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Research Article

Strong Convergence of Viscosity Methods for Continuous Pseudocontractions in Banach Spaces

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We define a viscosity method for continuous pseudocontractive mappings defined on closed and convex subsets of reflexive Banach spaces with a uniformly Gâteaux differentiable norm. We prove the convergence of these schemes improving the main theorems in the work by Y. Yao et al. (2007) and H. Zhou (2008).

1. Introduction

Let X be a real Banach space and let J be the normalized duality mapping from X into 2^{X^*} defined by

$$Jx = \{j(x) \in X^* : \langle x, j(x)\rangle = \|x\|^2, \|j(x)\| = \|x\|\}, \tag{1.1}$$

where X^* denotes the dual space of X and $\langle\cdot,\cdot\rangle$ the generalized duality pairing between X and X^*.

Recall that if

$$\lim_{t\to 0}\frac{\|x+th\|-\|x\|}{t} \tag{1.2}$$

exists for each x and h on the unit sphere S_X of X, the norm of X is *Gâteaux differentiable*. Moreover, if for each $h \in S_X$ the limit defined by (1.2) is uniformly attained for $x \in S_X$, we say that the norm of X is *uniformly Gâteaux differentiable*.

Definition 1.1. A mapping $T : D(T) \to X$ is said to be *k-pseudocontractive* ($k \in \mathbb{R}$) if, for every $x, y \in D(T)$, there exist some $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq k\|x-y\|^2. \tag{1.3}$$

In the inequality (1.3), if $0 < k < 1$, we say that T is *strongly pseudocontractive*. For $k = 1$, T is called *pseudocontractive* mapping.

Among classes of nonlinear mappings, the class of pseudocontractions is probably one of the most important classes of mappings. This happens because of the corresponding relation between the classes of pseudocontractions and accretive operators. In fact, a mapping $A : D(A) \to X$ is accretive (i.e., $\langle Ax - Ay, j(x-y)\rangle \geq 0$, for all $x, y \in D(A)$) if and only if $T := I - A$ is pseudocontractive.

Let T, V be two opportune mappings from C to C, where C is a closed and convex subset of a Banach space X. Consider the variational inequality problem of finding a fixed point x^* of T, with respect to another mapping V, to satisfy the inequality

$$\langle x^* - Vx^*, j(y - x^*)\rangle \geq 0, \quad \forall y \in \mathrm{Fix}(T). \tag{1.4}$$

A particular case occurs when $V = f$ with f a ρ-contraction (i.e., $\|f(x) - f(y)\| \leq \rho\|x-y\|$ for all $x, y \in C$). In this case, the method (implicit or explicit) that permits to solve the variational inequality problem is known as viscosity approximation method. It was first studied by Moudafi [1] in Hilbert spaces and further developed by Xu [2] in more general setting.

Next results, due to Morales [3] (2007), are the more general results concerning the convergence of implicit viscosity methods for continuous pseudocontractive mappings.

In particular, the author studies the convergence of the path defined as

$$x_t = tf(x_t) + (1-t)Tx_t, \quad t \in (0,1), \tag{1.5}$$

in more setting of Banach spaces and in more large class of mappings f including the ρ-contraction mappings.

Theorem 1.2 (see [3]). *Let C be a nonempty closed convex subset of a reflexive Banach space X with a uniformly Gâteaux differentiable norm. Let T and $f : C \to C$ be pseudocontractive and strongly pseudocontractive continuous mappings, respectively. Suppose that every closed, bounded, and convex subset of C has the fixed point property for nonexpansive self-mappings. If the sets*

$$E := \{x \in C : Tx = \lambda x + (1-\lambda)f(x) \text{ for some } \lambda > 1\} \tag{1.6}$$

and $f(E)$ are bounded, then the path $(x_t)_{t\in(0,1)}$ described by

$$x_t = (1-t)Tx_t + tf(x_t) \tag{1.7}$$

strongly converges, as $t \to 0$*, to a fixed point* q *of* T *which is the unique solution of the variational inequality*

$$\langle q - f(q), j(q - z)\rangle \geq 0 \quad \forall z \in Fix(T). \tag{1.8}$$

