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Research Article

Level Sets of Random Fields and Applications: Specular Points and Wave Crests

Esteban Flores[1] and José R. León R[2]

[1] *Departamento de Matemáticas, Facultad Experimental de Ciencias y Tecnología, Universidad de Carabobo, Valencia 2001, Venezuela*
[2] *Escuela de Matemática, Facultad de Ciencias, Universidad Central de Venezuela, A.P. 47197 Los Chaguaramos, Caracas 1041-A, Venezuela*

Correspondence should be addressed to José R. León R, jose.leon@ciens.ucv.ve

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We apply Rice's multidimensional formulas, in a mathematically rigorous way, to several problems which appear in random sea modeling. As a first example, the probability density function of the velocity of the specular points is obtained in one or two dimensions as well as the expectation of the number of specular points in two dimensions. We also consider, based on a multidimensional Rice formula, a curvilinear integral with respect to the level curve. It follows that its expected value allows defining the Palm distribution of the angle of the normal of the curve that defines the waves crest. Finally, we give a new proof of a general multidimensional Rice formula, valid for all levels, for a stationary and smooth enough random fields $X:\mathbb{R}^d \to \mathbb{R}^j (d > j)$.

1. Introduction

In 1944, Rice [1] proposed the model

$$\zeta(t) = \sum_n c_n \cos(\sigma_n t + \varepsilon_n), \tag{1.1}$$

to describe the noise in an electrical current. In this relation, $\sigma_n/(2\pi)$ denotes the different frequencies, c_n are Gaussian random variables, identically distributed and independent, and ε_n are random variables uniformly distributed in $[0, 2\pi]$.

Later, in 1957, Longuet-Higgins [2] defined the following multidimensional generalization of Rice's model:

$$\zeta(t, x, y) = \sum_n c_n \cos(u_n x + v_n y + \sigma_n t + \varepsilon_n). \tag{1.2}$$

Since then this model has been used to describe the movement of the sea.

The present work is aimed at studying functionals of random field level sets in order to understand certain phenomena occurring in random sea modeling such as the movement of the luminous points which appear over any water surface. These points are called specular points and originate when the light is reflected in agreement to Snell's Law from different zones which act as small mirrors. They can be modeled as level sets of certain derivatives of the original random field ζ. This type of phenomena leads us to study the size (cardinal, length, area, and volume) and other measurements of level sets for Gaussian random fields.

It is thus necessary to consider functionals over fields defined by (1.2) or their generalizations given in Section 3. Our study relates the expectation of such functionals with the moments of the spectral measure of this process. The latter is important for applications, as usually the spectral measure of the process as well as its moments may be estimated based on data measured by buoys or satellites. The main tools that we use are given by Rice's multidimensional formulas.

Our main results include the probability density function of the velocity of the specular points studied by Longuet-Higgins in [3–5]. First, we compute the probability density function of the Palm distribution of the speed of the specular points in an arbitrary, but fixed, direction. Then, using model (1.2) we are able to compute the density of the Palm distribution of the speed of the specular points in a 2D space (see [6] for applications of this type of densities). We are also interested in obtaining the expectation of the number of the specular points in two dimensions. We provide an expression for this expectation by using a multidimensional Rice formula recently proved in the books of Azaïs and Wschebor [7, page 163] and Adler and Taylor's (page 267).

Also based on a multidimensional Rice formula we are able to study a curvilinear integral with respect to the level curve whose expected value allows defining the Palm distribution of the angle of the normal of the curve that defines the waves crest in a fixed direction, such type of objects was recently introduced in [8].

All the expectations mentioned above can be rigorously computed by using the multidimensional Rice formula for Gaussian random fields $X : \mathbb{R}^d \to \mathbb{R}^d$, recently proved in [7] and by using another Rice formula for random fields $X : \mathbb{R}^d \to \mathbb{R}^j$ $(d > j)$ established by Cabaña in 1985 [9]. For the sake of completeness, we also include a simplified proof of the latter, which allows a generalization of the original results. Namely, we show that the formula holds true over the complete level set, instead of over the intersection of the level set with the set of regular points, that is, those where the derivative of the random field has rank equal to j.

This work can thus be viewed as the implementation of several applications suggested in the book mentioned in [7] as well as a continuation of the second author's articles [10, 11].

The paper is organized as follows. Section 2 studies the coarea formula and its application in the computation of the expectation of the Lebesgue measure of the level sets and some related surface integrals with respect to the measure over the level set, (see [9, 12, 13]). The formula holds true for all levels and this is a new result. Section 3 gives a stochastic integral representation of the Longuet-Higgins model and the relation between this model and the directional spectrum. Section 4 gives the probability density function for the speed of the specular points in a fixed but arbitrary direction. In Section 5, the multidimensional Rice formula (cf. [7, 14]) is used to obtain the expectation of the number of the specular points in two dimensions. Section 6 provides the probability density function associated with the velocity of the specular points in all directions. These velocities are computed both for Gaussian and non-Gaussian random fields, thus formalizing and generalizing, the deep and inspired work of Longuet-Higgins. Finally, Section 7 establishes an application of Rice's formula to study the asymptotic distribution of the normal angle to the crests.

