Research Article
Continuous Interpolation of Solution Sets of Lipschitzian Quantum Stochastic Differential Inclusions

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Given any finite set of trajectories of a Lipschitzian quantum stochastic differential inclusion (QSDI), there exists a continuous selection from the complex-valued multifunction associated with the solution set of the inclusion, interpolating the matrix elements of the given trajectories. Furthermore, the difference of any two of such solutions is bounded in the seminorm of the locally convex space of solutions.

1. Introduction

Establishment of continuous selections from the solution set multifunctions of differential inclusions defined on finite dimensional Euclidean spaces and their applications have been considered by many authors (see, e.g., Aubin and Cellina [1], Repovs and Semenov [2], Smirnov [3] and the references they contain). However, in the context of quantum stochastic differential inclusions (QSDI), research in these subjects has not enjoyed a comparable attention. In addition, theoretical and numerical aspects of QSDI have not enjoyed significant development in comparison with the classical cases although there are some recent results along these directions (see, e.g., [4–10]). This situation remains in spite of the numerous practical problems in quantum dynamical systems, quantum open systems, quantum measurement theory, quantum optics, and quantum stochastic control theory for which methods of quantum stochastic inclusions are applicable. In particular, it is well known that discontinuous quantum stochastic differential equations can be numerically and theoretically treated by reformulating them as regularized inclusions (see [8–13]).

Our present research effort in this field is motivated by the need to further explore the properties of solution spaces of quantum stochastic differential inclusions. The present paper is, therefore, concerned with the establishment of the existence of continuous selections from the complex valued multifunctions associated with the solution sets of QSDI interpolating the matrix elements of a given finite set of trajectories that start from distinct points. This work is a continuation of our work in [4] extending the case of a single trajectory to the case of a finite set of trajectories. In addition, we show that the difference of any two solutions of QSDI (1.1) below, that correspond to two distinct initial values is bounded in the seminorm of the locally convex space of solutions.

In what follows, we will be concerned with quantum stochastic differential inclusion in the integral form, given by

$$X(t) \in a + \int_0^t \big(E(s, X(s))d\wedge_\pi(s) + F(s, X(s))dA_f(s)$$
$$+ G(s, X(s))dA_g^+(s) + H(s, X(s))ds \big), \quad \text{almost all } t \in [0, T]. \tag{1.1}$$

We will employ in this paper the various spaces of quantum stochastic processes introduced in the works of Ekhaguere [8] and Ayoola [4]. As usual, our work is accomplished within the framework of the Hudson and Parthasarathy [11] formulation of quantum stochastic calculus employing the notations and the QSDI setup due to [8]. Corresponding to a pre-Hilbert space $\mathbb{D}$ with completion $\mathcal{R}$ and the Boson Fock space $\Gamma(L_\gamma^2(\mathbb{R}_+))$ with the dense subspace $\mathbb{E}$ generated by exponential vectors, we follow the fundamental concepts and structures as in the references by employing the locally convex space $\widetilde{\mathcal{A}}$ of noncommutative stochastic processes whose topology is generated by the family of seminorms $\{\|x\|_{\eta\xi} = |\langle \eta, x\xi \rangle|, \ x \in \mathcal{A}, \ \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$. The underlying elements of $\widetilde{\mathcal{A}}$ consists of linear maps from $\mathbb{D}\underline{\otimes}\mathbb{E}$ into $\mathcal{R} \otimes \Gamma(L_\gamma^2(\mathbb{R}_+))$ having domains of their adjoints containing $\mathbb{D}\underline{\otimes}\mathbb{E}$. In particular, the spaces $L_{\text{loc}}^P(\widetilde{\mathcal{A}})$, $L_{\gamma,\text{loc}}^\infty(\mathbb{R}_+)$, $L_{\text{loc}}^P(I \times \widetilde{\mathcal{A}})$ for a fixed Hilbert space γ are being adopted as in the above references. In the foregoing setup, the integral appearing in (1.1) is a set-valued quantum stochastic integral as defined in [8]. The coefficients E, F, G, H are elements of $L_{\text{loc}}^2([0, T] \times \widetilde{\mathcal{A}})_{\text{mvs}}$, where $\widetilde{\mathcal{A}}$ is a locally convex space and $(0, a) \in [0, T] \times \widetilde{\mathcal{A}}$ is a fixed point. The maps f, g, π appearing in (1.1) lie in some suitable function spaces. The integrators $\wedge_\pi$, A_g^+, and A_f are the gauge, creation, and annihilation processes associated with the basic field operators of quantum field theory.