Corollary 1.3 (see [3]). *Let* C *be a nonempty closed convex subset of a uniformly smooth Banach space* X*. Let* $T, f : C \to C$ *be a pseudocontractive and strongly pseudocontractive continuous mappings. If* f *is bounded and* T *admits at least a fixed point, then the path* $(x_t)_{t\in(0,1)}$ *described by*

$$x_t = (1 - t)Tx_t + tf(x_t) \tag{1.9}$$

strongly converges, as $t \to 0$*, to a fixed point* q *of* T *that is the unique solution of the variational inequality*

$$\langle q - f(q), j(q - z)\rangle \geq 0, \quad \forall z \in Fix(T). \tag{1.10}$$

Also in 2007, H. Zegeye et al. in [4] proved a convergence theorem of viscosity approximation methods for continuous pseudocontractive mappings in reflexive and strictly convex Banach spaces.

Theorem 1.4 (see [4]). *Let* C *be a nonempty closed and convex subset of a real Banach space* X *reflexive, strictly convex that has uniformly Gâteaux differentiable norm. Let* $T : C \to C$ *be a continuous pseudocontractive mapping and* $f : C \to C$ *be a* ρ*-contraction. Suppose further that* $Fix(T) \neq \emptyset$*. Then,* $(x_t)_{t\in(0,1)}$ *strongly converges, as* $t \to 0$*, to a fixed point* q *of* T *that is the unique solution of the variational inequality* (1.8).

Another interesting implicit-type Halpern algorithm has been recently introduced by Yao, Liou, and Chen in uniformly smooth Banach spaces.

Theorem 1.5 (see [5]). *Let* C *be a closed and convex subset of a real uniformly smooth Banach space* X*. Let* $T : C \to C$ *be a continuous pseudocontractive mapping. Let* $(\alpha_n)_{n\in\mathbb{N}}$*,* $(\beta_n)_{n\in\mathbb{N}}$*, and* $(\gamma_n)_{n\in\mathbb{N}}$ *be three real sequences in* $(0, 1)$ *satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to+\infty}(\alpha_n/\beta_n) = \lim_{n\to+\infty}\beta_n = 0$;

(iii) $\sum_{n\in\mathbb{N}}(\alpha_n/(\beta_n + \alpha_n)) = +\infty$.

Then, for arbitrary initial value $x_0 \in C$ *and a fixed* $u \in C$*, the sequence* $(x_n)_{n\in\mathbb{N}}$ *defined by*

$$x_n = \alpha_n u + \beta_n x_{n-1} + \gamma_n T x_n \tag{1.11}$$

strongly converges to a fixed point of T*.*

In our first result (Theorem 2.1), we prove the strong convergence of the viscosity implicit approximation method

$$x_n = \alpha_n f(x_n) + \beta_n x_{n-1} + \gamma_n T x_n, \tag{1.12}$$

where $f : C \to C$ is a continuous strongly pseudocontractive mapping. This result has as particular case Theorem 1.5 when f is a constant mapping.

On the other hand, on the idea of the implicit scheme (1.11), Zhou in [6] defines a Halpern explicit method for suitable continuous pseudocontractive mappings. Fixing an element $u \in C$ and an initial point $x_0 = x_0^0 \in C$, he constructs elements $(x_n^m)_{n\in N}$ as follows:

$$x_n^{m+1} = \alpha_n u + \beta_n x_n + \gamma_n T x_n^m, \quad m = 0, 1, 2, \ldots. \tag{1.13}$$

If for any $n \geq 0$ the continuous pseudocontractive mapping T admits an integer m that satisfies the following condition:

$$\left\| T x_n^{m+1} - T x_n^m \right\| \leq \gamma_n^{-1}(1-\gamma_n)\epsilon_n, \tag{1.14}$$

then he defines iteratively a sequence $(x_n)_{n\geq 0}$ as follows: called $N(n)$ the least positive integer m satisfying (1.14),

$$x_{n+1} = x_{n+1}^0 = x_n^{N(n)+1} = \alpha_n u + \beta_n x_n + \gamma_n T x_n^{N(n)}, \tag{1.15}$$

and he proves the convergence for this explicit method. Of course if $x_n^{N(n)} = x_n^{N(n)+1}$, one reobtains the implicit method (1.11).

