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Research Article

Fredholm Determinant of an Integral Operator Driven by a Diffusion Process

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This article aims to give a formula for differentiating, with respect to T, an expression of the form $\lambda(T,x) := \mathbb{E}_x[f(X_T)e^{-\int_0^T V(X_s)ds}(\det(I + K_{X,T}))^p]$, where $p \geq 0$ and X is a diffusion process starting from x, taking values in a manifold, and the expectation is taken with respect to the law of this process. $K_{X,T} : L^2([0,T) \to \mathbb{R}^N) \to L^2([0,T) \to \mathbb{R}^N)$ is a trace class operator defined by $K_{X,T}f(s) = \int_0^T H(s \wedge t)\Gamma(X(t))f(t)dt$, where H, Γ are locally Lipschitz, positive $N \times N$ matrices.

1. Introduction

Suppose we have a differentiable manifold M of dimension d. By the Whitney's embedding theorem, there exists an embedding $i : M \to \mathbb{R}^{N'}$ such that $i(M)$ is a closed subset of $\mathbb{R}^{N'}$. It turns out that $N' = 2d+1$ will do. We will identify M with the image $i(M)$ and assume that M itself is a closed submanifold of $\mathbb{R}^{N'}$. We will also assume that M does not have a boundary. Let $M \cup \{\partial_M\}$ be a one-point compactification of M.

Definition 1.1. An M-valued path ω with explosion time $e = e(\omega) > 0$ is a continuous map $\omega : [0,\infty) \to M \cup \{\partial_M\}$ such that $\omega_t \in M$ for $0 \leq t < e$ and $\omega_t = \partial_M$ for all $t \geq e$ if $e < \infty$. The space of M-valued paths with explosion time is called the path space of M and is denoted by $W(M)$.

Let $(\Omega, \mathcal{F}_*, \mathbb{P})$ be a filtered probability space and let L be a smooth second-order elliptic operator on M. Using the coordinates of the ambient space $\{x_1, \ldots, x_{N'}\}$, and extending L smoothly to $\widetilde{L}$ in the ambient space, we may write

$$\widetilde{L} = \frac{1}{2}\sum_{i,j=1}^{N'} A_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N'} b_i \frac{\partial}{\partial x_i}, \tag{1.1}$$

with $\sigma := \sqrt{A}$, A a positive matrix. Since A is smooth, its square root is locally Lipschitz. Construct a time homogeneous Itô diffusion process $X : \Omega \to W(M)$ which solves the following stochastic differential equation:

$$dX_s = b(X_s)ds + \sigma(X_s)dB_s, \quad s \geq t; \quad X_t = x \tag{1.2}$$

in the ambient space $\mathbb{R}^{N'}$, where B_s is N'-dimensional Euclidean-Brownian motion and $b : \mathbb{R}^{N'} \to \mathbb{R}^{N'}$, $\sigma : \mathbb{R}^{N'} \to \mathbb{R}^{N' \times N'}$ such that

$$\|b(x) - b(y)\| + \|\sigma(x) - \sigma(y)\| \leq D(R)|x - y| \tag{1.3}$$

for some constant $D(R)$ dependent on an open ball centered at x with radius R. Till the explosion time $e(X)$, $X_s \in M$ for $0 \leq s < e(X)$. On M, $\widetilde{L} = L$. Furthermore, $\mu^X := \mathbb{P} \circ X^{-1}$ is an L-diffusion measure on $W(M)$. As a result, we use μ^X to be the probability measure on $W(M)$. Refer to [1] for a more detailed description.

Fix some positive integer N. Let $M_N(\mathbb{R})$ and $\mathcal{S}_N(\mathbb{R})$ be the spaces of $N \times N$ real-valued matrices and symmetric $N \times N$ real-valued matrices, respectively. Also let $M_N^+(\mathbb{R}) \subseteq \mathcal{S}_N(\mathbb{R})$ be the space of nonnegative matrices. Suppose that $H : [0, \infty) \to M_N^+(\mathbb{R})$ is locally Lipschitz with $0 \leq H(s) < H(t)$ if $s < t$, $H(0) = 0$, and $\Gamma : M \to M_N^+(\mathbb{R})$ is locally Lipschitz. Also assume that $\sup_{x \in M}\|\Gamma(x)\|$ is bounded, where $\|\cdot\|$ is the operator norm. Define $M_{\Gamma(X)}$ as the multiplication operator with $\Gamma(X)$ and Υ_T as the integral operator with kernel $H(s \wedge t)$, that is, for any $f \in L^2([0,T] \to \mathbb{R}^N)$,

$$(\Upsilon_T f)(s) = \int_0^T H(s \wedge t) f(t)\, dt, \tag{1.4}$$

where $s \wedge t$ is the minimum of s and t. Note that under the assumptions on H, Υ_T is a positive operator and is trace class (see Proposition 3.1.).

Consider the following integral operator: $K_{X,T} : L^2([0,T] \to \mathbb{R}^N) \to L^2([0,T] \to \mathbb{R}^N)$,

$$(K_{X,T} f)(s) = (\Upsilon M_{\Gamma(X)} f)(s) = \int_0^T H(s \wedge t)\Gamma(X(t)) f(t)\, dt. \tag{1.5}$$

It is a fact that for any trace class operator K, if we let $\|K\|_1$ denote the trace of $|K|$, then

$$\|K_{X,T}\|_1 \leq \sup_{x \in M}\|\Gamma(x)\| \|\Upsilon\|_1 \leq C \int_0^T \operatorname{tr} H(s)\, ds, \tag{1.6}$$

for some constant C. Here, tr means taking the trace of a matrix. Thus $K_{X,T}$ is trace class. Therefore,

$$|\det(I + K_{X,T})| \leq \exp(\|K_{X,T}\|_1) \leq \exp\left(C \int_0^T \operatorname{tr} H(s)\, ds\right). \tag{1.7}$$

Hence the Fredholm determinant is bounded for each T.