As in Ayoola [4–7] we will consider the equivalent form of (1.1) given by

$$\frac{d}{dt}\langle \eta, X(t)\xi \rangle \in P(t, X(t))(\eta, \xi),$$
$$X(0) = a, \quad t \in [0, T]. \tag{1.2}$$

Inclusion (1.2) is a nonclassical ordinary differential inclusion and the map $(\eta, \xi) \to P(t, x)(\eta, \xi)$ is a multivalued sesquilinear form on $(\mathbb{D}\underline{\otimes}\mathbb{E})^2$ for $(t, x) \in [0, T] \times \widetilde{\mathcal{A}}$. We refer the reader to the works of Ekhaguere [8–10] for the explicit forms of the map and the existence results for solutions of QSDI (1.1) of Lipschitz, hypermaximal monotone and of evolution types.

The rest of the paper is organised as follows: In Section 2, we outline some fundamental definitions, notations and results needed for the establishment of the main result. Section 3 is devoted to the main results of the paper.

2. Preliminary results and assumptions

As in [4, 8], we let $\mathrm{clos}(\mathcal{N})$ denote the family of all nonempty closed subsets of a topological space $\mathcal{N}$. For $\mathcal{N} \in \{\widetilde{\mathcal{A}}, \mathbb{C}\}$, we adopt the Hausdorff topology on $\mathrm{clos}(\mathcal{N})$ as explained in the references above. We denote by $d(x, A)$, the distance from a point $x \in \mathbb{C}$ to a set $A \subseteq \mathbb{C}$. For $A, B \in \mathrm{clos}(\mathbb{C})$, $\rho(A, B)$ denote the Hausdorff distance between the sets.

For a real number $\delta > 0$, we let $B(x, \delta)$ denote the open ball of radius δ around a point $x \in \mathbb{C}$. As in the references above, we shall employ the space $\mathrm{wac}(\widetilde{\mathcal{A}})$ which is the completion of the locally convex topological space $(\mathrm{Ad}(\widetilde{\mathcal{A}})_{\mathrm{wac}}, \tau)$ of adapted weakly absolutely continuous stochastic processes $\Phi : [0, T] \to \widetilde{\mathcal{A}}$ whose topology τ is generated by the family of seminorms given by

$$|\Phi|_{\eta\xi} := \|\Phi(0)\|_{\eta\xi} + \int_0^T \left| \frac{d}{dt} \langle \eta, \Phi(t)\xi \rangle \right| dt, \quad \text{for } \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}. \tag{2.1}$$

Associated with space $\mathrm{wac}(\widetilde{\mathcal{A}})$, we will employ the space $\mathrm{wac}(\widetilde{\mathcal{A}})(\eta, \xi)$ consisting of absolutely continuous complex valued functions $\langle \eta, \Phi(\cdot)\xi \rangle := \Phi_{\eta\xi}(\cdot) : [0, T] \to \mathbb{C}$, where $\Phi \in \mathrm{wac}(\widetilde{\mathcal{A}})$ and for arbitrary pair of points $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. We will also denote by $S^{(T)}(a)$, the subset of $\mathrm{wac}(\widetilde{\mathcal{A}})$ consisting of the set of solutions of QSDI (1.1) corresponding to the initial value $a \in \widetilde{\mathcal{A}}$ and write $S^{(T)}(a)(\eta, \xi) = \{ \langle \eta, \Phi(\cdot)\xi \rangle : \Phi \in S^{(T)}(a) \}$.

We assume the following conditions in what follows.

$(\mathscr{S}_{(1)})$ The coefficients E, F, G, H appearing in QSDI (1.1) are continuous.

$(\mathscr{S}_{(2)})$ The multivalued map $(t, x) \to P(t, x)(\eta, \xi)$ has nonempty and closed values as subsets of the field $\mathbb{C}$ of complex numbers.

$(\mathscr{S}_{(3)})$ For each $x \in \widetilde{\mathcal{A}}$, the map $t \to P(t, x)(\eta, \xi)$ is measurable.

$(\mathscr{S}_{(4)})$ There exists a map $K_{\eta\xi}^P : [0, T] \to \mathbb{R}_+$ lying in $L_{\mathrm{loc}}^1([0, T])$ such that

$$\rho(P(t, x)(\eta, \xi), P(t, y)(\eta, \xi)) \le K_{\eta\xi}^P(t) \|x - y\|_{\eta\xi} \tag{2.2}$$

for $t \in [0, T]$, and for each pair $x, y \in \widetilde{\mathcal{A}}$.

$(\mathscr{S}_{(5)})$ There exists a stochastic process $Y : [0, T] \to \widetilde{\mathcal{A}}$ lying in $\mathrm{Ad}(\widetilde{\mathcal{A}})_{\mathrm{wac}}$ such that for each pair $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$,

$$d\left(\frac{d}{dt} \langle \eta, Y(t)\xi \rangle, P(t, Y(t))(\eta, \xi) \right) \le \rho_{\eta\xi}(t) \tag{2.3}$$

for almost all $t \in [0, T]$ and for some locally integrable map $\rho_{\eta\xi} : [0, T] \to R_+$.