In our second theorem, we improve Zhou's result [6] to the viscosity setting. In both proofs, we use Morales's Theorem 1.2.

Let us conclude this section by two lemmas that are useful in many convergence results.

Following the proof of Theorem 2.3 in [7], one can show the following.

Lemma 1.6. *Let C be a nonempty closed convex subset of a real Banach space X with a uniformly Gâteaux differentiable norm, let $T, f : C \to C$ be a pseudocontractive and strongly pseudocontractive continuous mappings with $Fix(T) \neq \emptyset$. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence such that* $\lim_{n\to+\infty}\|x_n - Tx_n\| = 0$. *Define, for all $t \in (0,1)$,*

$$x_t = tf(x_t) + (1-t)Tx_t, \tag{1.16}$$

and let us suppose that $q = \lim_{t\to 0} x_t$ exists.

Then,

$$\limsup_{n\to+\infty} \langle f(q) - q, j(x_n - q) \rangle \leq 0. \tag{1.17}$$

The following lemma on real sequences can be found in Liu [8].

Lemma 1.7. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers satisfying the following inequality:*

$$a_{n+1} \leq (1-t_n)a_n + o(t_n) + s_n, \quad \forall n \geq 0, \tag{1.18}$$

where $(t_n)_{n\in N}$ *is a sequence in* $]0,1[$ *such that* $\sum_{n\geq 0} t_n = +\infty$ *and* $(s_n)_{n\in N}$ *is a summable sequence of positive numbers.*

Then, $(a_n)_{n\in\mathbb{N}}$ *converges to zero.*

2. Convergence results

In this section, we prove the convergence's theorems on implicit and explicit viscosity method.

Theorem 2.1. *Let* C *be a nonempty closed convex subset of a reflexive Banach space* X *with a uniformly Gâteaux differentiable norm. Suppose that every closed, bounded, and convex subset of* C *has the fixed point property for nonexpansive self-mappings. Let* $T : C \to C$ *be a continuous pseudocontractive mapping and let* $f : C \to C$ *be a continuous strongly pseudocontractive mapping (with constant* $0 < k < 1$*) such that the sets*

$$E := \{x \in C : Tx = \lambda x + (1-\lambda) f(x) \text{ for some } \lambda > 1\} \tag{2.1}$$

and $f(E)$ *are bounded.*

Let $(\alpha_n)_{n\in\mathbb{N}}$*,* $(\beta_n)_{n\in\mathbb{N}}$*, and* $(\gamma_n)_{n\in\mathbb{N}}$ *be three real sequences in* $(0,1)$ *satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to+\infty}\alpha_n = \lim_{n\to+\infty}\beta_n = 0$;

(iii) $\sum_{n\in\mathbb{N}}(\alpha_n/(\beta_n + (1-k)\alpha_n)) = +\infty$.

For arbitrary initial point, $x_0 \in C$ *and a fixed* $n \geq 0$*, we construct elements* $(x_n)_{n\in\mathbb{N}}$ *as follows:*

$$x_n = \alpha_n f(x_n) + \beta_n x_{n-1} + \gamma_n T x_n. \tag{2.2}$$

Then, $(x_n)_{n\in\mathbb{N}}$ *strongly converges to* q*, where* $q \in \mathit{Fix}(T)$ *is the unique solution of* (1.8)

$$\langle q - f(q), j(q-p)\rangle \geq 0, \quad \forall p \in \mathit{Fix}(T). \tag{2.3}$$

Proof. First of all, from [3], it follows that $\mathrm{Fix}(T) \neq \emptyset$.