In what follows λ_d and σ_{d-m} will denote, respectively, the Lebesgue measure in the space $\mathbb{R}^d$ and the Hausdorff measure defined in the subspaces of dimension $d-m$, trivially by definition $\lambda_d = \sigma_d$.

2. The Coarea Formula and Its Application to Rice's Formula

Before proving our main result let us give an overview of the area formula and its probabilistic consequence, the Rice formula. Let $g : \mathbb{R}^d \to \mathbb{R}^d$ be a continuously differentiable function. If we define

$$N_A^g(\mathbf{y}) = \#\left\{\mathbf{s} \in A \subset \mathbb{R}^d : g(\mathbf{s}) = \mathbf{y}\right\}, \tag{2.1}$$

then if $d = 1$, one has

$$\int_A f(g(\mathbf{s}))\left|g'(\mathbf{s})\right| d\mathbf{s} = \int_{\mathbb{R}^d} f(\mathbf{y}) N_A^g(\mathbf{y}) d\mathbf{y}, \tag{2.2}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous and bounded function. This formula was obtained by Banach in 1925 [15].

If $\nabla g(\mathbf{x})$ is the Jacobian of g in $\mathbf{x}$ and $f : \mathbb{R}^d \to \mathbb{R}$ is a continuous bounded function, then the version of Banach's formula for $d \geq 1$ is

$$\int_A f(g(\mathbf{s}))\left|\det \nabla g(\mathbf{s})\right| d\mathbf{s} = \int_{\mathbb{R}^d} f(\mathbf{y}) N_A^g(\mathbf{y}) d\mathbf{y}. \tag{2.3}$$

This expression is usually called the area formula (cf. [12]).

Now, let $X : \mathbb{R}^d \to \mathbb{R}^d$ be a Gaussian random field with continuously differentiable trajectories. The random number of times that X takes the value $\mathbf{y}$ in the set A is defined as

$$N_A^X(\mathbf{y}) = \#\left\{\mathbf{s} \in A \subset \mathbb{R}^d : X(\mathbf{s}) = \mathbf{y}\right\}. \tag{2.4}$$

Let $p_{X(\mathbf{s})}$ denote the marginal density of $X(\mathbf{s})$. By using formula (2.3), Fubini's Theorem and duality we get for a.s. $\mathbf{y} \in \mathbb{R}^d$

$$\mathbb{E}\left[N_A^X(\mathbf{y})\right] = \int_A \mathbb{E}[|\det \nabla X(\mathbf{s})| X(\mathbf{s}) = \mathbf{y}] p_{X(\mathbf{s})}(\mathbf{y}) d\mathbf{s}. \tag{2.5}$$

The fact that this formula is true for all $\mathbf{y} \in \mathbb{R}^d$ is not trivial. The book by Azaïs and Wschebor [7, page 163] contains a definitive proof. The motivated reader can also read the interesting discussion given in Sections 11.2 and 11.4 of Adler and Taylor's recent book [14] and the references therein.

We will study below the more difficult case when function g has a domain whose dimension is greater than the dimension of the rank, namely, $g : \mathbb{R}^d \to \mathbb{R}^j$ $(d > j)$ is a continuously differentiable function, with Jacobian $\nabla g(\cdot)$ defining the level set

$$C_Q(\mathbf{y}) = \{\mathbf{x} \in Q : g(\mathbf{x}) = \mathbf{y}\} = g^{-1}(\mathbf{y}) \cap Q, \tag{2.6}$$

where Q is a compact set of $\mathbb{R}^d$. The following two results are well known as the Coarea formula (cf. Federer [12, pages 247–249] and Cabaňa [9]). The reader may consult the excellent set of lectures by Weizsäker and Geibler of the University of Kaiserslautern [13] for an up to data exposition.

Theorem 2.1. *Let $f : \mathbb{R}^j \to \mathbb{R}$ be a continuous and bounded function. Restricted to the set*

$$\left\{\mathbf{x} \in \mathbb{R}^d : \nabla g(\mathbf{x}) \text{ has rank } j\right\}, \tag{2.7}$$

the following formula holds:

$$\int_Q f(g(\mathbf{x})) \det\left(\nabla g(\mathbf{x}) \nabla g(\mathbf{x})^T\right)^{1/2} d\mathbf{x} = \int_{R^j} f(\mathbf{y}) \sigma_{d-j}(C_Q(\mathbf{y})) d\mathbf{y}. \tag{2.8}$$

Corollary 2.2. *Let $Y : \mathbb{R}^d \to \mathbb{R}$ be a continuous and bounded function under restriction* (2.7)*, then*

$$\int_{R^j} f(\mathbf{y}) \left[\int_{C_Q(\mathbf{y})} Y(\mathbf{x}) d\sigma_{d-j}(\mathbf{x})\right] dy = \int_{\mathbb{R}^d} f(g(\mathbf{x})) Y(\mathbf{x}) \det\left(\nabla g(\mathbf{x}) \nabla g(\mathbf{x})^T\right)^{1/2} d\mathbf{x}. \tag{2.9}$$

Remark 2.3. Formula (2.8) and (2.9) hold true without restriction (2.7). In fact it can be proved that for

$$A \subset \left\{\mathbf{x} \in \mathbb{R}^d : \nabla g(\mathbf{x}) \text{ has rank} < j\right\}, \tag{2.10}$$