$(\mathscr{S}_{(6)})$ The initial point a lies in a set $A \subseteq \widetilde{\mathcal{A}}$ such that the set of complex numbers $A(\eta, \xi) := \{ \langle \eta, a\xi \rangle : a \in A \}$ is compact in $\mathbb{C}$. For points $a_i \in A$, $i = 1, 2, \ldots$, and $Y_i \in S^{(T)}(a_i)$, we employ the notation $a_{\eta\xi,i} := \langle \eta, a_i\xi \rangle$ and $Y_{\eta\xi,i}(\cdot) := \langle \eta, Y_i(\cdot)\xi \rangle$ where $a \to S^{(T)}(a)$ is the multivalued solution map of QSDI (1.1) corresponding to the initial value $x = a$.

Under the conditions above, it is well known that the set $S^{(T)}(a)$ is not empty for arbitrary $a \in \widetilde{\mathcal{A}}$ (see Ekhaguere [8–10]).

Next, we recall from [4] a useful result in what follows.

PROPOSITION 2.1. *Let $V_0, V_1, \ldots, V_m$ be stochastic processes in $L^1_{loc}(\widetilde{\mathcal{A}})$ and for any pair of points $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, let $\{I_j(a_{\eta\xi})\}$ be a partition of the interval $I = [0, T]$ into a finite number of subintervals with endpoints depending continuously on the point $a_{\eta\xi} := \langle \eta, a\xi \rangle$, $a \in A$.*

Consider the map

$$W : a_{\eta\xi} \longrightarrow a_{\eta\xi} + \int_0^t \sum_{j=0}^m \chi_{I_j(a_{\eta\xi})}(s) \langle \eta, V_j(s)\xi \rangle \, ds. \tag{2.4}$$

Then there exists a map $R_{\eta\xi}(t)$ lying in $L^1_{loc}([0, T])$ such that for every $\epsilon > 0$, there exists $\partial > 0$ such that $|a_{\eta\xi} - a'_{\eta\xi}| < \partial$ implies that

$$\left| \frac{d}{dt} W(a_{\eta\xi})(t) - \frac{d}{dt} W(a'_{\eta\xi})(t) \right| \leq R_{\eta\xi}(t) \chi_E(t), \tag{2.5}$$

for some set $E \subseteq I$ with measure $\mu(E) < \epsilon$.

3. Main results

The main result of this paper is established by adapting to the present quantum stochastic calculus, a line of argument employed in the work of Broucke and Arapostathis [14], concerning classical differential inclusions where multifunctions take values in finite dimensional Euclidean spaces. In what follows, we establish a continuous selection that interpolates a finite number of trajectories extending the case of a single trajectory established in [4].

THEOREM 3.1. *Assume that the conditions $\mathcal{S}_{(1)} - \mathcal{S}_{(6)}$ are satisfied.*

Let $A \subseteq \widetilde{\mathcal{A}}$ such that for arbitrary pair $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, $A(\eta, \xi)$ is compact in $\mathbb{C}$ with diameter $D_{\eta\xi}$. Suppose further that a finite set of distinct initial conditions $\{a_i, i = 1, 2, \ldots, N\}$, from the set A and the corresponding solutions $\{Y_i(t), i = 1, 2, \ldots, N\}$ of QSDI (1.1), are given on a time interval $[0, T]$.

Then there exists a continuous map $W : A(\eta, \xi) \to wac(\widetilde{\mathcal{A}})(\eta, \xi)$, a selection from $S^{(T)}(a)(\eta, \xi)$ such that

$$W(a_{\eta\xi,i}) = Y_{\eta\xi,i}, \quad i = 1, 2, \ldots, N. \tag{3.1}$$

Proof. We define for any element $a \in A$ and arbitrary pair $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$

$$\delta(a_{\eta\xi}) = \begin{cases} \dfrac{1}{2} \min_{1 \leq j \leq N} |a_{\eta\xi} - a_{\eta\xi,j}|, & a_{\eta\xi} \neq a_{\eta\xi,i}, \ i = 1, 2, \ldots, N, \\[2mm] \dfrac{1}{2} \min_{i,j} |a_{\eta\xi,i} - a_{\eta\xi,j}|, & \text{otherwise.} \end{cases} \tag{3.2}$$

The collection of open sets $\{B(a_{\eta\xi}, \delta(a_{\eta\xi})), a_{\eta\xi} \in A(\eta, \xi)\}$ is a covering for the set $A(\eta, \xi)$. By the compactness of $A(\eta, \xi)$, let $\{B(b_{\eta\xi,j}, \delta(b_{\eta\xi,j}))\}_{j=1}^M$ be a finite open subcovering and

let $\{p_j\}_{j=1}^{M}$ be a partition of unity subordinated to it (see Ayoola [4] for the existence of such partition of unity).

