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Research Article

On the Optimality of (s, S) Inventory Policies: A Quasivariational Approach

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This paper revisits the classical discrete-time stationary inventory model. A new proof, based on the theory of quasivariational inequality (QVI), of the optimality of (s, S) policy is presented. This proof reveals a number of interesting properties of the optimal cost function. Further, the proof could be used as a tutorial for applications of QVI to inventory control.

1. Introduction

Consider an inventory model which consists in controlling the level of stock of a single product where the demands $D_1, D_2, \ldots$ for the product in periods $1, 2, \ldots$ are independently and identically distributed (i.i.d) random variables with density function ψ, and finite mean $\mu < \infty$.

Assume that at the beginning of each period the system is reviewed and we are allowed to increase the level of stock to any level we wish. Orders are assumed to be delivered immediately.

Let f be a real-valued function representing the holding and shortage cost with $f(0) = 0$ and $f(x) > 0$ for $x \neq 0$. The cost $c(x)$ of ordering an amount x is given by

$$c(x) = \begin{cases} k + cx, & x > 0, \\ 0, & x = 0, \end{cases} \tag{1.1}$$

where c is the unit cost of the item and k is the set-up cost ($c > 0,\ k > 0$). Costs are assumed to be additive and geometrically discounted at a rate $\alpha,\ 0 < \alpha < 1$, and that unmet demand is completely backlogged.

An admissible replenishment policy consists of a sequence (t_i, ξ_i), $i = 1, \dots,$ where t_i represents the ith time of ordering and $\xi_i > 0$ represents the quantity ordered at time t_i. Write

$$\begin{gathered} \mathcal{V}_n = \{(t_i, \xi_i),\ i = 1, \dots n\}, \\ \mathcal{V}_\infty = \lim_{n\to\infty} \mathcal{V}_n = \mathcal{V}. \end{gathered} \tag{1.2}$$

Let $x(n)$ denote the level of stock at time n, $n = 0, 1, \dots,$ and let $\mathcal{F}_n = \sigma\{x(s),\ s \le n\}$ be the σ-algebra generated by the history of the inventory level up to time n. Assume that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is $\mathcal{F}_n$-measurable. Then for a given initial inventory level x and an ordering policy $\mathcal{V}$, the infinite horizon discounted cost is defined by

$$y(x, \mathcal{V}) := E_{\mathcal{V}} \left\{ \sum_{t=0}^{\infty} \alpha^t f(x(t)) + \sum_{i=1}^{n} \alpha^{t_i} c(\xi_i) \right\}, \tag{1.3}$$

where the expectation is taken with respect to all possible realizations of the process $x(t)$ under policy $\mathcal{V}$. Set

$$y(x) := \inf_{\mathcal{V}} y(x, \mathcal{V}). \tag{1.4}$$

The objective is to find an admissible policy $\mathcal{V}^*$ such that $y(x, \mathcal{V}^*) = y(x)$.

Scarf [1] considered a finite horizon version of the problem described in (1.3). He showed using dynamic programming that if the one period expected holding plus shortage cost function is convex, then the optimal policy for period n is an (s_n, S_n) policy. The principal tool used by Scarf was a concept of K-convexity which he introduced in the same paper. Subsequently, Iglehart [2] extended Scarf's result to the infinite horizon case by showing that the property of K-convexity holds for the infinite period stationary model. Veinott [3] replaced the requirement of the convexity of the one period expected holding plus shortage cost by a quasiconvexity requirement and added other conditions. Again using dynamic programming, he showed the optimality of an (s, S) policy.

In this paper, we approach the problem of determining the optimal inventory policy as an impulse control problem, the theory which has been developed by Bensoussan and Lions [4]. Under this theory, the Bellman equation of dynamic programming for the inventory problem leads to a set of quasivariational inequalities (QVIs) whose solution leads to the optimal inventory policy. This approach leads to a new proof of the result which does not use K-convexity and is based on the examination of some properties of an integral equation. Previous applications of QVI to inventory control revolved around diffusion processes from which the machinery needed to prove optimality of (s, S) policy was not simple. This paper we hope can serve as a tutorial of applications of QVI to inventory control. Readers interested in applications of QVI to inventory control may consult [5–8].

