligned} \tag{2.1}$$

where

$$F_n(\theta) = \begin{cases} 2\alpha_{(n/2-1)}\cos\theta + 2\beta_{(n/2-1)}\sin\theta + \alpha_{n/2} + i\beta_{n/2} & \text{for } n \text{ even}, \\ 2\alpha_{(n-1)/2}\cos\theta + 2\beta_{(n-1)/2}\sin\theta & \text{for } n \text{ odd}. \end{cases} \tag{2.2}$$

For the above regrouping of terms, we used the self-reciprocating property of $\eta_j \equiv \overline{\eta_{n-j}}$. Indeed, since for n even not all the coefficients η_j in (1.1) can have a matching conjugate, the main interest, as far as random matrix theory is concerned, is for the case of n odd. The latter case, therefore, would be our main interest. However, in order to be complete, at this stage we present $P_n(z)$ for both n odd and n even. To this end, for n odd we have

$$P_n(\theta) = 2\sum_{j=1}^{(n-1)/2}\left\{\alpha_j\cos\left(\frac{n-2j}{2}\right)\theta + \beta_j\sin\left(\frac{n-2j}{2}\right)\theta\right\} + 2\cos\left(\frac{n\theta}{2}\right), \tag{2.3}$$

and for n even

$$P_n(\theta) = 2\sum_{j=1}^{n/2-1}\left\{\alpha_j\cos\left(\frac{n-2j}{2}\right)\theta + \beta_j\sin\left(\frac{n-2j}{2}\right)\theta\right\} + 2\cos\left(\frac{n\theta}{2}\right) + \alpha_{n/2} + i\beta_{n/2}. \tag{2.4}$$

Now from (2.3), that is, for n odd, the polynomial of interest has the form

$$P_n(\theta) = T_N(\theta) + \cos(N + \theta/2), \tag{2.5}$$

where $N = (n-1)/2$ and

$$T_N(\theta) = \sum_{j=0}^{N-1} \{\alpha_{N-j}\cos(j+1/2)\theta + \beta_{N-j}\sin(j+1/2)\theta\}. \tag{2.6}$$

The classical form of random trigonometric polynomials is defined as

$$Q_N(\theta) = \sum_{j=1}^{N} \alpha_j \cos j\theta. \tag{2.7}$$

The study of these types of random polynomials is initiated by Dunnage [11]. Although Dunnage studied the actual number of real zeros, his work showed that, for N large, the expected number of real zeros of $Q_N(\theta)$ is asymptotic to $2N/\sqrt{3}$. This result is later generalized to the case of a constant level crossing in [12] and to the case of nonstandard normal in [13, 14]. The other results concerning $Q_N(\theta)$ can be found in [7]. Further, as reported by Bharucha-Reid and Sambandham [15], Das [16] considered the expected number of zeros of a random trigonometric polynomial similar to (2.6). These types of random trigonometric polynomials, as we will see, have the advantage of being stationary with respect to θ. However, since we are interested in the expected number of zeros of $P_n(z)$ given in (2.3), we need to generalize the result to the number of level crossings of $T_N(\theta)$.

In what follows, we therefore study the number of level crossings of $T_N(\theta)$ with $\cos(N+\theta/2)$. As this level is a function of θ, it can be seen as a moving level crossing case, where there is no known formula for its expected number of crossings with $T_N(\theta)$. We will develop this in the following section. Denote this number in the interval (a,b) by $\mathcal{N}(a,b)$ and its expected value by $E\mathcal{N}(a,b)$. We prove the following theorem.