Now, we verify that the sequence $(x_n)_{n\in\mathbb{N}}$ exists.

We prove that, for fixed $\alpha, \beta, \gamma \in (0,1)$ with $\alpha + \beta + \gamma = 1$ and $z \in C$, the map

$$Sx = \alpha f(x) + \beta z + \gamma T x \tag{2.4}$$

has a unique fixed point. By Deimling [9], it is enough to show that $S : C \to C$ is strongly pseudocontractive and continuous. Now,

$$\begin{aligned}\langle Sx - Sy, j(x-y)\rangle &= \langle \alpha(f(x) - f(y)) + \gamma(Tx - Ty), j(x-y)\rangle \\ &= \alpha\langle f(x) - f(y), j(x-y)\rangle + \gamma\langle Tx - Ty, j(x-y)\rangle \\ &\leq \alpha k\|x-y\|^2 + \gamma\|x-y\|^2 = (\alpha k + \gamma)\|x-y\|^2.\end{aligned} \tag{2.5}$$

Since $(k\alpha+\gamma)<1$, then S is a strongly pseudocontractive. To prove the claim of the theorem, we show firstly that $(x_n)_{n\in\mathbb{N}}$ is bounded.

Picking $p\in \mathrm{Fix}(T)$, we have

$$\begin{aligned}\|x_n-p\|^2 &= \alpha_n\langle f(x_n)-p, j(x_n-p)\rangle + \beta_n\langle x_{n-1}-p, j(x_n-p)\rangle + \gamma_n\langle Tx_n-p, j(x_n-p)\rangle\\ &= \alpha_n\langle f(x_n)-f(p), j(x_n-p)\rangle + \alpha_n\langle f(p)-p, j(x_n-p)\rangle\\ &\quad+\beta_n\langle x_{n-1}-p, j(x_n-p)\rangle + \gamma_n\langle Tx_n-p, j(x_n-p)\rangle\\ &\le [k\alpha_n+(1-\alpha_n-\beta_n)]\|x_n-p\|^2+\alpha_n\|f(p)-p\|\|x_n-p\|\\ &\quad+\beta_n\|x_{n-1}-p\|\|x_n-p\|\end{aligned} \tag{2.6}$$

which implies that

$$\|x_n-p\|\le \frac{\alpha_n}{\alpha_n(1-k)+\beta_n}\|f(p)-p\|+\frac{\beta_n}{\alpha_n(1-k)+\beta_n}\|x_{n-1}-p\|. \tag{2.7}$$

By a simple induction, we get that

$$\|x_n-p\|\le \max\left\{\frac{\|f(p)-p\|}{1-k}, \|x_0-p\|\right\}. \tag{2.8}$$

Moreover, we have that $\lim_{n\to+\infty}\|x_n-Tx_n\|=0$.

In fact,

$$\begin{aligned}\|x_n-Tx_n\| &= \|\alpha_n f(x_n)+\beta_n x_{n-1}-(\alpha_n+\beta_n)Tx_n\|\\ &\le \alpha_n\|f(x_n)-Tx_n\|+\beta_n\|x_{n-1}-Tx_n\|,\end{aligned} \tag{2.9}$$

and by boundedness of $(x_n)_{n\in\mathbb{N}}$ and condition (ii), it follows the statement.

Let, for every $t\in(0,1)$,

$$x_t = tf(x_t)+(1-t)Tx_t. \tag{2.10}$$

By Morales's Theorem 1.2, this implicit method converges to a unique point $q\in\mathrm{Fix}(T)$ that is the unique solution of (1.8). Next, we show that $x_n\to q$.

By Lemma 1.6, we obtain

$$\limsup_{n\to\infty}\langle f(q)-q, j(x_n-q)\rangle\le 0; \tag{2.11}$$

then, if we define the real sequence

$$\sigma_n := \max\{0, \langle f(q)-q, j(x_n-q)\rangle\}, \tag{2.12}$$

we can show that $\sigma_n\ge 0$ and $\lim_{n\to+\infty}\sigma_n=0$.