We remark that by the definition of the covering, each $a_{\eta\xi,i}$ belongs to exactly one member of the subcovering since for each $k \neq i$, the inequality $|a_{\eta\xi,k} - a_{\eta\xi,i}| < \delta(a_{\eta\xi,i})$ is invalid.

For each $a_{\eta\xi} \in A(\eta,\xi)$ define the interval

$$I_j(a_{\eta\xi}) = \left[T\sum_{i=1}^{j-1} p_i(a_{\eta\xi}), \ T\sum_{j=1}^{j} p_i(a_{\eta\xi}) \right], \quad 1 \leq j \leq M. \tag{3.3}$$

As in [14], we employ the partition $\{J_1, J_2, J_3, \ldots, J_N\}$ of the set of positive integers $\{1, 2, 3, \ldots, M\}$ that indexed the subcoverings defined as follows: for $1 \leq k < N$,

$$J_k = \{j : |b_{\eta\xi,j} - a_{\eta\xi,k}| < |b_{\eta\xi,j} - a_{\eta\xi,l}|, \ l > k\}$$

$$\bigcap \{j : |b_{\eta\xi,j} - a_{\eta\xi,k}| \leq |b_{\eta\xi,j} - a_{\eta\xi,l}|, \ l \leq k\}; \tag{3.4}$$

$$J_N = \{1, 2, 3, \ldots, M\} \setminus \bigcup_{k=1}^{N-1} J_k.$$

Next we define the function

$$\alpha_k(a_{\eta\xi}, t) = \sum_{j \in J_k} \chi_{I_j(a_{\eta\xi})}(t), \quad k = 1, 2, \ldots, N. \tag{3.5}$$

Thus by definition, the set J_k is not empty and $\alpha_k(a_{\eta\xi,k}, t) = 1$ for all $k \in \{J_1, J_2, \ldots, J_N\}$.

Also we have

$$\sum_{k=1}^{N} \alpha_k(a_{\eta\xi}, t) = 1, \quad \forall a_{\eta\xi} \in A(\eta,\xi), \ t \in [0, T]. \tag{3.6}$$

Since each $Y_i \in S^{(T)}(a_i)$, then $Y_i \in \mathrm{Ad}(\tilde{\mathcal{A}})_{\mathrm{wac}}$, $i = 1, 2, \ldots, N$. By the properties of solutions of QSDI (1.1) (see [4, 8]), there exist stochastic processes $V_{0,i} \in L^1_{\mathrm{loc}}(\tilde{\mathcal{A}})$, $i = 1, 2, \ldots, N$ such that

$$Y_i(t) = a_i + \int_0^t V_{0,i}(s)ds, \quad \text{almost all } t \in [0, T] \tag{3.7}$$

and for arbitrary $\eta, \xi \in \mathbb{D} \otimes E$, we have

$$\langle \eta, V_{0,i}(t)\xi \rangle = \frac{d}{dt} \langle \eta, Y_i(t)\xi \rangle \in P(t, Y_i(t))(\eta, \xi), \quad t \in [0, T]. \tag{3.8}$$

For any point $a \in A$, we consider the family of maps $\{\Phi^0(a)\} \subseteq \mathrm{wac}(\tilde{\mathcal{A}})$ given by

$$\Phi^0(a)(t) = a + \int_0^t \sum_{1 \leq k \leq N} \alpha_k(a_{\eta\xi}, s) V_{0,k}(s)ds. \tag{3.9}$$

Associated with $\Phi^0(a)$, we define the map

$$W^0 : A(\eta, \xi) \longrightarrow \text{wac}(\tilde{\mathscr{A}})(\eta, \xi) \tag{3.10}$$

given by

$$W^0(a_{\eta\xi})(t) = a_{\eta\xi} + \int_0^t \sum_{1 \leq k \leq N} \alpha_k(a_{\eta\xi}, s)\langle \eta, V_{0,k}(s)\xi \rangle ds. \tag{3.11}$$