Theorem 2.1. *With the above assumption on the distributions of α_js and β_js and for all sufficiently large N,*

$$E\mathcal{N}(0,2\pi) \sim \frac{2N}{\sqrt{3}}. \tag{2.8}$$

3. Expected number of crossings

In what follows, we generalize the known result from constant level crossing to this moving level for the special form of $T_N(\theta)$ given in (2.6). We use a formula known as the Kac-Rice formula which is originally derived for the expected number of real zeros (axes crossings) of a random algebraic polynomial. It is known (see, e.g., [7, page 12]) that the expected number of real zeros of polynomial $T_N(\theta)$ in (2.3) in the interval (a,b) is given by

$$E\mathcal{N}(a,b) = \int_a^b d\theta \int_{-\infty}^{\infty} |y|\varphi(0,y)\,dy, \tag{3.1}$$

where $\varphi(x_1,x_2)$ is the joint probability density function of $T_N(\theta)+\cos(N+\theta/2)$ and its derivative $T'_N(\theta)-(1/2)\sin(N+\theta/2)$. Let

$$A_N^2(\theta)\equiv A^2=\operatorname{var}\left(T_N(\theta)\right),\qquad B_N^2(\theta)\equiv B^2=\operatorname{var}\left(T'_N(\theta)\right),\tag{3.2}$$

and let $C_N(\theta)\equiv C$ be the covariance of $T_N(\theta)$ and $T'_N(\theta)$. Since α_j's and β_j's are independent in themselves and from each other, we can easily show that $C=0$. Using the assumption of normality of the coefficients of the polynomial, we therefore obtain the required joint probability density function as

$$\varphi(x_1,x_2)=\frac{1}{2\pi AB}\exp\left(-\frac{B^2\left(x_1-\cos(N+\theta/2)\right)^2+A^2\left(x_2+\left(\sin(N+\theta/2)\right)/2\right)^2}{2A^2B^2}\right).\tag{3.3}$$

This will enable us to evaluate a part of the Kac-Rice formula given in (3.1) as

$$\begin{aligned}
&\int_{-\infty}^{\infty}|y|\varphi(0,y)\,dy\\
&\quad=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|y|}{AB}\exp\left(\frac{-B^2\cos^2(N+\theta/2)+A^2\left(y+\left(\sin(N+\theta/2)\right)/2\right)^2}{2A^2B^2}\right)dy\\
&\quad=\frac{1}{2\pi AB}\exp\left(-\frac{\cos^2(N+\theta/2)}{2A^2}\right)\int_{-\infty}^{\infty}|y|\exp\left(-\frac{y+\left(\sin(N+\theta/2)/2\right)^2}{2B^2}\right)dy\\
&\quad=\frac{1}{2\pi AB}\exp\left(-\frac{\cos^2(N+\theta/2)}{2A^2}-\frac{\sin^2(N+\theta/2)}{8B^2}\right)\\
&\qquad\times\int_{-\infty}^{\infty}|y|\exp\left(-\frac{\sin(N+\theta/2)}{2B^2}y-\frac{y^2}{2B^2}\right)dy.
\end{aligned}\tag{3.4}$$

Now we let $t=y/(B\sqrt{2})$, which enables us to proceed with the above integration as

$$\begin{aligned}
\int_{-\infty}^{\infty}|y|\varphi(0,y)\,dy=\frac{B}{\pi A}&\exp\left(-\frac{\cos^2(N+\theta/2)}{2A^2}-\frac{\sin^2(N+\theta/2)}{8B^2}\right)\\
&\times\int_{-\infty}^{\infty}|t|\exp\left(-\frac{\sin(N+\theta)}{\sqrt{2}B}t-t^2\right)dt.
\end{aligned}\tag{3.5}$$

Now we let $\lambda(N,\theta)\equiv\lambda=-(\sin(N+\theta/2))/\sqrt{2}B$ and $J(\lambda)=\int_0^{\infty}t\exp\left(\lambda t-t^2\right)dt$. Then the last integral that appears in (3.5) can be written as

$$\int_{-\infty}^{\infty}|t|\exp\left(\lambda t-t^2\right)dt=\int_0^{\infty}t\left(e^{\lambda t}+e^{-\lambda t}\right)e^{-t^2}dt=J(\lambda)+J(-\lambda).\tag{3.6}$$