So we conclude

$$\begin{aligned}
\|x_n - q\|^2 &= \alpha_n\langle f(x_n) - q, j(x_n - q)\rangle + \beta_n\langle x_{n-1} - q, j(x_n - q)\rangle + \gamma_n\langle Tx_n - q, j(x_n - q)\rangle \\
&\le \alpha_n\langle f(x_n) - f(q), j(x_n - q)\rangle + \alpha_n\langle f(q) - q, j(x_n - q)\rangle \\
&\quad + \beta_n\|x_{n-1} - q\|\|x_n - q\| + +\gamma_n\|x_n - q\|^2 \\
&\le \left(k\alpha_n + \frac{\beta_n}{2} + \gamma_n\right)\|x_n - q\|^2 + \frac{\beta_n}{2}\|x_{n-1} - q\|^2 + \alpha_n\sigma_n \\
&= \left[(k-1)\alpha_n - \frac{\beta_n}{2} + 1\right]\|x_n - q\|^2 + \frac{\beta_n}{2}\|x_{n-1} - q\|^2 + \alpha_n\sigma_n \\
&\le \left[(k-1)\frac{\alpha_n}{2} - \frac{\beta_n}{2} + 1\right]\|x_n - q\|^2 + \frac{\beta_n}{2}\|x_{n-1} - q\|^2 + \alpha_n\sigma_n,
\end{aligned} \tag{2.13}$$

which implies that

$$\begin{aligned}
\|x_n - q\|^2 &\le \frac{\beta_n}{(1-k)\alpha_n + \beta_n}\|x_{n-1} - q\|^2 + \frac{2\alpha_n\sigma_n}{(1-k)\alpha_n + \beta_n} \\
&= \left[1 - \frac{(1-k)\alpha_n}{(1-k)\alpha_n + \beta_n}\right]\|x_{n-1} - q\|^2 + \frac{2\alpha_n\sigma_n}{(1-k)\alpha_n + \beta_n}.
\end{aligned} \tag{2.14}$$

By Liu's Lemma 1.7 and condition (iii), we obtain that $x_n \to q$, as $n \to \infty$. □

In the next theorem, we consider a viscosity explicit method which extends (1.15) substituting the constant u with a ρ-contraction f, and we establish a convergence's result for this scheme.

Theorem 2.2. *Let C be a nonempty closed convex subset of a reflexive Banach space X with a uniformly Gâteaux differentiable norm. Suppose that every closed, bounded, and convex subset of C has the fixed point property for nonexpansive self-mappings. Let $T : C \to C$ be a continuous pseudocontractive mapping and let $f : C \to C$ be a ρ-contraction such that the set*

$$E := \{x \in C : Tx = \lambda x + (1-\lambda)f(x) \text{ for some } \lambda > 1\} \tag{2.15}$$

is bounded.

Let $(\alpha_n)_{n\in N}$, $(\beta_n)_{n\in N}$, and $(\gamma_n)_{n\in N}$ be three real sequences in $(0,1)$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to+\infty}\alpha_n = \lim_{n\to+\infty}\beta_n = 0$;

(iii) $\sum_{n\in\mathbb{N}}(\alpha_n/(1-\gamma_n)) = +\infty$.

Let $(\epsilon_n)_{n\ge 0}$ be a summable sequence of positive numbers.