We remark here that $\Phi^0(a_k)(t) = Y_k(t)$ and therefore we have

$$\langle \eta, \Phi^0(a_k)(t)\xi \rangle = Y_{\eta\xi,k}(t). \tag{3.12}$$

Next we have the following estimates:

$$d\left(\frac{d}{dt} W^0(a_{\eta\xi})(t), \, P(t, \Phi^0(a)(t))(\eta, \xi) \right)$$

$$= d\left(\sum_{1 \leq k \leq N} \alpha_k(a_{\eta\xi}, t)\langle \eta, V_{0,k}(t)\xi \rangle, \, P(t, \Phi^0(a)(t))(\eta, \xi) \right)$$

$$\leq \max_{1 \leq k \leq N} \rho\big(P(t, Y_k(t))(\eta, \xi), \, P(t, \Phi^0(a)(t))(\eta, \xi)\big)$$

$$\leq K^P_{\eta\xi}(t) \max_{1 \leq k \leq N} \|Y_k(t) - \Phi^0(a)(t)\|_{\eta\xi}$$

$$\leq K^P_{\eta\xi}(t) \max_{1 \leq k \leq N}\left[|a_{\eta\xi,k} - a_{\eta\xi}| + \int_0^t \left| \langle \eta, V_{0,k}(s)\xi \rangle - \sum_{1 \leq l \leq N} \alpha_l(a_{\eta\xi}, s)\langle \eta, V_{0,l}(s)\xi \rangle \right| ds \right]$$

$$\leq K^P_{\eta\xi}(t) L_{\eta\xi}, \tag{3.13}$$

where

$$L_{\eta\xi} = D_{\eta\xi} + \min_{i,j} \int_0^T |\langle \eta, V_{0,i}(s)\xi \rangle - \langle \eta, V_{0,j}(s)\xi \rangle| \, ds. \tag{3.14}$$

Inequality (3.13) holds since

$$\max_{1 \leq k \leq N}\left[|a_{\eta\xi,k} - a_{\eta\xi}| + \int_0^t \left| \langle \eta, V_{0,k}(s)\xi \rangle - \sum_{1 \leq l \leq N} \alpha_l(a_{\eta\xi}, s)\langle \eta, V_{0,l}(s)\xi \rangle \right| ds \right] \leq L_{\eta\xi}. \tag{3.15}$$

We remark also that for any pair of points $a, a' \in A$ with $a_{\eta\xi}, a'_{\eta\xi} \in A(\eta, \xi)$,

$$| W^0(a_{\eta\xi})(t) - W^0(a'_{\eta\xi})(t) |$$

$$= \left| (a_{\eta\xi} - a'_{\eta\xi}) + \int_0^t \sum_{1 \leq k \leq N} [\alpha_k(a_{\eta\xi}, s) - \alpha_k(a'_{\eta\xi}, s)]\langle \eta, V_{0,k}(s)\xi \rangle ds \right| \leq L_{\eta\xi}. \tag{3.16}$$

As in our previous work [4] (see also Ekhaguere [8], Aubin and Cellina [1]), we can choose $V^0(a)(t)(\eta,\xi)$ to be a measurable selection from $P(t,\Phi^0(a)(t))(\eta,\xi)$ such that

$$\left| \frac{d}{dt} W^0(a_{\eta\xi})(t) - V^0(a)(t)(\eta,\xi) \right| = d\left(\frac{d}{dt} W^0(a_{\eta\xi})(t), P(t,\Phi^0(a)(t))(\eta,\xi) \right). \quad (3.17)$$

As $(\eta,\xi) \to V^0(a)(t)(\eta,\xi)$ is a sesquilinear form, there exists a stochastic process $V^0(a) \in \mathrm{Ad}(\widetilde{\mathcal{A}})_{\mathrm{wac}}$ such that

$$V^0(a)(t)(\eta,\xi) = \langle \eta, V^0(a)(t)\xi \rangle, \quad t \in [0,T]. \quad (3.18)$$

Next we define the map

$$\Phi^1(a)(t) = a + \int_0^t \sum_{j=0}^M \chi_{I_j(a_{\eta\xi})}(s) V^0(b_j)(s) ds, \quad (3.19)$$

$$W^1(a_{\eta\xi})(t) = a_{\eta\xi} + \int_0^t \sum_{j=0}^M \chi_{I_j(a_{\eta\xi})}(s) \langle \eta, V^0(b_j)(s)\xi \rangle ds. \quad (3.20)$$

From (3.9), (3.11), (3.13), and (3.20), we have

$$\left| \frac{d}{dt} W^1(a_{\eta\xi})(t) - \frac{d}{dt} W^0(a_{\eta\xi})(t) \right|$$

$$= \left| \sum_{j=0}^M \chi_{I_j(a_{\eta\xi})}(t) \langle \eta, V^0(b_j)(t)\xi \rangle - \sum_{1 \le k \le N} \alpha_k(a_{\eta\xi},t) \langle \eta, V_{0,k}(t)\xi \rangle \right|$$

$$\le \max_{j,k} \rho(P(t,\Phi^0(b_j)(t))(\eta,\xi), P(t,Y_k(t))(\eta,\xi)) \quad (3.21)$$