$\sigma_{d-j}(g^{-1}(\mathbf{y}) \cap A) = 0$ holds for almost all $\mathbf{y}$. This also implies that

$$\left|\int_{A \cap C_Q(\mathbf{y})} Y d\sigma_{d-j}\right| \le \|Y\|_\infty \sigma_{d-j}\left(g^{-1}(\mathbf{y}) \cap A\right) = 0. \tag{2.11}$$

Let us define for a compact Q the functions

$$G(\mathbf{y}) = \sigma_{d-j}\left(g^{-1}(\mathbf{y}) \cap Q\right), \qquad F(\mathbf{y}) = \int_{C_Q(\mathbf{y})} Y(\mathbf{x}) d\sigma_{d-j}(\mathbf{x}). \tag{2.12}$$

The following lemma holds true.

Lemma 2.4. *Under hypothesis of Theorem 2.1, functions G and F are continuous.*

The following results are simple consequences of Theorem 2.1 and Corollary 2.2.

Let $X : \Omega \times \mathbb{R}^d \to \mathbb{R}^j$ $(d > j)$ be a stationary random field belonging to $\mathbf{C}^1(\mathbb{R}^d, \mathbb{R}^j)$ and suppose that for all $\mathbf{x} \in \mathbb{R}^d$, the density of $X(\mathbf{x}), p_{X(\mathbf{x})}(\cdot)$ exists (in the Gaussian case this holds whenever $\operatorname{Var} X(\mathbf{x}) > 0$). We have

(i) For almost all $\mathbf{y} \in \mathbb{R}^j$,

$$\mathbb{E}\left[\sigma_{d-j}(\mathcal{C}_Q(\mathbf{y}))\right] = \int_Q p_{X(\mathbf{x})}(\mathbf{y})\mathbb{E}\left[\det\left(\nabla X(\mathbf{x})\nabla X(\mathbf{x})^T\right)^{1/2} \mid X(\mathbf{x}) = \mathbf{y}\right]d\mathbf{x}. \tag{2.13}$$

(ii) For almost all $\mathbf{y} \in \mathbb{R}^j$,

$$\mathbb{E}\left[\int_{\mathcal{C}_Q(\mathbf{y})} Y(\mathbf{x})d\sigma_{d-j}(\mathbf{x})\right] = \int_Q p_{X(\mathbf{x})}(\mathbf{y})\mathbb{E}\left[Y(\mathbf{x})\det\left(\nabla X(\mathbf{x})\nabla X(\mathbf{x})^T\right)^{1/2} \mid X(\mathbf{x}) = \mathbf{y}\right]d\mathbf{x}. \tag{2.14}$$

Let us remark that formula (2.13) and (2.14) hold for almost all $\mathbf{y} \in \mathbb{R}^j$. However in applications, as we will see in the next sections, they are needed for a fixed $\mathbf{y}$. We will prove in what follows that the formulas hold for all $\mathbf{y}$.

Define the set $D^r = \{\mathbf{x} : \nabla X(\mathbf{x}) \text{ has rank } j\}$. We will establish the continuity of the left-hand side term in formulas (2.13) and (2.14) restricting ourselves first to this set. Thus let us define $\mathcal{C}_Q^{D^r}(\mathbf{y}) = \mathcal{C}_Q(\mathbf{y}) \cap D^r$. The following theorem was proved in 1985 by Cabaňa [9]. The article was written in Spanish and had a very limited diffusion. We give a new and slightly more general proof. We point out that Theorems 6.8 and 6.9 of [7] yield the same result as our Theorem 2.8. However, in this book the proofs of these results are only sketched.

Before stating the proof we include two useful conditions.

(i) A_1: for all $\mathbf{x} \in \mathbb{R}^d$, $p_{X(\mathbf{x})}(\cdot)$ exists and is continuous. For a continuous function $H : \mathbb{R}^d \times \mathbb{R}^j \to \mathbb{R}$ the following expression:

$$\int_Q p_{X(\mathbf{x})}(\mathbf{y})\mathbb{E}[H(\nabla X(\mathbf{x})) \mid X(\mathbf{x}) = \mathbf{y}]d\mathbf{x} \tag{2.15}$$

is a continuous function in the $\mathbf{y}$ variable.

(ii) A_2: the expression

$$\int_Q p_{X(\mathbf{x})}(\mathbf{y})\mathbb{E}[Y(\mathbf{x})H(\nabla X(\mathbf{x})) \mid X(\mathbf{x}) = \mathbf{y}]d\mathbf{x} \tag{2.16}$$

is a continuous function in the $\mathbf{y}$ variable. Let us note that if $Y(\mathbf{x}) = Y(\nabla X(\mathbf{x}))$, then A_1 is sufficient for A_2 to hold.