Let f, V be continuous bounded functions taking $x \in M \cup \{\partial M\} \mapsto \mathbb{R}$. Fix some number $p \geq 0$. Define a function $\lambda : [0,\infty) \times M \to \mathbb{R}$ by

$$\lambda(T,x) := \mathbb{E}_x\Big[f(X_T)e^{-\int_0^T V(X_s)\,ds}\big(\det\left(I + K_{X,T}\right)\big)^p\Big], \tag{1.8}$$

where the expectation is taken with respect to μ^X and the paths start from x. Note that λ is finite for any x, T from the above discussion. Let $h : [0,\infty) \times M \times M_N(\mathbb{R}) \to M_N(\mathbb{R})$ such that

$$h(t,x,v) = \left(v - H(t)\right)\Gamma(x)\left(v - H(t)\right). \tag{1.9}$$

The main result is as follows.

Theorem 1.2. *Let $(t,x,v) \in [0,\infty) \times M \times \mathcal{S}_N(\mathbb{R})$. Then*

$$\lambda(T,x) = \left(e^{T\widehat{H}}f\right)(0,x,0), \tag{1.10}$$

where

$$\widehat{H} = L + \frac{\partial}{\partial t} + \sum_{i,j=1}^{N} h(t,x,v)_{ij}\frac{\partial}{\partial v^{ij}} - V(x) + p\,\mathrm{tr}\left(H(t)\Gamma(x) - v\Gamma(x)\right). \tag{1.11}$$

Here, $h(t,x,v)_{ij}$ is the i, j component of the matrix h.

Clearly, λ is not in the Feynman-Kac formula form using the process X. The idea is to construct a diffusion process $W_s = (s, X_s, Z_s) \in [0,\infty) \times M \times \mathcal{S}_N(\mathbb{R})$ such that

$$\det\left(I + K_{X,T}\right) = \exp\left(\int_0^T G(W_s)ds\right), \tag{1.12}$$

and G is given by

$$G(t,x,v) = \mathrm{tr}\left(H(t)\Gamma(x) - v\Gamma(x)\right). \tag{1.13}$$

If we can achieve this, then our result follows from a simple application of the Feynman-Kac formula. Proving (1.12) requires the following 2 steps.

First, we have to prove an essential formula for the derivative of $\log\det(I + zK_{X,T})$ with respect to T, given by

$$\frac{d}{dT}\log\det\left(I + zK_{X,T}\right) = \mathrm{tr}\left(zH(T)\Gamma(X_T) - Z_{z,T}\Gamma(X_T)\right), \tag{1.14}$$

where z is a complex number and $Z_{z,T}$ is some adapted process. For a precise definition of $Z_{z,T}$, see (4.1), with $K_{X,s}$ replaced by $zK_{X,s}$. When $z = 1$, $Z_{1,T} = Z_T$. The goal is to show that the formula holds for $z = 1$ by analytic continuation.

Fix a time T. By making $|z|$ small such that $\|zK_{X,T}\| < 1$, we can use the perturbation formula and apply it to the determinant; see (2.3). Differentiating this equation with respect to T will give us (1.14). By analytic continuation, we can extend the formula in some domain $O \subseteq \mathbb{C}$ containing the origin, provided we avoid the poles of the resolvent of $zK_{X,s}$ for all $s \leq T$. If $1 \in O$, then (1.14) holds with $z = 1$. By integrating both sides and raising to the exponent, we will get (1.12). Note that if $\|K_{X,T}\| < 1$ for some time T, (1.14) and hence (1.12) hold. The details are given in Sections 2 and 3.

Now assume that (1.14) holds with $z = 1$. The second step consists of constructing a diffusion process W from X by using a stochastic differential equation. To do this, we differentiate Z_T with respect to T and show that it satisfies the differential equation

$$dZ_T = h(T, X_T, Z_T) = (Z_T - H(T))\Gamma(X_T)(Z_T - H(T))dT, \tag{1.15}$$

and hence $W_T = (T, X_T, Z_T)$ satisfies the following stochastic differential equation:

$$dW_T = (1, b(X_T), h(W_T))dT + (0, \sigma(X_T), 0)dB_T, \tag{1.16}$$

with explosion time $e(W)$. From this stochastic differential equation, it is clear that W is a diffusion process and by replacing the Fredholm determinant by the formula in (1.12), $\lambda(T, x)$ can be written as a Feynman-Kac form using this process W. However, if $e(W) < T < e(X)$, then (1.12) may fail to hold.

The positivity of H and Γ are used to show that $Z_{1,T} = Z_T$ exists for all time T and hence (1.14) holds at $z = 1$. This will also imply that $e(W) = e(X)$. In particular, only the positivity of H is required to show that $K_{X,T}$ is a trace class operator. To avoid $e(W) < e(X)$, we can restrict ourselves to small time T such that (2.24) holds true.

We can weaken our assumptions on H and Γ by not insisting that they are symmetric matrices. If we only assume that $K_{X,T}$ is trace class, then we can replace $\mathcal{S}_N(\mathbb{R})$ with $M_N(\mathbb{R})$. Under these weaker assumptions, we have the following result.

Theorem 1.3. *Suppose that, for a given locally Lipschitz H and Γ, $K_{X,T}$ is trace class. Assume that there exists some constant C such that*

$$\begin{gathered} \sup_{x \in M} \left\| \Gamma(x) \right\| \leq C < \infty, \\ \left\| H(s) \right\| \leq Cs, \quad s \geq 0. \end{gathered} \tag{1.17}$$

Let $(t, x, v) \in [0, \infty) \times M \times M_N(\mathbb{R})$. Then for all $T < 1/C^2$,

$$\lambda(T, x) = \left(e^{T\widehat{H}} f\right)(0, x, 0), \tag{1.18}$$

where

$$\widehat{H} = L + \frac{\partial}{\partial t} + \sum_{i,j=1}^{N} h(t, x, v)_{ij} \frac{\partial}{\partial v^{ij}} - V(x) + p \operatorname{tr}\left(H(t)\Gamma(x) - v\Gamma(x)\right). \tag{1.19}$$

From Lemma 2.4, using the assumptions on H and Γ, $\|K_{X,T}\| \le C^2 T$. If $T < 1/C^2$, then the norm is less than 1. Hence (1.14) holds and thus (1.12) holds true.