$$= \max_{j,k} \rho(P(t,\Phi^0(b_j)(t))(\eta,\xi), P(t,\Phi^0(a_k)(t))(\eta,\xi))$$

$$\le L_{\eta\xi} K^P_{\eta\xi}(t).$$

Since by definition, $\langle \eta, \Phi^0(a)(t)\xi \rangle = W^0(a_{\eta\xi})(t)$, then by (3.16) we have the following estimates:

$$d\left(\frac{d}{dt} W^1(a_{\eta\xi})(t), P(t,\Phi^0(a)(t))(\eta,\xi) \right)$$

$$\le \max_{1 \le j \le M} d(\langle \eta, V^0(b_j)(t)\xi \rangle, P(t,\Phi^0(a)(t))(\eta,\xi)) \quad (3.22)$$

$$\le \max_{1 \le j \le M} \rho(P(t,\Phi^0(b_j)(t))(\eta,\xi), P(t,\Phi^0(a)(t))(\eta,\xi))$$

$$\le L_{\eta\xi} K^P_{\eta\xi}(t).$$

Therefore, by (3.20) and (3.22), we have

$$
\begin{aligned}
d\left(\frac{d}{dt}W^1(a_{\eta\xi})(t),\, P(t,\Phi^1(a)(t))(\eta,\xi)\right) & \\
\le\, d\left(\frac{d}{dt}W^1(a_{\eta\xi})(t),\, P(t,\Phi^0(a)(t))(\eta,\xi)\right) & \\
+\, \rho(P(t,\Phi^0(a)(t))(\eta,\xi),\, P(t,\Phi^1(a)(t))(\eta,\xi)) & \\
\le\, L_{\eta\xi}K^P_{\eta\xi}(t) + K^P_{\eta\xi}(t)\big\|\Phi^0(a)(t) - \Phi^1(a)(t)\big\|_{\eta\xi}. &
\end{aligned}
\tag{3.23}
$$

Let $t \in I_j(a_{\eta\xi})$, then by (3.13) and (3.17), we have

$$
\begin{aligned}
\big\|\Phi^0(a)(t) - \Phi^1(a)(t)\big\|_{\eta\xi} = \big|\, W^0(a_{\eta\xi})(t) - W^1(a_{\eta\xi})(t)\,\big| & \\
\le \int_0^t \left|\frac{d}{ds}W^0(a_{\eta\xi})(s) - \langle\eta, V^0(b_j)(s)\xi\rangle\right| ds & \\
= \int_0^t d\left(\frac{d}{ds}W^0(a_{\eta\xi})(s),\, P(t,\Phi^0(b_j)(s))(\eta,\xi)\right) ds & \\
\le L_{\eta\xi}\int_0^t K^P_{\eta\xi}(s)ds = L_{\eta\xi}M_{\eta\xi}(t). &
\end{aligned}
\tag{3.24}
$$

Thus from (3.23),

$$
d\left(\frac{d}{dt}W^1(a_{\eta\xi})(t),\, P(t,\Phi^1(a)(t))(\eta,\xi)\right) \le L_{\eta\xi}K^P_{\eta\xi}(t) + L_{\eta\xi}K^P_{\eta\xi}(t)M_{\eta\xi}(t).
\tag{3.25}
$$

The estimate given by (3.25) is independent of any j and so holds on the whole interval $[0,T]$.

In general, by the method employed in [4], we can construct sequences of maps: $\Phi^n : A \to \mathrm{wac}(\widetilde{\mathcal{A}})$ and $W^n : A(\eta,\xi) \to \mathrm{wac}(\widetilde{\mathcal{A}})(\eta,\xi)$ such that W^n is continuous on $A(\eta,\xi)$ satisfying for each $a \in A$, $a_{\eta\xi} \in A(\eta,\xi)$, $t \in [0,T]$, $\Phi^n(a)(0) = a$, $\Phi^n(a_i)(t) = Y_i(t)$, $W^n(a_{\eta\xi,i})(t) = Y_{\eta\xi,i}(t)$, $i = 1,2,\ldots,N$. Moreover,

$$
\begin{aligned}
\int_0^t \left|\frac{d}{ds}W^n(a_{\eta\xi})(s) - \frac{d}{ds}W^{n-1}(a_{\eta\xi})(s)\right| ds & \\
\le L_{\eta\xi}\left[\frac{M^n_{\eta\xi}(t)}{n!} + \frac{4}{2^n}\sum_{i=1}^{n}\frac{(2M_{\eta\xi}(t))^i}{i!} + \frac{1}{2^n}\right], &
\end{aligned}
\tag{3.26}
$$

$$
\begin{aligned}
d\left(\frac{d}{dt}W^n(a_{\eta\xi})(t),\, P(t,\Phi^n(a)(t))(\eta,\xi)\right) & \\
\le L_{\eta\xi}K^P_{\eta\xi}(t)\left[\frac{M^n_{\eta\xi}(t)}{n!} + \frac{4}{2^n}\sum_{i=0}^{n}\frac{(2M_{\eta\xi}(t))^i}{i!}\right]. &
\end{aligned}
\tag{3.27}
$$