Before we embark on the proof we will first formulate the problem described in (1.3) as a QVI.

Recall that $x(t)$ refers to the level of stock at time t, and consider all possible actions at time t.

(i) If no order is made, then it follows from (1.3) and (1.4) that

$$y(x(t)) \le E[f(x(t) - D)] + \alpha E[y(x(t) - D)] \tag{1.5}$$

or

$$y(x(t)) - \alpha E[y(x(t) - D)] \le E[f(x(t) - D)], \tag{1.6}$$

where D refers to the demand in a period.

(ii) If an order of size ξ is made, then the level of stock jumps from $x(t)$ to $x(t) + \xi$, and

$$y(x(t)) \le k + \inf_{\xi > 0} [c\xi + y(x(t) + \xi)]. \tag{1.7}$$

For $x \in \mathbb{R}$, define the operators A and M by

$$\begin{aligned} (Ay)(x) &:= y(x) - \alpha E[y(x - D)], \\ (My)(x) &:= k + \inf_{\xi > 0} [c\xi + y(x + \xi)]. \end{aligned} \tag{1.8}$$

It follows that the problem of finding the optimal solution to (1.3) reduces to solving the following QVI problem:

$$\begin{gathered} Ay \le F, \\ y \le My, \\ (Ay - F)(y - My) = 0, \end{gathered} \tag{1.9}$$

where

$$F(x) := E[f(x - D)]. \tag{1.10}$$

To solve the QVI given in (1.9), we examine an integral equation problem related to the QVI. This is done in Section 2. The properties obtained of the integral equation are then used to show the optimality of (s, S) policy in Section 3.

2. An integral equation problem

Consider the space of continuous functions $C(\mathbb{R})$. Assume that we are given a nonnegative function h in $C(\mathbb{R})$.

Further, suppose that

(A1) there exists $\gamma_h, -\infty < \gamma_h < \infty$, such that h is decreasing on $(-\infty, \gamma_h]$ and nondecreasing on $[\gamma_h, \infty)$;

(A2) $h(x) \to \infty$, as $|x| \to \infty$.

For L in $C(\mathbb{R})$, define the convolution operator $*$ by

$$(\psi * L)(x) = \int_0^\infty L(x - t)\psi(t)dt. \tag{2.1}$$

Now, consider the integral equation

$$\begin{aligned} L(x) - \alpha(\psi * L)(x) &= h(x), \quad x > s, \\ L(x) &= \frac{1}{1 - \alpha} h(s), \quad x \le s. \end{aligned} \tag{2.2}$$

Here, $s < \gamma_h$, and it is a free parameter.

Under assumption that h is in $C(\mathbb{R})$, the integral equation (2.2) has a unique solution in $C(\mathbb{R})$ (see [9]). Let L_s denote this solution. In what follows, there is a list of properties of L_s which will prove useful in showing the optimality of the (s, S) policy:

$$L_s(x) = \frac{1}{1-\alpha} h(s) \quad \forall x \leq s, \tag{2.3}$$

$$L_s \text{ is decreasing on } (s, \gamma_h], \tag{2.4}$$

$$L_s(x) \geq \frac{1}{1-\alpha} h(\gamma_h) \quad \forall x \text{ in } \mathbb{R}, \tag{2.5}$$

$$L_s(x) \longrightarrow \infty \quad \text{as } x \longrightarrow \infty. \tag{2.6}$$

Property (2.3) is the boundary condition of (2.2).