2. Functional analytic tools

Notation 2.1. Suppose that K is an integral operator, acting on $L^2([0,T] \to \mathbb{R}^N) \mapsto L^2([0,T] \to \mathbb{R}^N)$. We will write Kf to mean

$$(Kf)(s) = \int_0^T K(s,t) f(t)\, dt, \tag{2.1}$$

where $T < \infty$. To distinguish the operator K from its kernel, we will write $K(s,t)$ to refer to its kernel. This may be confusing, but it is used to avoid too many symbols. In this article, our integral operator is always trace class and the kernel is a continuous $N \times N$ matrix-valued function. By abuse of notation, $K^n(s,t)$ refers to the kernel of the integral operator K^n.

Notation 2.2. We will use tr to denote the trace of a matrix and Tr to denote taking the trace of a trace class operator. $\|\cdot\|$ will denote the operator norm.

It is well known that for a trace class operator A and $z \in \mathbb{C}$, $\operatorname{Tr}\log(I + zA)$ is a meromorphic function and has singularities at points z such that $-z^{-1} \in \sigma(A)$. Define the determinant $\det(I + zA)$, given by

$$\det(I + zA) = e^{\operatorname{Tr}\log(I+zA)}. \tag{2.2}$$

However, this determinant, also known as the Fredholm determinant of A, is analytic in z because the singularities z such that $-z^{-1} \in \sigma(A)$ are removable; see [2, Lemma 16].

We want to differentiate the function $\log\det(I + K_T)$ with respect to T, where we write K_T to denote the dependence on the domain $[0,T]$. If the kernel of K_T is given by $K(s,t)$, then for small z such that $\|zK_T\| < 1$, using the perturbation formula,

$$\det\left(I + zK_T\right) = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \operatorname{Tr}\left(zK_T\right)^n\right). \tag{2.3}$$

If we let $r = \|K_T\| < 1$, then

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \operatorname{Tr} K_T^n \right| \le \sum_{n=1}^{\infty} \frac{r^{n-1}}{n} \left\|K_T\right\|_1 \le \frac{\left\|K_T\right\|_1}{1-r}. \tag{2.4}$$

Thus the series in the exponent converges absolutely.

We will define the resolvent R_T by $K_T(I + K_T)^{-1}$. Since we can write $K_T(I + K_T)^{-1} = K_T - K_T(I + K_T)^{-1}K_T$, the kernel of R_T can be written as

$$R_T(s,t) = K(s,t) - K_T\left(I + K_T\right)^{-1} K_T(s,t). \tag{2.5}$$

When we write $K_T(I+K_T)^{-1}K_T(s,t)$, we mean

$$K_T\left(I+K_T\right)^{-1}K_T(s,t)=\int_0^T K(s,u)\left(\left(I+K_T\right)^{-1}K(\cdot,t)\right)(u)\,du. \tag{2.6}$$

We will also write the resolvent of the operator zK_T, $z\in\mathbb{C}$ as $R_{T,z}$. One more point to note is that we assume that $K(\cdot,0)=K(0,\cdot)=0$.

Now the operator K_s is an operator defined on different Hilbert spaces $L^2[0,s]$. Therefore, we will now think of our operator K_s as acting on $L^2[0,T]$, defined as

$$\left(K_s f\right)(u)=\int_0^T K(u,v)\chi_{[0,s]}(v)f(v)\,dv, \tag{2.7}$$

where χ is the characteristic function. Hence now our operator K_s has a kernel $K(u,v)\chi_{[0,s]}(v)$ dependent on the parameter s. Note that our K_s is continuous in the u variable but is discontinuous at $v=s$. Thus when we write $K_s(u,s)$, we mean

$$K_s(u,s):=\lim_{v\uparrow s}K_s(u,v). \tag{2.8}$$

Definition 2.3. Let $K(\cdot,\cdot)$ be a continuous matrix-valued function and let $\|K(\cdot,\cdot)\|$ be the matrix norm of $K(\cdot,\cdot)$. Define C_T to be the maximum value of $\|K(\cdot,\cdot)\|$ on $[0,T]\times[0,T]$.

The next lemma allows us to control the operator norm of the operator by controlling the sup norm of the kernel.

Lemma 2.4. *For* $0\le s<s'\le T$,

$$\left\|K_{s'}-K_s\right\|\le C_T\sqrt{T(s'-s)}. \tag{2.9}$$

Proof. For f, $g\in L^2$ and any $s\in[0,T]$,

$$\begin{aligned}
\left|\left\langle\left(K_s-K_{s'}\right)f,g\right\rangle\right| &\le \iint_0^T \left\|g(u)^T\chi_{[s,s']}(v)K(u,v)f(v)\right\|dv\,du\\
&\le \iint_0^T \chi_{[s,s']}(v)C_T\left|g(u)\right|\left|f(v)\right|dv\,du\\
&\le C_T\left(\int_0^T\left|g(u)\right|du\right)\left(\int_0^T\chi_{[s,s']}(u)\left|f(u)\right|du\right)\\
&\le C_T\left(\int_0^T\left|g(u)\right|^2du\cdot\int_0^T 1\,du\right)^{1/2}\left(\int_0^T\left|f(u)\right|^2du\cdot\int_0^T\chi_{[s,s']}(u)du\right)^{1/2}\\
&\le C_T\sqrt{T(s'-s)}\|f\|_2\|g\|_2.
\end{aligned} \tag{2.10}$$