Theorem 2.5. *Consider $X : \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}^j (d > j)$ a random field belonging to $\mathbf{C}^1(\mathbb{R}^d, \mathbb{R}^j)$.*

(i) *Then under A_1 for all $\mathbf{y} \in \mathbb{R}^j$,*

$$\mathbb{E}\left[\sigma_{d-j}\left(\mathcal{C}_Q^{D^r}(\mathbf{y})\right)\right] = \int_Q p_{X(\mathbf{x})}(\mathbf{y}) \mathbb{E}\left[\det\left(\nabla X(\mathbf{x}) \nabla X(\mathbf{x})^T\right)^{1/2} \mid X(\mathbf{x}) = \mathbf{y}\right] d\mathbf{x}. \tag{2.17}$$

(ii) *If Υ is an almost sure continuous function under A_2, for all $\mathbf{y} \in \mathbb{R}^j$ one has*

$$\mathbb{E}\left[\int_{\mathcal{C}_Q^{D^r}(\mathbf{y})} \Upsilon(\mathbf{x}) d\sigma_{d-j}(\mathbf{x})\right] = \int_Q p_{X(\mathbf{x})}(\mathbf{y}) \mathbb{E}\left[\Upsilon(\mathbf{x}) \det\left(\nabla X(\mathbf{x}) \nabla X(\mathbf{x})^T\right)^{1/2} \mid X(\mathbf{x}) = \mathbf{y}\right] d\mathbf{x}. \tag{2.18}$$

Proof. We begin proving formula (2.17). Let the differentiable function $\psi : \mathbb{R}^+ \longrightarrow [0,1]$ be such that $\psi(t) = 1$ If $0 \le t \le 1$ and $\psi(t) = 0$ if $t \ge 2$. Let us define the function

$$\widetilde{\Upsilon}_m(\nabla X(\mathbf{x})) = \psi\left(\frac{1}{m} \det\left(\nabla X(\mathbf{x}) \nabla X(\mathbf{x})^T\right)\right) \psi\left(\frac{1}{m \det\left(\nabla X(\mathbf{x}) \nabla X(\mathbf{x})^T\right)}\right), \tag{2.19}$$

$\widetilde{\Upsilon}_m(\nabla X(\mathbf{x})) = 0$ if $\mathbf{x} \in \{\mathbf{y} : \det(\nabla X(\mathbf{x}) \nabla X(\mathbf{x}))^T > 2m\} \cup \{\mathbf{y} : 1/(\det(\nabla X(\mathbf{x}) \nabla X(\mathbf{x})^T)) > 2m\}$.

For $\mathbf{x}$ belonging to the complement of this set and defining $\lambda_1(\mathbf{x}) \le \lambda_2(\mathbf{x}) \le \cdots \le \lambda_j(\mathbf{x})$ the eigenvalues of $\nabla X(\mathbf{x}) \nabla X(\mathbf{x})^T$, it holds

$$\det(\nabla X(\mathbf{x}) \nabla X(\mathbf{x}))^T \le 2m, \qquad \frac{1}{\det\left(\nabla X(\mathbf{x}) \nabla X(\mathbf{x})^T\right)} \le 2m, \tag{2.20}$$

and moreover, defining $V = \ker(\nabla X(x))$ and $V^{\perp}$ its orthogonal subspace, we have

$$\left\| \left(\nabla X(\mathbf{x})|_{V\perp}\right)^{-1} \right\| = \frac{1}{\sqrt{\lambda_1(\mathbf{x})}} = \left(\frac{\prod_{i=2}^{j} \lambda_i(\mathbf{x})}{\det\left(\nabla X(\mathbf{x}) \nabla X(\mathbf{x})^T\right)}\right)^{1/2} \le \sqrt{2m} \times \|\nabla X(\mathbf{x})|_{V\perp}\|^{(j/2)}. \tag{2.21}$$

Observe that the hypothesis of continuity of $\nabla X(\mathbf{x})|_{V\perp}$ and the compactness of Q imply a uniform bound for the inverse. Lemma 2.4 implies that the following function:

$$\widetilde{F}_X^{n,m}(\mathbf{y}) := \psi\left(\frac{1}{n} \sigma_{d-j}\left(\mathcal{C}_Q^{D^r}(\mathbf{y})\right)\right) \int_{\mathcal{C}_Q^{D^r}(\mathbf{y})} \widetilde{\Upsilon}_m(\nabla X(\mathbf{x})) d\sigma_{d-j}(\mathbf{x}) \tag{2.22}$$

is a.s. continuous, also the sequence $\widetilde{F}_X^{n,m}$ is nondecreasing in both indexes. Moreover, the inequality $\widetilde{F}_X^{n,m}(\mathbf{y}) \le n$ yields that $E[\widetilde{F}_X^{n,m}(\mathbf{y})]$ is a continuous function. Using formula (2.9) applied to the field X and the function $\widetilde{F}_X^{n,m}$, we have

$$\int_{\mathbb{R}^j} \varphi\left(\frac{1}{n}\sigma_{d-j}\left(\mathcal{C}_Q^{D^r}(\mathbf{y})\right)\right)\int_{\mathcal{C}_Q^{D^r}(\mathbf{y})} \widetilde{Y}_m(\nabla X(\mathbf{x}))d\sigma_{d-j}(\mathbf{x})d\mathbf{y}$$
$$= \int_Q \varphi\left(\frac{1}{n}\sigma_{d-j}\,\mathcal{C}_Q^{D^r}(X(\mathbf{x}))\right)\widetilde{Y}_m(\nabla X(\mathbf{x}))\det\left(\nabla X(\mathbf{x})\nabla X(\mathbf{x})^T\right)^{1/2}d\mathbf{x}. \tag{2.23}$$