For arbitrary initial point $x_0 = x_0^0 \in C$ *and a fixed* $n \geq 0$, *we construct elements* $(x_n^m)_{n \in N}$ *as follows:*

$$x_n^{m+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T x_n^m, \quad m = 0, 1, 2, \ldots. \tag{2.16}$$

Suppose that there exists $N(n)$, *the least positive integer satisfying the following condition:*

$$\left\| T x_n^{N(n)+1} - T x_n^{N(n)} \right\| \leq \gamma_n^{-1} (1 - \gamma_n) \epsilon_n. \tag{2.17}$$

Then, $(x_n)_{n \in \mathbb{N}}$ *defined as*

$$x_{n+1} = x_{n+1}^0 = x_n^{N(n)+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T x_n^{N(n)} \tag{2.18}$$

strongly converges to q, *where* $q \in Fix(T)$ *is the unique solution of* (1.8),

$$\langle q - f(q), j(q - p) \rangle \geq 0 \quad \forall p \in Fix(T). \tag{2.19}$$

Proof. We divide the proof into three steps.

Step 1. $(x_n)_{n \in \mathbb{N}}$ is bounded.

Proof of Step 1. Picking $p \in \mathrm{Fix}(T)$, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \alpha_n \langle f(x_n) - p, j(x_{n+1} - p) \rangle + \beta_n \langle x_n - p, j(x_{n+1} - p) \rangle \\
&\quad + \gamma_n \langle T x_n^{N(n)} - p, j(x_{n+1} - p) \rangle \\
&= \alpha_n \langle f(x_n) - f(p), j(x_{n+1} - p) \rangle + \alpha_n \langle f(p) - p, j(x_{n+1} - p) \rangle \\
&\quad + \beta_n \langle x_n - p, j(x_{n+1} - p) \rangle + \gamma_n \langle T x_n^{N(n)} - T x_n^{N(n)+1}, j(x_{n+1} - p) \rangle \\
&\quad + \gamma_n \langle T x_{n+1} - p, j(x_{n+1} - p) \rangle \\
&\leq (\rho \alpha_n + \beta_n) \|x_n - p\| \|x_{n+1} - p\| + \alpha_n \|f(p) - p\| \|x_{n+1} - p\| \\
&\quad + (1 - \gamma_n) \epsilon_n \|x_{n+1} - p\| + \gamma_n \|x_{n+1} - p\|^2,
\end{aligned} \tag{2.20}$$

which implies that

$$\|x_{n+1} - p\| \leq \frac{\rho \alpha_n + \beta_n}{1 - \gamma_n} \|x_n - p\| + \frac{\alpha_n}{1 - \gamma_n} \|f(p) - p\| + \epsilon_n. \tag{2.21}$$

By a simple induction, we get that

$$\|x_n - p\| \leq \max \left\{ \|x_0 - p\|, \frac{\|f(p) - p\|}{1 - \rho} \right\} + \sum_{k=0}^{n-1} \epsilon_k. \tag{2.22}$$

□

Step 2. $\lim_{n\to+\infty}\|x_{n+1} - Tx_{n+1}\| = 0$.

Proof of Step 2. Since

$$\begin{aligned}\|x_{n+1} - Tx_{n+1}\| &\le \alpha_n\|f(x_n) - Tx_{n+1}\| + \beta_n\|x_n - Tx_{n+1}\| + \gamma_n\|Tx_n^{N(n)} - Tx_n^{N(n)+1}\| \\ &\le \alpha_n\|f(x_n) - Tx_{n+1}\| + \beta_n\|x_n - Tx_{n+1}\| + \epsilon_n,\end{aligned} \tag{2.23}$$

by the boundedness of $(x_n)_{n\in\mathbb{N}}$, condition (ii), and the summability of $(\epsilon_n)_{n\in\mathbb{N}}$, we obtain the claim. □

Step 3. $\lim_{n\to+\infty}x_n = q$.