Hence,

$$\|K_{s'} - K_s\| \leq C_T \sqrt{T(s' - s)} \tag{2.11}$$

for all $0 \leq s < s' \leq T$. □

Lemma 2.5. *Fix a $z \in \mathbb{C}$. For any T such that $|zC_T T| < 1$ and if $K(s,t)$ is continuous, then*

$$\frac{d}{dT}\left(\log\det\left(I + zK_T\right)\right) = \mathrm{tr}\ R_{T,z}(T,T). \tag{2.12}$$

Proof. Since z is fixed, we will replace zK_T by K_T and hence assume that $|C_T T| < 1$. Lemma 2.4 tells us that $\|K_T\| < 1$ and thus (2.3) holds true. Taking log on both sides of (2.3), we have

$$\log\det\left(I + K_T\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \mathrm{Tr}\, K_T^n. \tag{2.13}$$

Now

$$\mathrm{Tr}\, K_T^n = \int_0^T \cdots \int_0^T \mathrm{tr} \prod_{i=1}^{n} K\left(s_i, s_{i+1}\right) ds_1 \cdots ds_n, \tag{2.14}$$

where $s_{n+1} = s_1$. Differentiate with respect to T and using the fundamental theorem of calculus, we get

$$\begin{aligned}
&\frac{d}{dT}\int_0^T \cdots \int_0^T \mathrm{tr} \prod_{i=1}^{n} K\left(s_i, s_{i+1}\right) ds_1 \cdots ds_n \\
&\quad = n\,\mathrm{tr} \int_0^T \cdots \int_0^T K\left(T, s_2\right)\left(\prod_{i=2}^{n-1} K\left(s_i, s_{i+1}\right)\right) K\left(s_n, T\right) ds_2 \cdots ds_n \\
&\quad = n\,\mathrm{tr}\, K_T^n(T,T).
\end{aligned} \tag{2.15}$$

Let $C_T T = \alpha$. Thus,

$$\sum_{n=1}^{\infty} \left|\mathrm{tr}\, K_T^n(T,T)\right| \leq N \sum_{n=1}^{\infty} \left\|K_T^n(T,T)\right\| \leq N \sum_{n=1}^{\infty} C_T^n T^{n-1} \leq N C_T \sum_{n=0}^{\infty} \alpha^n = \frac{N C_T}{1 - \alpha} < \infty. \tag{2.16}$$

Thus

$$\begin{aligned}
\frac{d}{dT}\left(\log\det\left(I+K_T\right)\right) &= \sum_{n=1}^{\infty}\frac{d}{dT}\left(\frac{(-1)^{n+1}}{n}\operatorname{tr}\int_0^T\cdots\int_0^T\prod_{i=1}^{n}K\left(s_i,s_{i+1}\right)ds_1\cdots ds_n\right)\\
&= \operatorname{tr}\sum_{n=1}^{\infty}(-1)^{n+1}K_T^n(T,T)\\
&= \operatorname{tr}K_T(I+K_T)^{-1}(T,T)\\
&= \operatorname{tr}R_T(T,T).
\end{aligned}\tag{2.17}$$

□

Lemma 2.6. *Let $K(\cdot,\cdot)$ be continuous. For all $z\in\mathbb{C}$ such that $-z^{-1}\notin\sigma(K_s)$ for all $s\in[0,T]$, then* $\operatorname{tr}R_{\cdot,z}(\cdot,\cdot)$ *is continuous.*

Proof. Fix a z and write zK_s as K_s. By assumption, $I+K_s$ is invertible for all s. By Lemma 2.4, $K_s\to K_{s_0}$ as $s\to s_0$. Note that

$$I+K_s=I+K_{s_0}+K_s-K_{s_0}=\left(I+K_{s_0}\right)\left(I+\left(I+K_{s_0}\right)^{-1}\left(K_s-K_{s_0}\right)\right),\tag{2.18}$$

and if we let $G_s=K_s-K_{s_0}$, then

$$\left(I+K_s\right)^{-1}=\left(I+\left(I+K_{s_o}\right)^{-1}G_s\right)^{-1}\left(I+K_{s_o}\right)^{-1}.\tag{2.19}$$

By the open mapping theorem, because $I+K_{s_o}$ is a surjective continuous map, it is an open map. Therefore, its inverse is a bounded operator. Since $G_s=K_s-K_{s_0}\to 0$, thus $(I+(I+K_{s_o})^{-1}G_s)^{-1}\to I$ and hence $(I+K_s)^{-1}$ converges to $(I+K_{s_0})^{-1}$ as $s\to s_0$. This shows that $(I+K_s)^{-1}$ is continuous in s. Note that

$$R_s(s,s)=K(s,s)-K_s\left(I+K_s\right)^{-1}K(s,s).\tag{2.20}$$

Since $K(s,s)$, K_s and $(I+K_s)^{-1}$ are continuous in s, hence $R_s(s,s)$ is continuous in s. □

Lemma 2.7. *Let $K(\cdot,\cdot)$ be continuous. If there exists an open-connected set O containing 0 such that for $z\in O$, $(I+zK_s)^{-1}$ is analytic for all $s\in[0,T]$, then*

$$\log\det\left(I+zK_T\right)=\int_0^T\operatorname{tr}R_{s,z}(s,s)\,ds\tag{2.21}$$

for all $z\in O$.

Proof. For all $z\in O$, $I+zK_s$ is invertible for all $s\in[0,T]$ and hence $\operatorname{tr}R_{s,z}(s,s)$ is analytic in O. Therefore, it follows that $\int_0^T\operatorname{tr}R_{s,z}(s,s)\,ds$ is analytic in O, because

$$\frac{d}{dz}\int_0^T\operatorname{tr}R_{s,z}(s,s)\,ds=\int_0^T\frac{d}{dz}\operatorname{tr}R_{s,z}(s,s)\,ds.\tag{2.22}$$

By Lemma 2.4, $\|K_s\| \leq TC_T$ for all $s \in [0,T]$. Thus if we choose $U := \{z \,|\, |z| < 1/(TC_T)\}$, then U is an open set containing 0 and for $z \in U$, $\|zK_s\| < 1$ for all $s \in [0,T]$. From Lemma 2.5, for $z \in U$,

$$\log\det\left(I + zK_T\right) = \int_0^T \operatorname{tr} R_{s,z}(s,s)\, ds. \tag{2.23}$$

Since $\log\det(I + zK_T)$ is also analytic in O and agrees with $\int_0^T \operatorname{tr} R_{s,z}(s,s)\, ds$ in U, it follows that both functions are equal for all $z \in O$. □

The proof in the previous theorem gives us the existence of a small neighbourhood containing 0 such that (2.21) holds. Hence we have the following corollary.

