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Research Article

Modified Iterative Algorithms for Nonexpansive Mappings

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Let H be a real Hilbert space, let S, T be two nonexpansive mappings such that $F(S) \cap F(T) \neq \emptyset$, let f be a contractive mapping, and let A be a strongly positive linear bounded operator on H. In this paper, we suggest and consider the strong converegence analysis of a new two-step iterative algorithms for finding the approximate solution of two nonexpansive mappings as $x_{n+1} = \beta_n x_n + (1-\beta_n)Sy_n$, $y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n$, $n \geq 0$, where $\gamma > 0$ is a real number and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $(0,1)$ satisfying the following control conditions: (C1) $\lim_{n\to\infty} \alpha_n = 0$, (C3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, then $\|x_{n+1} - x_n\| \to 0$. We also discuss several special cases of this iterative algorithm.

1. Introduction

Let H be a real Hilbert space. Recall that a mapping $f : H \to H$ is a *contractive mapping* on H if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad x, y \in H. \tag{1.1}$$

We denote by Π the collection of all contractive mappings on H, that is,

$$\prod = \{f : H \longrightarrow H \text{ is a contractive mapping}\}. \tag{1.2}$$

Let $T : H \to H$ be a *nonexpansive mapping*, namely,

$$\|Tx - Ty\| \leq \|x - y\|, \quad x, y \in H. \tag{1.3}$$

Iterative algorithms for nonexpansive mappings have recently been applied to solve convex minimization problems (see [1–4] and the references therein).

A typical problem is to minimize a quadratic function over the closed convex set of the fixed points of a nonexpansive mapping T on a real Hilbert space H:

$$\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.4}$$

where C is a closed convex set of the fixed points a nonexpansive mapping T on H, b is a given point in H and A is a linear, symmetric and positive operator.

In [5] (see also [6]), the author proved that the sequence $\{x_n\}$ defined by the iterative method below with the initial point $x_0 \in H$ chosen arbitrarily

$$x_{n+1} = (1 - \alpha_n A)Tx_n + \alpha_n b, \quad n \geq 0, \tag{1.5}$$

converges strongly to the unique solution of the minimization problem (1.4) provided the sequence $\{\alpha_n\}$ satisfies certain control conditions.

On the other hand, Moudafi [3] introduced the viscosity approximation method for nonexpansive mappings (see also [7] for further developments in both Hilbert and Banach spaces). Let f be a contractive mapping on H. Starting with an arbitrary initial point $x_0 \in H$, define a sequence $\{x_n\}$ in H recursively by

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n f(x_n), \quad n \geq 0, \tag{1.6}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, which satisfies some suitable control conditions.

Recently, Marino and Xu [8] combined the iterative algorithm (1.5) with the viscosity approximation algorithm (1.6), considering the following general iterative algorithm:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{1.7}$$

where $0 < \gamma < \overline{\gamma}/\alpha$.

In this paper, we suggest a new iterative method for finding the pair of nonexpansive mappings. As an application and as special cases, we also obtain some new iterative algorithms which can be viewed as an improvement of the algorithm of Xu [7] and Marino and Xu [8]. Also we show that the convergence of the proposed algorithms can be proved under weaker conditions on the parameter $\{\alpha_n\}$. In this respect, our results can be considered as an improvement of the many known results.

2. Preliminaries

In the sequel, we will make use of the following for our main results:

Lemma 2.1 (see [4]). *Let $\{s_n\}$ be a sequence of nonnegative numbers satisfying the condition*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\beta_n, \quad n \geq 0, \tag{2.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of real numbers such that

(i) $\{\alpha_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$,

(ii) $\lim_{n\to\infty}\beta_n \leq 0$ *or* $\sum_{n=0}^{\infty}\alpha_n\beta_n$ *is convergent.*

Then $\lim_{n\to\infty}s_n = 0$.

Lemma 2.2 (see [9, 10]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ be a sequence in $[0,1]$ with*

$$0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1. \tag{2.2}$$

Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \leq 0$. Then $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 2.3 (see [2] (demiclosedness Principle)). *Assume that T is a nonexpansive self-mapping of a closed convex subset C of a Hilbert space H. If T has a fixed point, then $I-T$ is demiclosed, that is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I-T)x = y$, where I is the identity operator of H.*

Lemma 2.4 (see [8]). *Let $\{x_t\}$ be generated by the algorithm $x_t = t\gamma f(x_t) + (I - tA)Tx_t$. Then $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point x^* of T which solves the variational inequality*

$$\langle (A-\gamma f)x^*, x^* - x\rangle \leq 0, \quad x \in F(T). \tag{2.3}$$

Lemma 2.5 (see [8]). *Assume A is a strong positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

3. Main Results

Let H be a real Hilbert space, let A be a bounded linear operator on H, and let S, T be two nonexpansive mappings on H such that $F(S) \cap F(T) \neq \emptyset$. Throughout the rest of this paper, we always assume that A is strongly positive.