From this we have that for almost all $\mathbf{y} \in \mathbb{R}^j$,

$$\mathbb{E}\left[\widetilde{F}_X^{n,m}(\mathbf{y})\right]$$
$$= \int_Q p_{X(\mathbf{x})}(\mathbf{y})\mathbb{E}\left[\varphi\left(\frac{1}{n}\sigma_{d-j}\mathcal{C}_Q^{D^r}(X(\mathbf{x}))\right)\widetilde{Y}_m(\nabla X(\mathbf{x}))\det\left(\nabla X(\mathbf{x})\nabla X(\mathbf{x})^T\right)^{1/2} \mid X(\mathbf{x}) = \mathbf{y}\right]d\mathbf{x}. \tag{2.24}$$

Thus,

$$\mathbb{E}\left[\widetilde{F}_X^{n,m}(\mathbf{y})\right] \le \int_Q \mathbb{E}\left[\det\left(\nabla X(\mathbf{x})\nabla X(\mathbf{x})^T\right)^{1/2} \mid X(\mathbf{x}) = \mathbf{y}\right]p_{X(\mathbf{x})}(\mathbf{y})d\mathbf{x}. \tag{2.25}$$

Condition A_1 implies that the function in the right-hand side is continuous, hence the inequality holds for all $\mathbf{y}$. Taking limits as $n \to \infty$ and $m \to \infty$ and using Beppo-Levi's Theorem, we have that

$$\mathbb{E}\left[\sigma_{d-j}\left(\mathcal{C}_Q^{D^r}(\mathbf{y})\right)\right] \le \int_Q \mathbb{E}\left[\det\left(\nabla X(\mathbf{x})\nabla X(\mathbf{x})^T\right)^{1/2} \mid X(\mathbf{x}) = \mathbf{y}\right]p_{X(\mathbf{x})}(\mathbf{y})d\mathbf{x}. \tag{2.26}$$

To prove the other inequality, let $\mathbf{y}_N \to \mathbf{y}$ be such that for all N equality (2.24) holds. This is possible because the equality is satisfied for almost all $\mathbf{y}$. Thus by applying Fatou's Lemma, we obtain

$$\mathbb{E}\left[\widetilde{F}_X^{n,m}(\mathbf{y})\right] = \lim_{N\to\infty}\mathbb{E}\left[\widetilde{F}_X^{n,m}(\mathbf{y}_N)\right]$$
$$= \lim_{N\to\infty}\int_Q \mathbb{E}\left[\varphi\left(\frac{1}{n}\sigma_{d-j}\left(\mathcal{C}_Q^{D_r}(\mathbf{y}_N)\right)\right)\right.$$
$$\left.\times\mathbb{E}\left[\widetilde{Y}_m(\nabla X(\mathbf{x}))\det\left(\nabla X(\mathbf{x})\nabla X(\mathbf{x})^T\right)^{1/2} \mid X(\mathbf{x}) = \mathbf{y}_N\right]\right]p_{X(\mathbf{x})}(\mathbf{y}_N)d\mathbf{x}$$
$$\ge \int_Q \mathbb{E}\left[\varphi\left(\frac{1}{n}\sigma_{d-j}\left(\mathcal{C}_Q^{D_r}(\mathbf{y})\right)\right)\right.$$
$$\left.\times\mathbb{E}\left[\widetilde{Y}_m(\nabla X(\mathbf{x}))\det\left(\nabla X(\mathbf{x})\nabla X(\mathbf{x})^T\right)^{1/2} \mid X(\mathbf{x}) = \mathbf{y}\right]\right]p_{X(\mathbf{x})}(\mathbf{y})d\mathbf{x}. \tag{2.27}$$

By using Fatou's Lemma again and that $\psi(\cdot/n)$ is a nondecreasing sequence, we obtain

$$\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{E}\left[\widetilde{F}_X^{n,m}(\mathbf{y})\right]=\int_Q\mathbb{E}\left[\det\left(\nabla X(\mathbf{x})\nabla X(\mathbf{x})^T\right)^{1/2}\mid X(\mathbf{x})=\mathbf{y}\right]p_{X(\mathbf{x})}(\mathbf{y})d\mathbf{x}. \tag{2.28}$$

Finally

$$F_X^{n,m}(\mathbf{y})\leq\int_{\mathcal{C}_Q^{D_r}(\mathbf{y})}\widetilde{Y}_m(\nabla X(\mathbf{x}))d\sigma_{d-j}(\mathbf{x})\leq\sigma_{d-j}\left(\mathcal{C}_Q^{D_r}(\mathbf{y})\right)<\infty\quad\text{a.s.} \tag{2.29}$$

Moreover $F_X^{n,m}(\mathbf{y})\uparrow\int_{\mathcal{C}_Q^{D_r}(\mathbf{y})}\widetilde{Y}_m(\nabla X(\mathbf{x}))d\sigma_{d-j}(\mathbf{x})$ when $n\to\infty$. Clearly applying Beppo-Levi's Theorem we get

$$\begin{aligned}\mathbb{E}\left[\sigma_{d-j}\left(\mathcal{C}_Q^{D_r}(\mathbf{y})\right)\right]&=\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{E}\left[\widetilde{F}_X^{n,m}(\mathbf{y})\right]\\&\geq\int_Q\mathbb{E}\left[\det\left(\nabla X(\mathbf{x})\nabla X(\mathbf{x})^T\right)^{1/2}\mid X(\mathbf{x})=\mathbf{y}\right]p_{X(\mathbf{x})}(\mathbf{y})d\mathbf{x}.\end{aligned} \tag{2.30}$$

Obtaining formula (2.17), formula (2.18) follows by approximating Y uniformly by a nondecreasing sequence of simple functions. □

The following two propositions of Azaïs and Wschebor [7, pages 132–134] provide the arguments to improve Cabaña's result. In the book, however, the hypothesis is a little different.