Corollary 2.8. *Fix $T > 0$ and $C_T = \sup_{s,t\in[0,T]}\|K(s,t)\|$. There exists an open set $U_T := \{z \,|\, |z| < 1/(TC_T)\}$ such that* (2.21) *holds for all $z \in U_T$.*

Corollary 2.9. *Let O be an open-connected set as in Lemma 2.7 such that $1 \in O$. Then for $s \in (0,T)$,*

$$\frac{d}{ds}\log\det\left(I + K_s\right) = \operatorname{tr} R_s(s,s). \tag{2.24}$$

Proof. The corollary follows from differentiating (2.21). By Lemma 2.6, $\operatorname{tr} R_s(s,s)$ is continuous and hence the fundamental theorem of calculus applies. □

3. Fredholm determinant

The kernel we are interested in is $K_{X,T} = H(s\wedge t)\Gamma(X(t))$, for some process X. More generally, the kernel we are interested in is of the form $K_T(s,t) = H(s \wedge t)\Lambda(t)$ for some continuous matrix-valued Λ. The Hilbert space is $L^2([0,T] \to \mathbb{R}^N)$ for some positive number T. Without any ambiguity, we will in future write this space as L^2. We will also use $\|\cdot\|_2$ to denote the L^2 norm.

Proposition 3.1. *If H is continuous, $0 \leq H(s) \leq H(t)$ for any $s \leq t$ and Λ continuous, then Υ_T as defined in Section 1 is a trace class operator.*

To prove this result, we need the following theorem, which is [3, Theorem 2.12].

Theorem 3.2. *Let μ be a Baire measure on a locally compact space X. Let K be a function on $X \times X$ which is continuous and Hermitian positive, that is,*

$$\sum_{i,j=1}^{J} \overline{z_i} z_j K\left(x_i, x_j\right) \geq 0 \tag{3.1}$$

for any $x_1, \dots, x_J \in X$, $z_1, \dots, z_J \in \mathbb{C}^J$ and for any J. Then $K(x,x) \geq 0$ for all x. Suppose that, in addition,

$$\int K(x,x)\, d\mu(x) < \infty. \tag{3.2}$$

Then there exists a unique trace class integral operator A such that

$$(Af)(x) = \int K(x,y)f(y)\,d\mu(y),$$
$$\|A\|_1 = \int K(x,x)\,d\mu. \tag{3.3}$$

Proof of Proposition 3.1. Let $X = [0,T]$ and μ be Lebesgue measure. Using Theorem 3.2, it suffices to show that $H(s \wedge t)$ is Hermitian positive. Let $z_1, z_2, \ldots, z_J$ be any complex column vectors. Note that there are N entries in each column and the entries are complex valued. Let $s_1, \ldots, s_J \in [0,T]$. The proof is obtained using induction. Clearly, when $J = 1$, it is trivial. Suppose it is true for all values from $k = 1, 2, \ldots, J-1$. By relabelling, we can assume that $s_1 \le s_k$, $k = 2, \ldots, J$. Hence $s_1 \wedge s_k = s_1$ for any k. Let $\langle \cdot, \cdot \rangle$ be the usual dot product. Then

$$\begin{aligned}\sum_{j=1}^{J}\langle H(s_1\wedge s_j)z_j,\overline{z_1}\rangle + \sum_{j=1}^{J}\langle H(s_j\wedge s_1)z_1,\overline{z_j}\rangle &= \sum_{j=1}^{J}\langle H(s_1)z_j,\overline{z_1}\rangle + \sum_{j=1}^{J}\langle H(s_1)z_1,\overline{z_j}\rangle \\ &= \left\langle \sum_{j=1}^{J} H(s_1)z_j, \sum_{j=1}^{J}\overline{z_j}\right\rangle - \sum_{i,j=2}^{J}\langle H(s_1)z_j,\overline{z_i}\rangle.\end{aligned} \tag{3.4}$$

Therefore,

$$\begin{aligned}\sum_{i,j=1}^{J}\langle H(s_i\wedge s_j)z_j,\overline{z_i}\rangle &= \left\langle \sum_{j=1}^{J} H(s_1)z_j, \sum_{j=1}^{J}\overline{z_j}\right\rangle - \sum_{i,j=2}^{J}\langle H(s_1)z_j,\overline{z_i}\rangle + \sum_{i,j=2}^{J}\langle H(s_i\wedge s_j)z_j,\overline{z_i}\rangle \\ &= \left\langle \sum_{j=1}^{J} H(s_1)z_j, \sum_{j=1}^{J}\overline{z_j}\right\rangle + \sum_{i,j=2}^{J}\langle (H(s_i\wedge s_j) - H(s_1))z_j,\overline{z_i}\rangle.\end{aligned} \tag{3.5}$$

Since $s_i \wedge s_j \ge s_1$, $i = 2, \ldots, J$ and hence $H(s_i \wedge s_j) \ge H(s_1)$ by assumption on H. Thus by induction hypothesis, (replace H by $H(\cdot) - H(s_1)$),

$$\sum_{i,j=2}^{J}\langle (H(s_i\wedge s_j) - H(s_1))z_j,\overline{z_i}\rangle \ge 0 \tag{3.6}$$

and hence

$$\sum_{i,j=1}^{J}\langle H(s_i\wedge s_j)z_j,\overline{z_i}\rangle \ge 0. \tag{3.7}$$

□

Notation 3.3. By abuse of notation, $\langle \cdot, \cdot \rangle_T$ will denote integration over $[0, T]$,

$$\langle f, g \rangle_T := \int_0^T f(u) \cdot g(u)\, du, \tag{3.8}$$

where $\cdot$ should be interpreted as matrix multiplication or inner product, depending on the context. To ease the notation, we will write in future $K(T) := K(T, T)$.