Proof of Property (2.4). To show (2.4) argue by contradiction. Assume that L_s initially does not decrease. In other words, there exists Δ, $s < \Delta < \gamma_h$ such that L_s is nondecreasing on $[s, \Delta)$. It follows that for x and t satisfying $s \leq x \leq \Delta$ and $t \geq 0$, $L_s(x) - L_s(x-t) \geq 0$; but (2.2) gives

$$(1-\alpha)L_s(\Delta) + \alpha \int_0^\infty (L_s(\Delta) - L_s(\Delta - t))\psi(t)dt = h(\Delta). \tag{2.7}$$

Therefore, $(1-\alpha)L_s(\Delta) \leq h(\Delta)$ or $L_s(\Delta) \leq (1/(1-\alpha))h(\Delta)$. This leads to $L_s(\Delta) < (1/(1-\alpha))h(s)$ by Assumption (A1). Property (2.3) then implies that $L_s(\Delta) < L_s(s)$, which leads to a contradiction.

To complete the proof, we again argue by contradiction and assume that there exists η, and Δ, $s < \eta < \Delta < \gamma_h$, such that L_s is decreasing on $[s, \eta]$ and nondecreasing on $[\eta, \Delta)$. Let x be such that $\eta < x < \Delta$, and $L_s(x) < L_s(s)$. We claim that for $t \geq 0$,

$$L_s(x) - L_s(x-t) \geq L_s(\eta) - L_s(\eta - t). \tag{2.8}$$

We have by (2.2)

$$(1-\alpha)L_s(x) + \alpha \int_0^\infty (L_s(x) - L_s(x-t))\psi(t)dt = h(x). \tag{2.9}$$

Now, use (2.8) and the fact that $L_s(x) \geq L_s(\eta)$ to get from (2.9) that

$$h(x) \geq (1-\alpha)L_s(\eta) + \alpha \int_0^\infty (L_s(\eta) - L_s(\eta - t))\psi(t)dt = h(\eta), \tag{2.10}$$

but $h(x) < h(\eta)$ by Assumption (A1). This leads to a contradiction. This ends the proof. □

Proof of Property (2.5). Assume that Property (2.5) is not true. Using Property (2.4) and the fact that L_s is continuous, let x^* be the first (smallest) solution of $L_s(x^*) = (1/(1-\alpha))h(\gamma_h)$. Clearly, $x^* > s$, and L_s attains its minimum at x^* on $(-\infty, x^*]$. Using (2.2), Assumption (A1), and recalling that ψ is a density function, we get

$$L_s(x^*) = h(x^*) + \alpha \int_0^\infty L_s(x^* - t)\psi(t)dt > h(x^*) + \alpha L_s(x^*). \tag{2.11}$$

Therefore, $L_s(x^*) > (1/(1-\alpha))h(\gamma_h)$. This leads to a contradiction. Whence Property (2.4) holds. □

Proof of Property (2.6). Using (2.2), we get

$$L_s(x) = h(x) + \alpha \int_0^\infty L_s(x-t)\psi(t)dt \geq h(x) + \frac{\alpha}{1-\alpha} h(\gamma_h). \tag{2.12}$$

The last inequality follows from Property (2.5). The result is then immediate from Assumption (A2) by taking the limit as $x \to \infty$. This completes the proof. □

We will next present further properties of L_s.

Theorem 2.1. *For a given $s < \gamma_h$, there exists an $S(s)$, $\gamma_h < S(s) < \infty$, which minimizes $L_s(x)$ for x in $\mathbb{R}$.*

Proof. The proof follows from Properties (2.3)–(2.6) and the continuity of L_s. □

We remark here that $S(s)$ may not be unique.

For $s < \gamma_h$, define

$$K(s) = L_s(s) - \min_{x \in \mathbb{R}} L_s(x). \tag{2.13}$$

Clearly, K is a well-defined function on $(-\infty, \gamma_h)$ and is nonnegative.