Now, let $f \in \Pi$ with the contraction coefficient $0 < \alpha < 1$ and let A be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$ satisfying $0 < \gamma < \overline{\gamma}/\alpha$. We consider the following modified iterative algorithm:

$$\begin{aligned} x_{n+1} &= \beta_n x_n + (1-\beta_n) S y_n, \\ y_n &= \alpha_n \gamma f(x_n) + (I - \alpha_n A) T x_n, \quad n \geq 0, \end{aligned} \tag{3.1}$$

where $\gamma > 0$ is a real number and$\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $(0,1)$.

First, we prove a useful result concerning iterative algorithm (3.1) as follows.

Lemma 3.1. *Let $\{x_n\}$ be a sequence in H generated by the algorithm* (3.1) *with the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying the following control conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$,

(C3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

Then $\|x_{n+1} - x_n\| \to 0$.

Proof. From the control condition (C1), without loss of generality, we may assume that $\alpha_n \leq \|A\|^{-1}$. First observe that $\|I - \alpha_n A\| \leq 1 - \alpha_n \overline{\gamma}$ by Lemma 2.5.

Now we show that $\{x_n\}$ is bounded. Indeed, for any $p \in F(S) \cap F(T)$,

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n(\gamma f(x_n) - Ap) + (I - \alpha_n A)(Tx_n - p)\| \\ &\leq \alpha_n \|\gamma f(x_n) - \gamma f(p)\| + \alpha_n \|\gamma f(p) - Ap\| + (1 - \alpha_n \overline{\gamma})\|Tx_n - p\| \\ &\leq \alpha_n \gamma \alpha \|x_n - p\| + \alpha_n \|\gamma f(p) - Ap\| + (1 - \alpha_n \overline{\gamma})\|x_n - p\| \\ &= [1 - (\overline{\gamma} - \gamma\alpha)\alpha_n]\|x_n - p\| + \alpha_n \|\gamma f(p) - Ap\|. \end{aligned} \tag{3.2}$$

At the same time,

$$\begin{aligned} \|x_{n+1} - p\| &= \|\beta_n(x_n - p) + (1-\beta_n)(Sy_n - p)\| \\ &\leq \beta_n \|x_n - p\| + (1-\beta_n)\|Sy_n - p\| \\ &\leq \beta_n \|x_n - p\| + (1-\beta_n)\|y_n - p\|. \end{aligned} \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\begin{aligned} \|x_{n+1} - p\| \leq{}& \beta_n \|x_n - p\| + (1-\beta_n)[1 - (\overline{\gamma} - \gamma\alpha)\alpha_n]\|x_n - p\| \\ &+ \alpha_n (1-\beta_n)\|\gamma f(p) - Ap\| \\ ={}& [1 - (\overline{\gamma} - \gamma\alpha)\alpha_n(1-\beta_n)]\|x_n - p\| \\ &+ (\overline{\gamma} - \gamma\alpha)\alpha_n(1-\beta_n)\frac{\|\gamma f(p) - Ap\|}{\overline{\gamma} - \gamma\alpha}, \end{aligned} \tag{3.4}$$

which implies that

$$\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|\gamma f(p) - Ap\|}{\overline{\gamma} - \gamma\alpha}\right\}, \quad n \geq 0. \tag{3.5}$$

Hence $\{x_n\}$ is bounded and so are $\{ATx_n\}$ and $\{f(x_n)\}$.

From (3.1), we observe that

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|\alpha_{n+1}\gamma f(x_{n+1}) + (I - \alpha_{n+1}A)Tx_{n+1} - \alpha_n\gamma f(x_n) - (I - \alpha_n A)Tx_n\| \\
&= \|\alpha_{n+1}\gamma(f(x_{n+1}) - f(x_n)) + (\alpha_{n+1} - \alpha_n)\gamma f(x_n) \\
&\quad + (I - \alpha_{n+1}A)(Tx_{n+1} - Tx_n) + (\alpha_n - \alpha_{n+1})ATx_n\| \\
&\leq \alpha_{n+1}\gamma\|f(x_{n+1}) - f(x_n)\| + (1 - \alpha_{n+1}\overline{\gamma})\|Tx_{n+1} - Tx_n\| \\
&\quad + |\alpha_{n+1} - \alpha_n|(\|\gamma f(x_n)\| + \|ATx_n\|) \\
&\leq \alpha_{n+1}\gamma\alpha\|x_{n+1} - x_n\| + (1 - \alpha_{n+1}\overline{\gamma})\|x_{n+1} - x_n\| \\
&\quad + |\alpha_{n+1} - \alpha_n|(\|\gamma f(x_n)\| + \|ATx_n\|) \\
&= [1 - (\overline{\gamma} - \gamma\alpha)\alpha_{n+1}]\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|(\|\gamma f(x_n)\| + \|ATx_n\|).
\end{aligned} \tag{3.6}$$