Remark 3.4. If we assume that $H(s) - H(t)$ is strictly positive if $s > t$, which is the case we are interested in this article, then the proof of Proposition 3.1 shows that the operator Υ_T with kernel $H(s \wedge t)$ is strictly positive, that is, $\langle \Upsilon_T f, f \rangle_T > 0$ if $\langle f, f \rangle_T > 0$. This follows using a Riemann lower sum approximation on a double integral and that for any complex vectors $z_1, \ldots, z_J$,

$$\sum_{i,j=1}^{J} \langle H(s_i \wedge s_j) z_j, \overline{z_i} \rangle \frac{T^2}{J^2} > 0 \tag{3.9}$$

under the strict positivity assumptions.

The next proposition is a crucial statement. For the time being, we will assume that $(I + K_T)^{-1}$ exists for any time T without any further justification. Later on, we will prove that for our operator $K_{X,T}$, this is true; (see Proposition 5.2.). Writing in our new notation, we obtain the next proposition from Corollary 2.9.

Proposition 3.5. *Let K_T be an integral operator with kernel $K_T(s, t) = H(s \wedge t)\Lambda(t)$ for some continuous matrix-valued Λ:*

$$\frac{d}{dT}\big(\log\det(I + K_T)\big) = \operatorname{tr}\left[K(T) - \left\langle H\Lambda, (I + K_T)^{-1} H \right\rangle_T \Lambda(T)\right]. \tag{3.10}$$

Proof. We will use (2.5). So

$$\begin{aligned} R_T(T, T) &= K(T, T) - \left\langle K(T, \cdot),\ (I + K_T)^{-1} K(\cdot, T) \right\rangle_T \\ &= K(T, T) - \left\langle H(\cdot)\Lambda(\cdot),\ (I + K_T)^{-1} H(\cdot)\Lambda(T) \right\rangle_T \\ &= K(T) - \left\langle H\Lambda,\ (I + K_T)^{-1} H \right\rangle_T \Lambda(T). \end{aligned} \tag{3.11}$$

Taking trace completes the proof. □

Definition 3.6. Let

$$Z_T = \left\langle H\Lambda,\ (I + K_T)^{-1} H \right\rangle_T. \tag{3.12}$$

To ease the notation, we will now write $L := (I + K_T)^{-1}$ and $L(s, t)$ to be the kernel of L. Note that in future we will drop the subscript T from the operator K and it should be understood that K is dependent on T. Operators with a prime will denote its derivative with respect to T. Our task now is to differentiate Z_T.

Define a distributional kernel

$$\rho(s,t) = R(s,t) - \delta(s-t), \tag{3.13}$$

where δ is the Dirac delta function and R is the resolvent.

For any operator depending smoothly on some parameter T, we have the differentiation formula

$$L'_T = \frac{d}{dT}(I+K_T)^{-1} = -(I+K_T)^{-1}K'_T(I+K_T)^{-1}. \tag{3.14}$$

For the integral operator K', its kernel is given, by the fundamental theorem of calculus, by

$$K'(s,t) = K(s,T)\delta(t-T); \tag{3.15}$$

and hence combining with (3.14), we have

$$\begin{aligned} L'(s,t) &= -LK'L(s,t) \\ &= -LK'(s,t) - \langle LK(s,T)\delta(\cdot - T), -KL(\cdot,t)\rangle_T \\ &= -LK(s,T)\delta(t-T) + LK(s,T)\langle \delta(\cdot - T), KL(\cdot,t)\rangle_T \\ &= -R(s,T)\delta(t-T) + R(s,T)R(T,t) \\ &= R(s,T)\rho(T,t). \end{aligned} \tag{3.16}$$

Notation 3.7. Let K be an integral operator with kernel $K(s,t)$. We define the adjoint K^* by

$$(K^* f)(t) = \int_0^T f(s)K(s,t)\,dt. \tag{3.17}$$

Here, f is an $N \times N$ matrix-valued function. We will also write $\Lambda_T = \Lambda(T)$ and $H_T = H(T)$.

The following lemma defines the relationship between R and L.

Lemma 3.8. *It holds that*

$$\begin{aligned} R(s,T) &= (LH)(s)\Lambda_T, \\ R(T,t) &= (L^* H\Lambda)(t). \end{aligned} \tag{3.18}$$

Proof. We write the identity operator I as $If(s) = f(s) = \int_0^T \delta(s-t)f(t)dt$, where δ is Dirac delta function. Then

$$\begin{gathered}R(s,T) = \langle L(s,\cdot), K(\cdot,T)\rangle_T = \langle L(s,\cdot), H(\cdot\wedge T)\Lambda_T\rangle_T = \langle L(s,\cdot), H(\cdot)\Lambda_T\rangle_T = (LH)(s)\Lambda_T, \\ R(T,t) = \langle K(T,\cdot), L(\cdot,t)\rangle_T = \langle H(T\wedge\cdot)\Lambda(\cdot), L(\cdot,t)\rangle_T = (L^*H\Lambda)(t).\end{gathered} \tag{3.19}$$

□

Theorem 3.9. *Z_T satisfies the following differential equation:*

$$Z_T' = (Z_T - H_T)\Lambda_T(Z_T - H_T). \tag{3.20}$$

Proof. Now by definition of Z_T,

$$Z_T = \langle H\Lambda, (I+K_T)^{-1}H\rangle_T = \langle H\Lambda, LH\rangle_T = \langle L^*H\Lambda, H\rangle_T. \tag{3.21}$$