Lemma 2.2. *The function K is decreasing in s.*

Proof. Let $t < s < \gamma_h$, and for x in $\mathbb{R}$, define

$$D(x) := L_t(x) - L_s(x). \tag{2.14}$$

It is easy to show that D is a solution of (2.2) with the right-hand side changed to

$$g(x) = \begin{cases} h(t) - h(s), & x \leq t, \\ h(x) - h(s), & t \leq x \leq s, \\ 0, & x > s. \end{cases}$$

The function g is constant on $(-\infty, t]$, decreasing on $[t, s]$, and is equal to zero for $x > s$. Therefore, a similar argument to that used to show properties (2.4) and (2.5) shows that D is decreasing on $\mathbb{R}$ and is nonnegative. Since $S(s) > s > t$, it follows from Theorem 2.1 that

$$L_t(t) - L_s(t) \geq L_t(S(s)) - L_s(S(s)) \geq L_t(S(t)) - L_s(S(s)), \tag{2.15}$$

but $L_s(t) = L_s(s)$. Therefore, $L_t(t) - L_t(S(t)) \geq L_s(s) - L_s(S(s))$, which leads to the required result. □

Lemma 2.3. *The function K is continuous.*

Proof. Fix $\epsilon > 0$. Since h is continuous, there exists $\delta > 0$ such that $|h(s) - h(t)| < ((1-\alpha)/2)\epsilon$ whenever $|s - t| < \delta$. Pick $t < \gamma_h$ such that $|s - t| < \delta$. To make things simple, assume $t < s$. It was shown in the proof of Theorem 2.1 that $L_t(S(t)) - L_s(S(s)) \leq L_t(t) - L_s(s)$. Now, use the definition of $L_t(t)$ and $L_s(s)$ to get that $L_t(S(t)) - L_s(S(s)) \leq (1/(1-\alpha))(h(t) - h(s)) < \epsilon/2$; but

$$|K(t) - K(s)| \leq |L_t(S(t)) - L_s(S(s))| + |L_t(t) - L_s(s)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{2.16}$$

Therefore, K is continuous. □

Lemma 2.4. *(i) $K(s) \to 0$ as $s \to \gamma_h$.*

(ii) $K(s) \to \infty$ as $s \to -\infty$.

Proof. (i) Recall that $K(s) \geq 0$ and that $L_s(x) \geq (1/(1-\alpha))h(\gamma_h)$ by Property (2.5). In particular, $L_s(S(s)) \geq (1/(1-\alpha))h(\gamma_h)$. It follows that $L_s(s) - (1/(1-\alpha))h(\gamma_h) \geq K(s) \geq 0$ or $(1/(1-\alpha))(h(s) - h(\gamma_h)) \geq K(s) \geq 0$. The result is then immediate from the continuity of h and Assumption (A2) by letting $s \to \gamma_h$.

(ii) Define

$$\widetilde{L}_s(x) := g_s(x), \tag{2.17}$$

where

$$g_s(x) = \begin{cases} h(x) + \dfrac{\alpha}{1-\alpha}h(s), & x > s, \\[2ex] \dfrac{1}{1-\alpha}h(s), & x \leq s. \end{cases} \tag{2.18}$$

The function $\widetilde{L}_s$ is decreasing on $(-\infty, \gamma_h]$. Therefore, for x in $(-\infty, \gamma_h]$,

$$\widetilde{L}_s(x) < \int_0^{\infty} \widetilde{L}_s(x-t)\psi(t)dt. \tag{2.19}$$

Write

$$G_s := \widetilde{L}_s - L_s. \tag{2.20}$$

It is not difficult to show that G_s satisfies the following:

$$\begin{gathered} G_s(x) - \alpha(\psi * G_s)(x) = \frac{\alpha}{1-\alpha}h(s) - \alpha(\psi * \widetilde{L}_s)(x), \quad x > s, \\ G_s(x) = 0, \quad x \leq s. \end{gathered} \tag{2.21}$$

Again, a similar argument used to prove Property (2.4) can be used to show that G_s is increasing on $(-\infty, \gamma_h)$. Therefore, $G_s(\gamma_h) \geq G_s(s) = 0$. This in turn leads to $\widetilde{L}_s(\gamma_h) \geq L_s(\gamma_h)$. Now,

$$K(s) = L_s(s) - L_s(S(s)) \geq L_s(s) - L_s(\gamma_h) \geq \widetilde{L}_s(s) - \widetilde{L}_s(\gamma_h). \tag{2.22}$$