Proposition 2.6. *Let $X:U\subset\mathbb{R}^d\to\mathbb{R}^{m+k}$ be a random field and I a subset of U, and let $\mathbf{u}\in\mathbb{R}^{m+k}$. One supposes that X satisfies the following conditions:*

(1) *the random field ∇X is α-Hölder continuous with $m/(m+k)<\alpha\leq 1$;*

(2) *for each $\mathbf{x}\in U$, the random vector $X(\mathbf{x})$ has a density $p_{X(\mathbf{x})}(\mathbf{y})$ such that $p_{X(\mathbf{x})}(\mathbf{y})\leq C$, for $\mathbf{x}\in I$ and $\mathbf{y}$ in some neighborhood of $\mathbf{u}$;*

(3) *the Hausdorff dimension of I is smaller than or equal to m.*

Then, almost surely, there is no point $\mathbf{x}\in I$ such that $X(\mathbf{x})=\mathbf{u}$.

Proposition 2.7. *Let $X:U\subset\mathbb{R}^d\to\mathbb{R}^j$ be a random field and U an open set of $\mathbb{R}^d$ and $\mathbf{y}\in\mathbb{R}^j$. Suppose that ∇X is a.s. α-Hölder continuous with $1-(1/(d+j))<\alpha\leq 1$ and moreover for all $\mathbf{x}\in U$ the random vector $(X(\mathbf{x}),\nabla X(\mathbf{x}))$ has a bounded continuous density $p_{X(\mathbf{x}),\nabla X(\mathbf{x})}(\mathbf{u},\dot{\mathbf{y}})$, for $\mathbf{u}$ in a neighborhood of $\mathbf{y}$ and $(\mathbf{x},\dot{\mathbf{y}})$ varying in a compact set of $U\times\mathbb{R}^{d\times j}$. Then*

$$\mathbb{P}\{\omega:\exists\mathbf{x}\ X(\mathbf{x})=\mathbf{y},\ \text{rank}\ \nabla X(\mathbf{x})<j\}=0. \tag{2.31}$$

Theorem 2.8. *Under the hypotheses of Theorem 2.5 and that* $\nabla X(\cdot)$ *is a.s* α*-Hölder continuous with* $1-(1/(d+j))<\alpha\le 1$*, one has the following.*

(1) *Under* A_1*, for all* $\mathbf{y}\in\mathbb{R}^j$*,*

$$\mathbb{E}\left[\sigma_{d-j}\left(\mathcal{C}_Q(\mathbf{y})\right)\right]=\int_Q p_{X(\mathbf{x})}(\mathbf{y})\mathbb{E}\left[\det\left(\nabla X(\mathbf{x})\nabla X(\mathbf{x})^T\right)^{1/2}\mid X(\mathbf{x})=\mathbf{y}\right]d\mathbf{x}. \tag{2.32}$$

(2) *Under* A_2 *and if* Y *is an almost sure continuous function, for all* $\mathbf{y}\in\mathbb{R}^j$ *one has*

$$\mathbb{E}\left[\int_{\mathcal{C}_Q(\mathbf{y})}Y(\mathbf{x})d\sigma_{d-j}(\mathbf{x})\right]=\int_Q p_{X(\mathbf{x})}(\mathbf{y})\mathbb{E}\left[Y(\mathbf{x})\det\left(\nabla X(\mathbf{x})\nabla X(\mathbf{x})^T\right)^{1/2}\mid X(\mathbf{x})=\mathbf{y}\right]d\mathbf{x}. \tag{2.33}$$

In what follows we give two examples under which the hypotheses A_1 and A_2 hold.

(1) Suppose that $X(\mathbf{x})$ is a Gaussian field verifying the hypothesis of Theorem 2.8 and that $\operatorname{Var}(X(\mathbf{x}))>0$ for each $\mathbf{x}$. By considering the regression model,

$$\nabla X(\mathbf{x})=\alpha(\mathbf{x})X(\mathbf{x})+\Gamma(\mathbf{x})\xi(\mathbf{x}) \tag{2.34}$$

with a Gaussian $\xi(\mathbf{x})\perp X(\mathbf{x})$, where

$$\begin{aligned}\alpha(\mathbf{x})&=\mathbb{E}\left[\nabla X(\mathbf{x})X(\mathbf{x})^T\right]\left(\mathbb{E}\left[X(\mathbf{x})X(\mathbf{x})^T\right]\right)^{-1},\\ \Gamma(\mathbf{x})\Gamma(\mathbf{x})^T&=\mathbb{E}\left[\nabla X(\mathbf{x})\nabla X(\mathbf{x})^T\right]-\mathbb{E}\left[\nabla X(\mathbf{x})X(\mathbf{x})^T\right]\left(\mathbb{E}\left[X(\mathbf{x})X(\mathbf{x})^T\right]\right)^{-1}\mathbb{E}\left[X(\mathbf{x})\nabla X(\mathbf{x})^T\right],\end{aligned} \tag{2.35}$$

the following equality in law is satisfied:

$$\mathcal{L}(\nabla X(\mathbf{x})\mid X(\mathbf{x})=\mathbf{y})=\mathbf{y}^T\alpha(\mathbf{x})+\Gamma(\mathbf{x})\xi(\mathbf{x}). \tag{2.36}$$