It follows that

$$\begin{aligned}
&\|Sy_{n+1} - Sy_n\| - \|x_{n+1} - x_n\| \\
&\quad \leq \|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \\
&\quad = (\overline{\gamma} - \gamma\alpha)\alpha_{n+1}\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|(\|\gamma f(x_n)\| + \|ATx_n\|),
\end{aligned} \tag{3.7}$$

which implies, from (C1) and the boundedness of $\{x_n\}$, $\{f(x_n)\}$, and $\{ATx_n\}$, that

$$\limsup_{n\to\infty}(\|Sy_{n+1} - Sy_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{3.8}$$

Hence, by Lemma 2.2, we have

$$\|Sy_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.9}$$

Consequently, it follows from (3.1) that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1 - \beta_n)\|Sy_n - x_n\| = 0. \tag{3.10}$$

This completes the proof. □

Remark 3.2. The conclusion $\|x_{n+1} - x_n\| \to 0$ is important to prove the strong convergence of the iterative algorithms which have been extensively studied by many authors, see, for example, [3, 6, 7].

If we take $S = I$ in (3.1), we have the following iterative algorithm:

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) y_n,$$
$$y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A) T x_n, \quad n \geq 0. \tag{3.11}$$

Now we state and prove the strong convergence of iterative scheme (3.11).

Theorem 3.3. *Let $\{x_n\}$ be a sequence in H generated by the algorithm* (3.11) *with the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying the following control conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $\lim_{n\to\infty}\alpha_n = \infty$,

(C3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of T which solves the variational inequality

$$\langle (A - \gamma f)x^*, x^* - x \rangle \leq 0, \quad x \in F(T). \tag{3.12}$$

Proof. From Lemma 3.1, we have

$$\|x_{n+1} - x_n\| \longrightarrow 0. \tag{3.13}$$

On the other hand, we have

$$\begin{aligned}
\|x_n - Tx_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - Tx_n\| \\
&= \|x_{n+1} - x_n\| + \left\|\beta(x_n - Tx_n) + (1 - \beta_n)(y_n - Tx_n)\right\| \\
&\leq \|x_{n+1} - x_n\| + \beta_n\|x_n - Tx_n\| + (1 - \beta_n)\left\|y_n - Tx_n\right\| \\
&\leq \|x_{n+1} - x_n\| + \beta_n\|x_n - Tx_n\| \\
&\quad + (1 - \beta_n)\alpha_n\left(\left\|\gamma f(x_n)\right\| + \|ATx_n\|\right),
\end{aligned} \tag{3.14}$$

that is,

$$\|x_n - Tx_n\| \leq \frac{1}{1 - \beta_n}\|x_{n+1} - x_n\| + \alpha_n\left(\left\|\gamma f(x_n)\right\| + \|ATx_n\|\right), \tag{3.15}$$

this together with (C1), (C3), and (3.13), we obtain

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{3.16}$$

Next, we show that, for any $x^* \in F(T)$,

$$\limsup_{n\to\infty} \langle y_n - x^*, \gamma f(x^*) - Ax^* \rangle \leq 0. \tag{3.17}$$

In fact, we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle x_n - x^*, \gamma f(x^*) - Ax^* \rangle = \lim_{k\to\infty} \langle x_{n_k} - x^*, \gamma f(x^*) - Ax^* \rangle. \tag{3.18}$$

Since $\{x_n\}$ is bounded, we may assume that $x_{n_k} \rightharpoonup z$, where "$\rightharpoonup$" denotes the weak convergence. Note that $z \in F(T)$ by virtue of Lemma 2.3 and (3.16). It follows from the variational inequality (2.3) in Lemma 2.4 that

$$\limsup_{n\to\infty} \langle x_n - x^*, \gamma f(x^*) - Ax^* \rangle = \langle z - x^*, \gamma f(x^*) - Ax^* \rangle \leq 0. \tag{3.19}$$