Using (3.16) and from Lemma 3.8,

$$\begin{aligned}(L'H)(s) &= \langle R(s,T)\rho(T,\cdot), H(\cdot)\rangle_T \\ &= \langle R(s,T)R(T,\cdot), H(\cdot)\rangle_T - R(s,T)H_T \\ &= (LH)(s)\Lambda_T\langle L^*H\Lambda, H\rangle - (LH)(s)\Lambda_T H_T \\ &= (LH)(s)\Lambda_T Z_T - (LH)(s)\Lambda_T H_T.\end{aligned} \tag{3.22}$$

Hence differentiating with respect to T, using the fundamental theorem and (3.22), gives

$$\begin{aligned}\frac{d}{dT}Z_T &= (H\Lambda LH)(T) + \langle H\Lambda, L'H\rangle_T \\ &= (H\Lambda LH)(T) + \langle (H\Lambda)(\cdot), (LH)(\cdot)\Lambda_T Z_T - (LH)(\cdot)\Lambda_T H_T\rangle_T \\ &= (H\Lambda LH)(T) - Z_T\Lambda_T H_T + Z_T\Lambda_T Z_T.\end{aligned} \tag{3.23}$$

But

$$\begin{aligned}(KLH)(T) &= \left(K(I+K)^{-1}H\right)(T) \\ &= \langle H(T\wedge\cdot)\Lambda(\cdot), (LH)(\cdot)\rangle_T \\ &= \langle H\Lambda, LH\rangle_T = Z_T.\end{aligned} \tag{3.24}$$

Hence

$$\begin{aligned}(H\Lambda LH)(T) &= (H\Lambda H)(T) - (H\Lambda KLH)(T) \\ &= (H\Lambda H)(T) - (H\Lambda)(T)Z_T.\end{aligned} \tag{3.25}$$

Therefore

$$\frac{d}{dT}Z_T = (H\Lambda H)(T) - (H\Lambda)(T)Z_T - Z_T\Lambda_T H_T + Z_T\Lambda_T Z_T \\ = (Z_T - H_T)\Lambda_T(Z_T - H_T). \tag{3.26}$$

This completes the proof. □

4. Integral operator driven by a diffusion process

Now back to the integral operator $K_{X,T}$ defined in Section 1. Define a process $Z_s : \Omega \to M_N(\mathbb{R})$,

$$Z_s = \left\langle H\Gamma_X, \left((I + K_{X,s})^{-1}H\right)\right\rangle_s, \tag{4.1}$$

where $\Gamma_X := \Gamma(X)$ (see Notation 3.3 for the definition of the angle brackets). From the definition, it is clear that Z_s is $\mathcal{F}_*$ adapted. In fact, Z_s is a symmetric matrix under the usual assumptions on H and Γ.

Proposition 4.1. *If H and Γ are symmetric matrices, then Z is symmetric as a matrix.*

Proof. Since s is fixed, we will drop the subscript s. Also fix an $\omega \in \Omega$, so we will also drop the subscript X. Let K^* be the adjoint of K with kernel $\Gamma(s)H(s \wedge t)$. By assumption of symmetry and by definition,

$$\begin{aligned} Z^T &= \left\langle H\left((I + K^*)^{-1}\right), \Gamma H\right\rangle \\ &= \langle H\Gamma, H\rangle - \left\langle HK^*\left((I + K^*)^{-1}\right), \Gamma H\right\rangle \\ &= \langle H\Gamma, H\rangle - \left\langle H\Gamma K\left((I + K)^{-1}\right), H\right\rangle \\ &= \left\langle H\Gamma\left((I + K)^{-1}\right), H\right\rangle \\ &= \left\langle H\Gamma, \left((I + K)^{-1}\right) H\right\rangle = Z. \end{aligned} \tag{4.2}$$

□

Theorem 4.2. *Let X_s be an L-diffusion process satisfying* (1.2) *and $H : [0,\infty) \to M_N^+(\mathbb{R})$ and let $\Gamma : M \to M_N^+(\mathbb{R})$ be continuous. Further assume that $H(s) \ge H(t) \ge 0$ for $s \ge t$. Let $K_{X,s}$ be an integral operator defined by* (1.5) *and*

$$Z_s = \left\langle H\Gamma_X, \left((I + K_{X,s})^{-1}H\right)\right\rangle_s. \tag{4.3}$$

Let $e(Z)$ be the explosion time of Z. Then for $s < e(Z)$, $W_s = (s, X_s, Z_s) : \Omega \to [0,\infty) \times M \times \mathcal{S}_N(\mathbb{R})$ satisfies the following stochastic differential equation:

$$dW_s = (1, b(X_s), h(W_s))ds + (0, \sigma(X_s), 0)dB_s, \quad s \ge t; \quad W_t = (t, x, v), \tag{4.4}$$

where

$$h(t,x,v) = \big(v - H(t)\big)\Gamma(x)\big(v - H(t)\big). \tag{4.5}$$

Proof. In the ambient space $\mathbb{R}^{N'}$, (s, X_s) is a diffusion satisfying the stochastic differential equation of the form

$$d\big(s, X_s\big) = \big(1, b\big(X_s\big)\big)ds + \big(0, \sigma\big(X_s\big)\big)dB_s. \tag{4.6}$$

Now by Theorem 3.9, Z_s satisfies the following differential equation:

$$dZ_s = h\big(s, X_s, Z_s\big)ds, \tag{4.7}$$

where h is defined by (4.5) and by Proposition 4.1, $Z_s \in \mathcal{S}_N(\mathbb{R})$. Thus we can write

$$d\big(s, X_s, Z_s\big) = \big(1, b\big(X_s\big), h\big(s, X_s, Z_s\big)\big)ds + \big(0, \sigma\big(X_s\big), 0\big)dB_s, \tag{4.8}$$

which is (4.4). The existence of Z_s for small time is guaranteed by Lemma 2.4. □

Lemma 4.3. *If* Γ *is locally Lipschitz, then* W_s *is the unique solution (path-wise) to* (4.4).