The right-hand side of (2.22) is equal to $h(s) - h(\gamma_h)$ with limit ∞ as $s \to -\infty$. This completes the proof of the lemma. □

Now consider the problem of finding a solution s to the problem

$$K(s) = k. \tag{2.23}$$

Theorem 2.5. *There exits a unique number $s < \gamma_h$ such that $K(s) = k$.*

Proof. The proof is immediate from Theorem 2.1 and Lemmas 2.2–2.4. □

3. Optimality of (s,S) policy

Recall the definitions of the functions y and F in (1.9) and let

$$\begin{gathered} L(x) := y(x) + cx, \\ h(x) := (1-\alpha)cx + F(x) + \alpha c\mu. \end{gathered} \tag{3.1}$$

It is an easy exercise to see that for $x > s$, $Ay = F$ is equivalent to $AL = h$, which is the integral equation (2.2) for $x > s$.

Assume that h satisfies Assumptions (A1) and (A2) and let $s < 0$ be the unique solution of (2.23). This value of s leads to a value of $S(s)$ which minimizes L_s (this may not be unique). Further, let S denote the generic value of $S(s)$. We will next show that the policy which asserts that if the level of stock $x < s$, order up to level S: else do not order, solves the QVI given by (1.9). The proof of optimality relies on the concept of non-k-decreasing functions which may be found in [10, page 137].

Definition 3.1. A function $v : \mathbb{R} \to \mathbb{R}$ is non-k-decreasing if $x \le y$ implies that

$$v(x) \le k + v(y). \tag{3.2}$$

Note that the concept of non-k-decreasing is weaker than the concept of k-convexity which is a standard tool for showing optimality of (s,S) policy; see [10] for more details.

Our objective is to show that the function L_s is non-k-decreasing. Note from Properties (2.3)–(2.6) that L_s is constant on $(-\infty, s]$, then decreases at least down to γ_h, reaches its minimum at some S, and eventually goes to ∞ as $x \to \infty$. Non-k-decreasing means that the function L_s cannot have a drop bigger thank beyond S. Let

$$\Delta_h = \min\{x > \gamma_h,\ L_s(x) = L_s(s)\}. \tag{3.3}$$

Note that Δ exists and is unique. Set

$$\mathcal{K}(s) = L_s(s) - \min_{x \le \Delta_h} L_s(x). \tag{3.4}$$

Theorem 3.2. *For $s < \gamma_h$, the solution L_s of* (2.2) *satisfies*

$$L_s(x) - L_s(y) \le \mathcal{K}(s) \quad \forall x \le y. \tag{3.5}$$

Proof. The proof is by contradiction and only a sketch of the proof will be given. Consider the set

$$\mathcal{R}(\mathcal{K}(s)) = \{x \ge \Delta_h,\ L_s(x) - L_s(y) > \mathcal{K}(s),\ \text{for some } y \ge x\}. \tag{3.6}$$

If $\mathcal{R}(s)$ is empty, there is nothing to prove and theorem is true. Assume that $\mathcal{R}(\mathcal{K}(s))$ is not empty, in which case it can be shown that there exists a triplet (S_1, S_2, S_3) such that $\gamma_h < S_1 < \Delta < S_2 < S_3$ such that on the interval $[s, S_3]$, L_s attains its minimum at S_1, and its maximum at S_2 (as shown in Figure 1) with

$$L_s(s) - L_s(S_1) = L_s(S_2) - L_s(S_3) := \mathcal{K}(s). \tag{3.7}$$

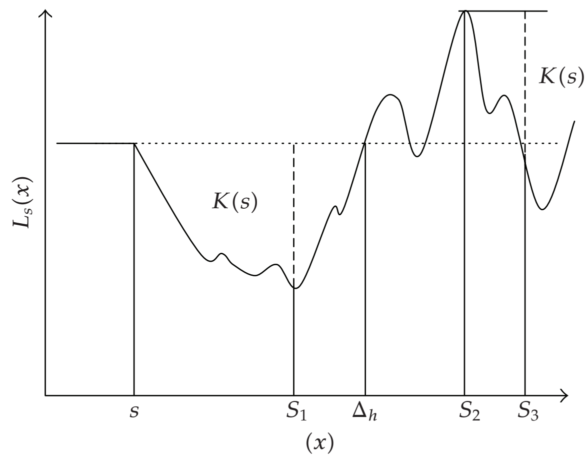

Figure 1: Plot of the function L_s.