By Lemma 3.1 (noting $S = I$), we have

$$\|y_n - x_n\| \longrightarrow 0. \tag{3.20}$$

Hence, we get

$$\limsup_{n\to\infty} \langle y_n - x^*, \gamma f(x^*) - Ax^* \rangle \leq 0. \tag{3.21}$$

Finally, we prove that $\{x_n\}$ converges to the point x^*. In fact, from (3.2) we have

$$\|y_n - x^*\| \leq \|x_n - x^*\| + \alpha_n \|\gamma f(x^*) - Ax^*\|. \tag{3.22}$$

Therefore, from (3.16), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\beta_n(x_n - x^*) + (1-\beta_n)(y_n - x^*)\|^2 \\
&\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 \\
&= \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|\alpha_n(\gamma f(x_n) - Ax^*) + (I - \alpha_n A)(Tx_n - x^*)\|^2 \\
&\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\left[(1-\alpha_n\overline{\gamma})^2\|x_n - x^*\|^2 + 2\alpha_n\langle \gamma f(x_n) - Ax^*, y_n - x^*\rangle\right] \\
&= \left[1 - 2\alpha_n\overline{\gamma} + (1-\beta_n)\alpha_n^2\overline{\gamma}^2\right]\|x_n - x^*\|^2 + 2\alpha_n\langle \gamma f(x_n) - \gamma f(x^*), y_n - x^*\rangle \\
&\quad + 2\alpha_n\langle \gamma f(x^*) - Ax^*, y_n - x^*\rangle
\end{aligned}$$

$$\begin{aligned}
&\leq \left[1-2\alpha_n\overline{\gamma}+(1-\beta_n)\alpha_n^2\overline{\gamma}^2\right]\|x_n-x^*\|^2 \\
&\quad+2\alpha_n\gamma\alpha\|x_n-x^*\|\left\|y_n-x^*\right\|+2\alpha_n\langle\gamma f(x^*)-Ax^*,y_n-x^*\rangle \\
&\leq [1-2\alpha_n(\overline{\gamma}-\gamma\alpha)]\|x_n-x^*\|^2+(1-\beta_n)\alpha_n^2\overline{\gamma}^2\|x_n-x^*\|^2 \\
&\quad+2\alpha_n^2\gamma\alpha\|x_n-x^*\|\left\|\gamma f(x^*)-Ax^*\right\|+2\alpha_n\langle\gamma f(x^*)-Ax^*,y_n-x^*\rangle.
\end{aligned} \tag{3.23}$$

Since $\{x_n\}$, $f(x^*)$ and Ax^* are all bounded, we can choose a constant $M>0$ such that

$$\frac{1}{\overline{\gamma}-\gamma\alpha}\left\{\frac{(1-\beta_n)\overline{\gamma}^2}{2}\|x_n-x^*\|^2+\gamma\alpha\|x_n-x^*\|\left\|\gamma f(x^*)-Ax^*\right\|\right\}\leq M,\quad n\geq 0. \tag{3.24}$$

It follows from (3.23) that

$$\|x_{n+1}-x^*\|^2\leq\left[1-2(\overline{\gamma}-\alpha\gamma)\alpha_n\right]\|x_n-x^*\|^2+2(\overline{\gamma}-\alpha\gamma)\alpha_n\delta_n, \tag{3.25}$$

where

$$\delta_n=\alpha_nM+\frac{1}{\overline{\gamma}-\gamma\alpha}\langle\gamma f(x^*)-Ax^*,y_n-x^*\rangle. \tag{3.26}$$

By (C1) and (3.17), we get

$$\limsup_{n\to\infty}\beta_n\leq 0. \tag{3.27}$$

Now, applying Lemma 2.1 and (3.25), we conclude that $x_n\to x^*$. This completes the proof. □

Taking $T=I$ in (3.1), we have the following iterative algorithm:

$$\begin{aligned}
x_{n+1}&=\beta_nx_n+(1-\beta_n)Sy_n, \\
y_n&=\alpha_n\gamma f(x_n)+(I-\alpha_nA)x_n,\quad n\geq 0.
\end{aligned} \tag{3.28}$$

Now we state and prove the strong convergence of iterative scheme (3.28).