Proof. Now by the definition of X_s, b and σ are locally Lipschitz. However, since Γ is locally Lipschitz on the manifold M with bounded operator norm, it follows that h is locally Lipschitz. Therefore (4.4) has a unique solution and is given by W_s. □

5. Long-time existence of W_s

We had addressed the existence and uniqueness of the solution to (4.5), given by $W_s = (s, X_s, Z_s)$, with $e(W) \le e(X)$. We will now give sufficient conditions for $e(W) = e(X)$.

Proposition 5.1. *Suppose that the integral operator* Υ_T *with kernel* $H(s \wedge t)$ *is a strictly positive operator and* Γ *is a symmetric nonnegative matrix. Then for* $z \in \mathbb{C}$ *such that* $\mathrm{Re}(z) \ge 0$, $(I + zK_{X,T})^{-1}$ *exists for all* $T < e(X)$.

Proof. When $z = 0$ is trivial. So assume $z \ne 0$. Fix an $\omega \in \Omega$ and any $T < e(X(\omega))$. Since $K_X = K_{X,T}$ is a compact operator, it suffices to show that the kernel of $I + zK_X$ is 0. Write $\Gamma(X(\omega)) = \Gamma$ and $K_{X(\omega)} = K$. Let $0 \ne v \in L^2$ such that $\langle v, v\rangle > 0$ and $\Gamma v = 0$. Then $(I + zK_X)v = v$ is nonzero. Hence we can assume that $\langle \Gamma v, \Gamma v\rangle > 0$. Note that $K = \Upsilon M_\Gamma$ and $(M_\Gamma + M_\Gamma \Upsilon M_\Gamma)$ is a symmetric operator (see Section 1 for definitions of Υ and M_Γ.) Therefore,

$$\begin{aligned}\big\langle \big(M_\Gamma + zM_\Gamma\Upsilon M_\Gamma\big)v, v\big\rangle &= \big\langle M_\Gamma v, v\big\rangle + z\big\langle M_\Gamma\Upsilon M_\Gamma v, v\big\rangle \\ &= \langle \Gamma v, v\rangle + z\langle \Upsilon\Gamma v, \Gamma v\rangle.\end{aligned} \tag{5.1}$$

Since Γ is a nonnegative matrix, $\langle \Gamma v, v\rangle \geq 0$, and because Υ is a strictly positive operator, $\langle \Upsilon\Gamma v, \Gamma v\rangle > 0$. If $\mathrm{Re}(z) > 0$,

$$\mathrm{Re}\big(\langle M_\Gamma v, v\rangle + z\langle \Upsilon M_\Gamma v, M_\Gamma v\rangle\big) > 0. \tag{5.2}$$

Otherwise, $\mathrm{Im}(z) \neq 0$ and hence we have

$$\mathrm{Im}(z)\langle \Upsilon M_\Gamma v, M_\Gamma v\rangle \neq 0. \tag{5.3}$$

Either way, if $\mathrm{Re}(z) \geq 0$,

$$(M_\Gamma + zM_\Gamma \Upsilon M_\Gamma)v = M_\Gamma(I + z\Upsilon M_\Gamma)v \tag{5.4}$$

is nonzero, and therefore $(I + z\Upsilon M_\Gamma)v$ is nonzero. Thus for any nonzero $L^2[0,T]$ function v, $(I + z\Upsilon M_\Gamma)v$ is never zero and since ω is arbitrary, hence $I + zK_X$ is invertible for any $\omega \in \Omega$. □

Proposition 5.2. *Suppose that the usual assumptions on* Γ *and* H *hold. Then* $(I + K_{X,s})^{-1}$ *exists for all* $0 \leq s < e(X)$. *Furthermore,* (3.10) *holds for all* $0 \leq s < e(X)$.

Proof. Under the assumptions on Γ and H, Proposition 3.1 and Remark 3.4 will imply Proposition 5.1. Fix an $\omega \in \Omega$, a $T < e(X(\omega))$ and let C_T be as defined in Lemma 2.4. Then on $U = \{z \mid |z| < 1/C_T\}$, $(I + zK_{X,s})$ is invertible for all $s \in [0,T]$. Hence $O = U \cup \{z \mid \mathrm{Re}(z) > 0\}$ is an open-connected set containing 0, and $(I + zK_{X,s})^{-1}$ exists for all $s \in [0,T]$. In particular, at $z = 1$. Then the assumptions in Corollary 2.9 are met, and hence (3.10) holds. □

6. Proof of main result

The proof of Theorem 1.2 now follows from Theorem 4.2 and Proposition 5.2. Integrating (3.10), we have for $T < e(X)$,

$$\log\det(I + K_{X,T}) = \int_0^T \mathrm{tr}\,(H(s)\Gamma(X_s)\,ds - Z_s\Gamma(X_s))\,ds. \tag{6.1}$$

For $(t,x,v) \in [0,\infty) \times M \times \mathcal{S}_N(\mathbb{R})$, define

$$\Psi(T,t,x,v) = \mathbb{E}_{(t,x,v)}\left[f(X_T)\exp\left(-\int_0^T V(X_s)\; - p\,\mathrm{tr}\,(H(s)\Gamma(X_s) - Z_s\Gamma(X_s))\,ds\right)\right] \tag{6.2}$$

and observe that

$$\Psi(T,0,x,0) = \mathbb{E}_x\left[f(X_T)e^{-\int_0^T V(X_s)\,ds}(\det(I + K_{X,T}))^p\right] = \lambda(T,x). \tag{6.3}$$

By the Feynman-Kac formula, Ψ satisfies the following partial differential equation:

$$\frac{\partial}{\partial T}\Psi(T,t,x,v) = \widehat{H}\Psi(T,s,x,v). \tag{6.4}$$

Thus

$$\Psi(T,t,x,v) = \left(e^{T\widehat{H}} f\right)(t,x,v), \tag{6.5}$$

and therefore from (6.3),

$$\lambda(T,x) = e^{T\widehat{H}} f(0,x,0). \tag{6.6}$$

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