This result entails that the expression

$$\mathbb{E}\left[\det\left(\nabla X(\mathbf{x})\nabla X(\mathbf{x})^T\right)^{1/2}\mid X(\mathbf{x})=\mathbf{y}\right]=\mathbf{y}^T\alpha(\mathbf{x})\alpha(\mathbf{x})^T\mathbf{y}+\Gamma(\mathbf{x})\Gamma(\mathbf{x})^T \tag{2.37}$$

is a continuous function of variable $\mathbf{y}$. Moreover, the hypothesis $\operatorname{Var}(X(\mathbf{x}))>0$ yields the continuity in y of $p_{X(\mathbf{x})}(\mathbf{y})$.

(1) Finally let us consider the case of the real envelope of the stationary Gaussian field $\zeta(t,x,y)$. As in [16], we define the envelope of the Gaussian field $\zeta(t,x,y)$ as follows. Let us consider the random spectral measure $M(\lambda_1,\lambda_2,\omega)$ restricted to the Airy manifold Λ defined below in (3.1). In this manner if we restrict the stochastic integral to the set $\Lambda^+ = \{(\lambda_1,\lambda_2,\omega) : \omega \geq 0, |\vec{k}| = (\omega^2/g)\}$. By using polar coordinates we can write

$$\zeta(t,x,y) = 2\int_0^\infty \int_{-\pi}^{\pi} \cos\left(\left|\vec{k}\right| \cos\Theta x + \left|\vec{k}\right| \sin\Theta y + \omega t\right) dc(\omega,\Theta). \tag{2.38}$$

We define the Hilbert transform of ζ as the Gaussian field

$$\widehat{\zeta}(t,x,y) = 2\int_0^\infty \int_{-\pi}^{\pi} \sin\left(\left|\vec{k}\right| \cos\Theta x + \left|\vec{k}\right| \sin\Theta y + \omega t\right) dc(\omega,\Theta). \tag{2.39}$$

The real envelope $E(t,x,y)$ is defined as

$$E(t,x,y) = \sqrt{\zeta^2(t,x,y) + \widehat{\zeta}^2(t,x,y)}. \tag{2.40}$$

It holds

$$\mathbb{E}\left[\left\|\nabla E(t,x,y)\right\| E(t,x,y) = \mathbf{u}\right] = \frac{1}{\mathbf{u}}\mathbb{E}\left[\left\|\nabla\zeta(0,0,0) + \nabla\widehat{\zeta}(0,0,0)\right\|\right]. \tag{2.41}$$

This expression is continuos whenever $\mathbf{u} > 0$. Moreover the density of $E(0,0,0)$, in the point $\mathbf{u}$, is the Rayleigh density $(1/\sigma_\zeta^2)\mathbf{u}e^{-\mathbf{u}^2/2\sigma_\zeta^2}$, that exists and is continuous if $\sigma_\zeta^2 = \mathrm{Var}(\zeta(0,0,0)) > 0$.

3. Representation with Integrals and the Directional Spectrum

In this section, we study a generalization of the Gaussian random fields defined in (1.2) that model the waves of the sea. We use its representation as a stochastic integral which also yields the spectral representation of a stationary mean zero Gaussian random field. The approach will be somewhat informal in order to make the reading easier. The interested reader can consult Krée and Soize "Mecanique Aleatorie," [17, pages 366–376], or the very readable article [18] in which. Lindgren gives a definitive treatment for this type of spectral stochastic integral models.

Another way of looking at Longuet-Higgins' model is

$$\zeta(t,x,y) = \int_\Lambda e^{i(\lambda_1 x + \lambda_2 y + \omega t)} dM(\lambda_1,\lambda_2,\omega), \quad \text{where } \Lambda \text{ is the Airy manifold} \left\{\lambda_1^2 + \lambda_2^2 = \frac{\omega^4}{g^2}\right\}, \tag{3.1}$$

g is the gravitational constant and M is a random Gaussian orthogonal measure defined on Λ. Defining $\vec{k} = (\lambda_1, \lambda_2)$ and the following change of variable $|\vec{k}| = (\omega^2/\mathrm{g})$ and $\lambda_1 = (\omega^2/\mathrm{g})\ \cos\Theta$ and $\lambda_2 = (\omega^2/\mathrm{g})\ \sin\Theta$ (see [17]), we obtain

$$\begin{aligned}\zeta(t,x,y) &= \int_{-\infty}^{\infty}\int_{-\pi}^{\pi} e^{i(|\vec{k}|\cos\Theta\ x+|\vec{k}|\sin\Theta\ y+\omega t)} dc(\omega,\Theta) \\ &= \int_{-\infty}^{\infty}\int_{-\pi}^{\pi} \exp\left[i\left(\vec{k}\cdot\vec{x}+\omega t\right)\right] dc(\omega,\Theta),\end{aligned} \tag{3.2}$$

where $dc(\omega,\Theta)$ is a random measure. The covariance function, $K(\tau,X,Y)$, is defined as

$$K(\tau,X,Y) = E\left[\zeta(t,x,y)\bar{\zeta}(t+\tau,x+X,y+Y)\right]. \tag{3.3}$$