Theorem 3.4. *Let $\{x_n\}$ be a sequence in H generated by the algorithm* (3.28) *with the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying the following control conditions:*

(C1) $\lim_{n\to\infty}\alpha_n=0$,

(C2) $\lim_{n\to\infty}\alpha_n=\infty$,

(C3) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of S which solves the variational inequality

$$\langle (A-\gamma f)x^*, x^*-x\rangle \leq 0, \quad x \in F(S). \tag{3.29}$$

Proof. From Lemma 3.1, we have

$$\|x_n - Sy_n\| \longrightarrow 0. \tag{3.30}$$

Thus, we have

$$\begin{aligned}\|x_n - Sx_n\| &\leq \|x_n - Sy_n\| + \|Sy_n - Sx_n\| \\ &\leq \|x_n - Sy_n\| + \|y_n - x_n\| \\ &\leq \|x_n - Sy_n\| + \alpha_n(\|\gamma f(x_n)\| + \|Ax_n\|) \longrightarrow 0.\end{aligned} \tag{3.31}$$

By the similar argument as (3.17), we also can prove that

$$\limsup_{n\to\infty}\langle y_n - x^*, \gamma f(x^*) - Ax^*\rangle \leq 0. \tag{3.32}$$

From (3.28), we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\beta_n(x_n - x^*) + (1-\beta_n)(Sy_n - x^*)\|^2 \\ &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|Sy_n - x^*\|^2 \\ &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 \\ &= \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|\alpha_n(\gamma f(x_n) - Ax^*) + (I-\alpha_n A)(x_n - x^*)\|^2 \\ &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\left\{(1-\alpha_n\overline{\gamma})^2\|x_n - x^*\|^2 + 2\langle \gamma f(x_n) - Ax^*, y_n - x^*\rangle\right\}.\end{aligned} \tag{3.33}$$

The remainder of proof follows from the similar argument of Theorem 3.3. This completes the proof. □

From the above results, we have the following corollaries.

Corollary 3.5. *Let $\{x_n\}$ be a sequence in H generated by the following algorithm*

$$\begin{aligned}x_{n+1} &= \beta_n x_n + (1-\beta_n)y_n, \\ y_n &= \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n \geq 0,\end{aligned} \tag{3.34}$$

where the sequences $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy the following control conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $\lim_{n\to\infty}\alpha_n = \infty$,

(C3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

Then $\{x_n\}$ *converges strongly to a fixed point* x^* *of* T *which solves the variational inequality*

$$\langle (I-f)x^*, x^* - x\rangle \leq 0, \quad x \in F(T). \tag{3.35}$$

Corollary 3.6. *Let* $\{x_n\}$ *be a sequence in* H *generated by the following algorithm*

$$\begin{aligned} x_{n+1} &= \beta_n x_n + (1-\beta_n) S y_n, \\ y_n &= \alpha_n f(x_n) + (1-\alpha_n)x_n, \quad n \geq 0, \end{aligned} \tag{3.36}$$

where the sequences $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy the following control conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $\lim_{n\to\infty}\alpha_n = \infty$,

(C3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

Then $\{x_n\}$ *converges strongly to a fixed point* x^* *of* S *which solves the variational inequality*

$$\langle (I-f)x^*, x^* - x\rangle \leq 0, \quad x \in F(S). \tag{3.37}$$

Remark 3.7. Theorems 3.3 and 3.4 provide the strong convergence results of the algorithms (3.11) and (3.28) by using the control conditions (C1) and (C2), which are weaker conditions than the previous known ones. In this respect, our results can be considered as an improvement of the many known results.

References

[1] F. Deutsch and I. Yamada, "Minimizing certain convex functions over the intersection of the fixed point sets of nonexpansive mappings," *Numerical Functional Analysis and Optimization*, vol. 19, no. 1-2, pp. 33–56, 1998.

[2] K. Geobel and W. A. Kirk, *Topics on Metric Fixed Point Theory*, Cambridge University Press, Cambridge, UK, 1990.

[3] A. Moudafi, "Viscosity approximation methods for fixed-points problems," *Journal of Mathematical Analysis and Applications*, vol. 241, no. 1, pp. 46–55, 2000.

[4] H.-K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society*, vol. 66, no. 1, pp. 240–256, 2002.

[5] H.-K. Xu, "An iterative approach to quadratic optimization," *Journal of Optimization Theory and Applications*, vol. 116, no. 3, pp. 659–678, 2003.

[6] I. Yamada, "The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings," in *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications (Haifa, 2000)*, D. Butnariu, Y. Censor, and S. Reich, Eds., vol. 8 of *Studies in Computational Mathematics*, pp. 473–504, North-Holland, Amsterdam, The Netherlands, 2001.

[7] H.-K. Xu, "Viscosity approximation methods for nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 279–291, 2004.
[8] G. Marino and H.-K. Xu, "A general iterative method for nonexpansive mappings in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 43–52, 2006.
[9] T. Suzuki, "Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 227–239, 2005.
[10] T. Suzuki, "Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces," *Fixed Point Theory and Applications*, vol. 2005, no. 1, pp. 103–123, 2005.