However, from the above definition of $J(\lambda)$, it is easy to see that

$$J(\lambda) = -\frac{1}{2}\int_0^\infty e^{\lambda t} d(e^{-t^2}) = \frac{1}{2} + \frac{\lambda}{2}\exp\left(\frac{\lambda^2}{4}\right)\int_0^\infty \exp\left(-\left(t-\frac{\lambda}{2}\right)^2\right)dt$$
$$= \frac{1}{2} + \frac{\lambda\sqrt{\pi}}{4}\exp\left(\frac{\lambda^2}{4}\right) + \frac{\lambda}{2}\exp\left(\frac{\lambda^2}{4}\right)\operatorname{erf}\left(\frac{\lambda}{2}\right), \tag{3.7}$$

where $\operatorname{erf}(x) = \int_0^x \exp(-u^2)du$. In the derivation of (3.7), use has been made of the following:

$$\int_0^\infty \exp\left(-\left(t-\frac{\lambda}{2}\right)^2\right) = \int_{-\lambda/2}^0 e^{-u^2}\,du + \int_0^\infty e^{-u^2}\,du = \frac{\sqrt{\pi}}{2} + \operatorname{erf}\left(\frac{\lambda}{2}\right). \tag{3.8}$$

Therefore, from (3.6) and (3.7) we have

$$\int_{-\infty}^\infty |t|\exp(\lambda t - t^2)\,dt = 1 + |\lambda|\exp\left(\frac{\lambda^2}{4}\right)\operatorname{erf}\left(\frac{|\lambda|}{2}\right). \tag{3.9}$$

This together with (3.5) gives

$$\int_{-\infty}^\infty |y|\varphi(0,y)\,dy = \frac{B}{\pi A}\exp\left\{-\frac{\cos^2(N+\theta/2)}{2A^2} - \frac{\sin^2(N+\theta/2)}{8B^2}\right\}$$
$$+ \frac{|\sin(N+\theta/2)|}{\pi A\sqrt{2}}\exp\left\{-\frac{\cos^2(N+\theta/2)}{2A^2}\right\}\operatorname{erf}\left\{\left|\frac{\sin(N+\theta/2)}{2\sqrt{2}B}\right|\right\}. \tag{3.10}$$

Therefore, $E\mathcal{N}(a,b)$ can be obtained by integrating (3.10) with respect to θ in the interval (a,b). To this end, we need the following characteristics of the polynomial $T_N(\theta)$ and its derivative $T_N'(\theta)$. As mentioned above, $\operatorname{cov}(T_N(\theta), T_N'(\theta)) = 0$. Also,

$$A^2 = \operatorname{var}(P_N(\theta)) = \operatorname{var}(T_N(\theta)) = \sigma^2 N,$$
$$B^2 = \operatorname{var}(P_N'(\theta)) = \operatorname{var}(T_N'(\theta)) = \sigma^2\sum_{j=0}^{N-1}(j+1/2)^2$$
$$= \sigma^2\left\{\frac{N(N-1)(2N-1)}{6} + \frac{N(N-1)}{2} + \frac{N}{4}\right\}$$
$$= \frac{N\sigma^2}{3}\left(N^2 - \frac{1}{4}\right). \tag{3.11}$$

Therefore, from the Kac-Rice formula, (3.1), (3.6), (3.11), and for all sufficiently large N, we obtain

$$E\mathcal{N}(0,2\pi) = \int_0^{2\pi}\left\{\frac{B}{\pi A}\exp\left(-\frac{\cos^2(N+\theta/2)}{2A^2}-\frac{\sin^2(N+\theta/2)}{8B^2}\right)\right.$$
$$\left.+\frac{|\sin(N+\theta)|}{\pi A\sqrt{2}}\exp\left(-\frac{\cos^2(N+\theta/2)}{2A^2}\right)\operatorname{erf}\left(\left|\frac{\sin(N+\theta/2)}{2\sqrt{2}B}\right|\right)\right\}d\theta$$
$$\sim\frac{2N}{\sqrt{3}}. \tag{3.12}$$

Note that the first equality for $E\mathcal{N}(0,2\pi)$ in the above formula as well as (3.11) is valid for all N, which is a much stronger result than the one we stated here. However, the gain in stating such an untidy result does not justify the advantage of the generalization.

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