Proof of Step 3. As in Theorem 2.1, set

$$q := \lim_{t\to 0} x_t, \tag{2.24}$$

where $x_t = tf(x_t) + (1-t)Tx_t$, and

$$\sigma_n := \max\{0, \langle f(q) - q, j(x_{n+1} - q)\rangle\}. \tag{2.25}$$

We known that $\lim_{n\to+\infty}\sigma_n = 0$; now we show that $x_n \to q$. In fact,

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \alpha_n\langle f(x_n) - q, j(x_{n+1} - q)\rangle + \beta_n\langle x_n - q, j(x_{n+1} - q)\rangle + \gamma_n\langle Tx_n^{N(n)} - q, j(x_{n+1} - q)\rangle \\ &\le \alpha_n\langle f(x_n) - f(q), j(x_{n+1} - q)\rangle + \alpha_n\langle f(q) - q, j(x_{n+1} - q)\rangle \\ &\quad + \beta_n\langle x_n - q, j(x_{n+1} - q)\rangle + \gamma_n\langle Tx_n^{N(n)} - Tx_n^{N(n)+1}, j(x_{n+1} - q)\rangle + \gamma_n\|x_{n+1} - q\|^2 \\ &\le (\rho\alpha_n + \beta_n)\|x_n - q\|\|x_{n+1} - q\| + \alpha_n\sigma_n + (1-\gamma_n)\epsilon_n\|x_{n+1} - q\| + \gamma_n\|x_{n+1} - q\|^2 \\ &\le \frac{\rho\alpha_n + \beta_n}{2}\|x_n - q\|^2 + \frac{\alpha_n + \beta_n}{2}\|x_{n+1} - q\|^2 + \alpha_n\sigma_n \\ &\quad + (1-\gamma_n)\epsilon_n\|x_{n+1} - q\| + \gamma_n\|x_{n+1} - q\|^2 \\ &= \frac{\rho\alpha_n + \beta_n}{2}\|x_n - q\|^2 + \frac{1 + \gamma_n}{2}\|x_{n+1} - q\|^2 + \alpha_n\sigma_n + (1-\gamma_n)\epsilon_n\|x_{n+1} - q\|^2\end{aligned} \tag{2.26}$$

which implies that

$$\begin{aligned}\|x_{n+1} - q\|^2 &\le \frac{(1-\gamma_n) - (1-\rho)\alpha_n}{1-\gamma_n}\|x_n - q\|^2 + \frac{2\alpha_n\sigma_n}{1-\gamma_n} + 2d\epsilon_n, \quad d := \sup_{n\in\mathbb{N}}\|x_n - q\| \\ &= \left(1 - \frac{(1-\rho)\alpha_n}{1-\gamma_n}\right)\|x_n - q\|^2 + \frac{2\alpha_n\sigma_n}{1-\gamma_n} + 2d\epsilon_n.\end{aligned} \tag{2.27}$$

□

By Liu's Lemma 1.7, we obtain that $x_n \to q$, as $n \to \infty$. □

Remark 2.3. We can prove that if T is a nonexpansive mapping and x_n^m is defined as (2.32) of Theorem 2.2, then there always exists a positive integer $N(n)$ satisfying

$$\left\|Tx_n^{N(n)+1} - Tx_n^{N(n)}\right\| \leq \frac{(1-\gamma_n)}{\gamma_n}\epsilon_n. \tag{2.28}$$

In fact, fixed $n \in \mathbb{N} \cup \{0\}$, for every $k \in \mathbb{N} \cup \{0\}$, we have

$$\left\|Tx_n^{k+1} - Tx_n^k\right\| \leq \gamma_n^k \left\|x_n^1 - x_n^0\right\|. \tag{2.29}$$

If $x_n^1 = x_n^0$, we are done. Otherwise, since $0 < \gamma_n < 1$, it follows that there exists a sufficiently large $k = N(n) \in \mathbb{N} \cup \{0\}$ such that

$$\gamma_n^k \leq \frac{(1-\gamma_n)\epsilon_n}{\gamma_n \left\|x_n^1 - x_n^0\right\|}. \tag{2.30}$$

It is also well known [3, 5, 10] that if $T : C \to C$ is a continuous pseudocontractive mapping, defining the mapping $g : C \to C$ as $g(x) = (2I - T)^{-1}(x)$, we can observe that the following hold:

(1) g is a nonexpansive mapping;

(2) $\mathrm{Fix}(T) = \mathrm{Fix}(g)$;

(3) $\|x - g(x)\| \leq \|x - Tx\|$, for all $x \in C$.