We will next show that this cannot happen. Using (2.2), we get

$$L_s(S_2) = h(S_2) + \alpha \int_0^\infty L_s(S_2 - t)\varphi(t)dt, \tag{3.8}$$

$$L_s(S_3) = h(S_3) + \alpha \int_0^\infty L_s(S_3 - t)\varphi(t)dt. \tag{3.9}$$

It follows that for $t \geq 0$,

$$L_s(S_2 - t) - L_s(S_3 - t) \leq \mathcal{K}(s). \tag{3.10}$$

Using (3.7)–(3.9), we get

$$\begin{aligned} \mathcal{K}(s) &= h(S_2) - h(S_3) + \alpha \int_0^\infty (L_s(S_2 - t) - L_s(S_3 - t))\varphi(t)dt \\ &\leq h(S_2) - h(S_3) + \alpha\mathcal{K}(s). \end{aligned} \tag{3.11}$$

This leads to $(1-\alpha)\mathcal{K}(s) \leq h(S_2) - h(S_3) < 0$ since h is increasing on (γ_h, ∞) by Assumption (A1). Therefore, we have a contradiction that $\mathcal{K}(s) > 0$. Therefore, $R(s)$ is empty. This completes the proof. □

As a corollary of Theorem 3.2, we have the following.

Corollary 3.3. *For $s < \gamma_h$, a solution $S(s)$ of the equation $L_s(s) - L_s(S(s)) = k$ is a global minimum of the function L_s.*

Note that the proof of Theorem 3.2 revealed that the value of S belongs to some interval (γ_h, Δ_h). Further, the results of the previous section should make a numerical search for the value (s, S) an easy exercise.

Theorem 3.4. *The function L_s defined from the pair (s, S) which solves* (2.23) *solves* (1.9).

Proof. We need to show that $y < My$ for $x \geq s$ and $Ay \leq F$ for $x \leq s$. To show $Ay \leq F$ for $x \leq s$, let $x \leq s$; therefore, $L_s(x) = L_s(s)$, and $Ay(x) \leq F(x)$ is equivalent to $AL_s \leq h(x)$; but $L_s(s) = (1/(1-\alpha))h(s)$. Therefore, $AL_s(s) \leq h(x)$ is equivalent to $h(s) \leq h(x)$, which is true since h is decreasing for $x \leq \gamma_h$.

To show that $y \leq My(x)$ for $x \geq s$, note that

$$\begin{aligned} My(x) &= k + c(S-x) + y(S) \quad \text{for } s \leq x \leq S, \\ My(x) &= k + y(x) \quad \text{for } x \geq S. \end{aligned} \tag{3.12}$$

If $s \leq x \leq S$, then $y(x) \leq My(x)$ can be written as $L_s(x) \leq k + L_s(S)$. This is true since L_s is non-k-decreasing. This completes the proof. □

It is worth noting that Assumption (A1) is equivalent to saying that h is quasiconvex. Also, Assumption (A2) can be weakened by replacing it by $\lim_{|x|\to\infty} h(x) > h(\gamma_h) + k$. The limit when $x \to -\infty$ can be inferred from (2.22) and the limit when $x \to \infty$ can be obtained from the proof of Property (2.6). The optimality of (s, S) policy remains true.

In this short paper, an alternative proof of the optimality of (s, S) policy was given. The proof also revealed that finding optimal values of (s, S) is a simple exercise in numerical analysis. It is hoped that this new proof will lead to new insights in the examination of some stochastic inventory models.

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