The construction is established by induction on n. For $n = 1$, our claim and the estimates (3.26) and (3.27) hold by (3.21) and (3.25), respectively. For $n \ge 2$, we employ

Proposition 2.1 and select $\delta_n > 0$ such that $|a_{\eta\xi} - a'_{\eta\xi}| < \delta_n$ implies that

$$\int_0^T \left| \frac{d}{dt} W^{n-1}(a_{\eta\xi})(t) - \frac{d}{dt} W^{n-1}(a'_{\eta\xi})(t) \right| dt \leq \frac{L_{\eta\xi}}{2^n}. \tag{3.28}$$

Next we define for $a \in A$, $a_{\eta\xi} \in A(\eta,\xi)$,

$$\delta_n(a_{\eta\xi}) = \begin{cases} \min\left\{ \dfrac{L_{\eta\xi}}{2^n}, \delta_n, \dfrac{1}{2} \min_{1 \leq j \leq N} |a_{\eta\xi} - a_{\eta\xi,j}| \right\}, & a_{\eta\xi} \neq a_{\eta\xi,j}, \\[4mm] \min\left\{ \dfrac{L_{\eta\xi}}{2^n}, \delta_n, \dfrac{1}{2} \min_{i,j} |a_{\eta\xi,i} - a_{\eta\xi,j}| \right\}, & \text{otherwise.} \end{cases} \tag{3.29}$$

As before, we cover the set $A(\eta,\xi)$ with the balls $B(a_{\eta\xi}, \delta_n(a_{\eta\xi}))$, where $a_{\eta\xi} \in A(\eta,\xi)$, and by compactness of the set, we let $\{B(b^n_{\eta\xi,j}, \delta_n(b^n_{\eta\xi,j}))\}$, $j = 1,2,\dots,M_n$ be a finite subcovering where $b^n_{\eta\xi,j} := \langle \eta, b^n_j \xi \rangle$, $b^n_j \in A \,\forall\, j$.

Let $\{p^n_j\}_{j=1}^{M_n}$ be a partition of unity subordinated to the subcovering. Each point $a_{\eta\xi,i}$ belongs only to one member of the subcovering. Next for $a_{\eta\xi} \in A(\eta,\xi)$, we define the interval

$$I^n_j(a_{\eta\xi}) = \left[T \sum_{i=1}^{j-1} p^n_i(a_{\eta\xi}), T \sum_{i=1}^{j} p^n_i(a_{\eta\xi}) \right], \quad 1 \leq j \leq M_n. \tag{3.30}$$

Similar to the case (3.17)–(3.20), we can choose $V^{n-1}(a)(t)(\eta,\xi)$ to be measurable selection from $P(t, \Phi^{n-1}(a)(t))(\eta,\xi)$ such that

$$\left| \frac{d}{dt} W^{n-1}(a_{\eta\xi})(t) - V^{n-1}(a)(t)(\eta,\xi) \right| = d\left(\frac{d}{dt} W^{n-1}(a_{\eta\xi})(t), P(t, \Phi^{n-1}(a)(t))(\eta,\xi) \right). \tag{3.31}$$

As $(\eta,\xi) \to V^{n-1}(a)(t)(\eta,\xi)$ is a sesquilinear form, there exists a stochastic process $V^{n-1}(a) \in \mathrm{Ad}(\widetilde{\mathscr{A}})_{\mathrm{wac}}$ such that

$$V^{n-1}(a)(t)(\eta,\xi) = \langle \eta, V^{n-1}(a)(t)\xi \rangle, \quad t \in [0, T]. \tag{3.32}$$

Next we define the maps

$$\Phi^n(a)(t) = a + \int_0^t \sum_{j=0}^{M_n} \chi_{I^n_j(a_{\eta\xi})}(s) V^{n-1}(b^n_j)(s) ds,$$

$$W^n(a_{\eta\xi})(t) = a_{\eta\xi} + \int_0^t \sum_{j=0}^{M_n} \chi_{I^n_j(a_{\eta\xi})}(s) \langle \eta, V^{n-1}(b^n_j)(s)\xi \rangle ds. \tag{3.33}$$

The estimates (3.26)-(3.27) can then be established by induction in the same way as in Ayoola [4].