By Remark 2.3 and Theorem 2.2, we have the following.

Corollary 2.4. *Let C be a nonempty closed convex subset of a real reflexive Banach space X with a uniformly Gâteaux differentiable norm. Suppose that every closed, bounded, and convex subset of C has the fixed point property for nonexpansive self-mappings. Let $T : C \to C$ be a continuous pseudocontractive mapping and let $f : C \to C$ be a ρ-contraction such that the set*

$$E := \{x \in C : Tx = \lambda x + (1-\lambda) f(x) \text{ for some } \lambda > 1\} \tag{2.31}$$

is bounded.

Let $(\alpha_n)_{n\in N}$, $(\beta_n)_{n\in N}$, and $(\gamma_n)_{n\in N}$ be three real sequences in $(0,1)$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to+\infty}\alpha_n = \lim_{n\to+\infty}\beta_n = 0$;

(iii) $\sum_{n\in\mathbb{N}}(\alpha_n/(1-\gamma_n)) = +\infty$.

Let $(\epsilon_n)_{n\geq 0}$ be a summable sequence of positive numbers.

For arbitrary initial point $x_0 = x_0^0 \in C$ and a fixed $n \geq 0$, we construct elements $(x_n^m)_{n\in N}$ as follows (here, as above, $g = (2I - T)^{-1}$):

$$x_n^{m+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n g(x_n^m), \quad m = 0, 1, 2, \dots, N(n), \tag{2.32}$$

and we define $(x_n)_{n\in\mathbb{N}}$ *as*

$$x_{n+1} = x_{n+1}^0 = \alpha_n f(x_n) + \beta_n x_n + \gamma_n g(x_n^{N(n)}), \tag{2.33}$$

where for every $n \in \mathbb{N}$, $N(n)$ *is the positive integer such that*

$$\|g(x_n^{N(n)+1}) - g(x_n^{N(n)})\| \leq \frac{(1-\gamma_n)}{\gamma_n}\epsilon_n. \tag{2.34}$$

Then, $(x_n)_{n\in\mathbb{N}}$ *converges strongly to* $q \in Fix(T)$, *where* q *is the unique solution of* (1.8),

$$\langle q - f(q), j(q-p)\rangle \geq 0 \quad \forall p \in Fix(T). \tag{2.35}$$

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References

[1] A. Moudafi, "Viscosity approximation methods for fixed-points problems," *Journal of Mathematical Analysis and Applications*, vol. 241, no. 1, pp. 46–55, 2000.

[2] H.-K. Xu, "Viscosity approximation methods for nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 279–291, 2004.

[3] C. H. Morales, "Strong convergence of path for continuous pseudo-contractive mappings," *Proceedings of the American Mathematical Society*, vol. 135, no. 9, pp. 2831–2838, 2007.

[4] H. Zegeye, N. Shahzad, and T. Mekonen, "Viscosity approximation methods for pseudocontractive mappings in Banach spaces," *Applied Mathematics and Computation*, vol. 185, no. 1, pp. 538–546, 2007.

[5] Y. Yao, Y.-C. Liou, and R. Chen, "Strong convergence of an iterative algorithm for pseudocontractive mapping in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 12, pp. 3311–3317, 2007.

[6] H. Zhou, "Viscosity approximation methods for pseudocontractive mappings in Banach spaces," *Nonlinear Analysis*, 2008.

[7] Y. Song and R. Chen, "Convergence theorems of iterative algorithms for continuous pseudocontractive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 2, pp. 486–497, 2007.

[8] L. S. Liu, "Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 194, no. 1, pp. 114–125, 1995.

[9] K. Deimling, "Zeros of accretive operators," *Manuscripta Mathematica*, vol. 13, no. 4, pp. 365–374, 1974.

[10] W. Takahashi, *Nonlinear Functional Analysis. Fixed Point Theory and Its Applications*, Yokohama Publishers, Yokohama, Japan, 2000.