By (3.26), the sequences $\{\Phi^n(a)\}$ is uniformly Cauchy in wac($\widetilde{\mathcal{A}}$) and thus converges uniformly to a map $\Phi : A \to \text{wac}(\widetilde{\mathcal{A}})$. Also we have

$$\lim_{n \to \infty} W^n(a_{\eta\xi})(t) = \lim_{n \to \infty} \langle \eta, \Phi^n(a)(t)\xi \rangle = \langle \eta, \Phi(a)(t)\xi \rangle. \tag{3.34}$$

The map $a_{\eta\xi} \to \langle \eta, \Phi(a)(t)\xi \rangle$ is continuous and $\Phi(a) \in \text{Ad}(\widetilde{\mathcal{A}})_{\text{wac}} \cap L^1_{\text{loc}}(\widetilde{\mathcal{A}})$. By (3.27),

$$d\left(\frac{d}{dt}\langle \eta, \Phi(a)(t)\xi \rangle, P(t, \Phi(a)(t))(\eta, \xi) \right) = 0. \tag{3.35}$$

Hence, $\Phi(a) \in S^{(T)}(a)$, $\langle \eta, \Phi(a)(\cdot)\xi \rangle \in S^{(T)}(a)(\eta, \xi)$, and $\Phi(a_i)(t) = Y_i(t)$, $i = 1, 2, \ldots,$ N, $t \in [0, T]$.

Finally, we present a result which shows that the difference of any two solutions of QSDI (1.1) that start from distinct points in the set A is bounded in the seminorm of wac($\widetilde{\mathcal{A}}$). □

THEOREM 3.2. *Suppose that the assumptions of Theorem 3.1 are satisfied. Let $a_0 \in A$ and $\Phi_0 \in S^{(T)}(a_0)$. Then there exists a selection $\Phi(a) \in S^{(T)}(a)$ such that the map $a_{\eta\xi} \to \langle \eta, \Phi(a)\xi \rangle$ is continuous and*

$$|\Phi(a) - \Phi(a_0)|_{\eta\xi} \le 3D_{\eta\xi} e^{2M_{\eta\xi}(T)}, \quad a \ne a_0. \tag{3.36}$$

Proof. As established in [4], there exists a sequence of approximate trajectories $\{\Phi^n(a)\}_{n=0}^{\infty}$ which form a Cauchy sequence in the locally convex space wac($\widetilde{\mathcal{A}}$) and converges uniformly to a map $\Phi(a) \in S^{(T)}(a)$. In particular, the sequence can be established such that

$$|\Phi^n(a) - \Phi^{n-1}(a)|_{\eta\xi} \le D_{\eta\xi}\left(\frac{M^n_{\eta\xi}(T)}{n!} + \frac{e^{2M_{\eta\xi}(T)}}{2^{n+1}} \right), \tag{3.37}$$

where $D_{\eta\xi}$ is the diameter of the set $A(\eta, \xi)$.

Thus we have

$$|\Phi(a) - \Phi^0(a)|_{\eta\xi} \le D_{\eta\xi}\left(e^{M_{\eta\xi}(T)} + e^{2M_{\eta\xi}(T)} \right), \tag{3.38}$$

for some $V_0 \in L^1_{\text{loc}}(\widetilde{\mathcal{A}})$, where

$$\Phi^0(a)(t) = a + \int_0^t V_0(s)\,ds,$$
$$\langle \eta, V_0(s)\xi \rangle = \frac{d}{ds}\langle \eta, \Phi(a_0)(s)\xi \rangle \in P(s, \Phi(a_0)(s))(\eta, \xi). \tag{3.39}$$

Consequently, we have the following estimates:

$$
\begin{aligned}
\left| \Phi(a) - \Phi(a_0) \right|_{\eta\xi} &= \left\| \Phi(a)(0) - \Phi(a_0)(0) \right\|_{\eta\xi} \\
&\quad + \int_0^T \left| \frac{d}{dt}\langle \eta, \Phi(a)(t)\xi \rangle - \frac{d}{dt}\langle \eta, \Phi(a_0)(t)\xi \rangle \right| dt \\
&= \|a - a_0\|_{\eta\xi} + \int_0^T \left| \frac{d}{dt}\langle \eta, \Phi(a)(t)\xi \rangle - \frac{d}{dt}\langle \eta, \Phi^0(a)(t)\xi \rangle \right| dt \\
&\leq D_{\eta\xi} + \left| \Phi(a) - \Phi^0(a) \right|_{\eta\xi} \leq D_{\eta\xi}\left(e^{M_{\eta\xi}(T)} + e^{2M_{\eta\xi}(T)} + 1 \right) \\
&\leq 3D_{\eta\xi} e^{2M_{\eta\xi}(T)},
\end{aligned}
$$

$$(3.40)$$

on account of (3.38).   □

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