Then by using that

$$\mathbb{E}\left[dc(\omega,\Theta)\overline{dc}(\omega',\Theta')\right] = \widehat{S}(\omega,\Theta)\delta(\omega-\omega')\delta(\Theta-\Theta')d\omega\, d\omega'\, d\Theta\, d\Theta', \tag{3.4}$$

where $\widehat{S}(\omega,\Theta)$ is the two-dimensional spectrum of the wave surface and δ represents Dirac's delta function. Then,

$$K(\tau,X,Y) = \int_{-\infty}^{\infty}\int_{-\pi}^{\pi} \exp\left[i\left(\vec{k}\cdot\vec{X}+\omega\tau\right)\right]\widehat{S}(\omega,\Theta)d\omega\, d\Theta. \tag{3.5}$$

This procedure is justified formally in [17, 18]. If in (3.5) we let $X = 0$ and $Y = 0$, then we obtain

$$K(\tau) := K(\tau,0,0) = \int_{-\infty}^{\infty}\int_{-\pi}^{\pi} \widehat{S}(\omega,\Theta)e^{i\omega\tau} d\omega\, d\Theta, \tag{3.6}$$

or equivalently $K(\tau) = \int_{-\infty}^{\infty} \widehat{S}(\omega)e^{i\omega\tau}d\omega$, where $\widehat{S}(\omega) = \int_{-\pi}^{\pi} \widehat{S}(\omega,\Theta)d\Theta$. Function $\widehat{S}(\omega)$ represents the frequency spectrum of the sea surface. This spectrum contains the distribution of the wave energy in the frequency domain. The autocorrelation function $K(\tau)$ for the elevation surface $\zeta(t)$, in a fixed location, is a real even function.

Definition 3.1. The spectral moments of order ijk are defined as

$$m_{ijk} = \int_{0}^{\infty}\int_{-\pi}^{\pi} u^i v^j \omega^k S(\omega,\Theta)d\Theta\, d\omega, \tag{3.7}$$

where $u = (\omega^2/\mathrm{g})\cos(\Theta-\Theta_0)$, $v = (\omega^2/\mathrm{g})\sin(\Theta-\Theta_0)$, and g is the gravitational constant. If in (3.7) $i = j = 0$, then

$$m_{00k} = \int_{0}^{\infty}\int_{-\pi}^{\pi} \omega^k S(\omega,\Theta)d\Theta\, d\omega, \tag{3.8}$$

and this can be rewritten as $m_{00k} = \int_0^\infty \omega^k S(\omega)d\omega$. The previous relation corresponds to the one-dimensional moment of order k, $m_k = \int_0^\infty \omega^k S(\omega)d\omega$. Also m_{ij1} and m_{ij2} will be denoted by

$$m'_{ij} = \int_0^\infty \int_{-\pi}^{\pi} \omega u^i v^j S(\omega,\Theta)d\Theta\, d\omega, \qquad m''_{ij} = \int_0^\infty \int_{-\pi}^{\pi} \omega^2 u^i v^j S(\omega,\Theta)d\Theta\, d\omega, \tag{3.9}$$

respectively.

4. Velocity of the Specular Points in an Arbitrary Direction

We are now able to study the dynamical behavior of the specular points. Thus let $\zeta(t,x,y)$ be the random field (3.2) representing the sea height and suppose that it belongs a.s. to $C^3(\mathbb{R}^3,\mathbb{R})$. We observe the random field $(\partial\zeta/\partial x)(t,x,y)$ in a fixed direction, $y = 0$, for instance. The place where reflection occurs, when the surface $\zeta(t,x,0)$ is illuminated by a light source, placed in $(0,h_1)$ and observed in $(0,h_2)$, for each fixed t is the level curve

$$\frac{\partial\zeta}{\partial x}(t,x,0) = \zeta_x(t,x) = kx, \tag{4.1}$$

where $k = (1/2)[(1/h_1) + (1/h_2)]$. This condition is approximately true, whenever $k\zeta$ and ζ_x are both small quantities, see [4, page 845].

A consequence of the implicit function theorem is

$$(\zeta_{xx} - k)dx + \zeta_{xt}dt = 0, \tag{4.2}$$

that is,

$$c = \frac{dx}{dt} = -\frac{\zeta_{xt}}{(\zeta_{xx} - k)}. \tag{4.3}$$

This expression defines the velocity of the specular points. Thus let us define the number of specular points in $[0,M]$ having a speed in $[\alpha_1,\alpha_2]$ as $\widetilde{N}_{sp}(s,0,\alpha_1,\alpha_2)$, where

$$\widetilde{N}_{sp}(s,u,\alpha_1,\alpha_2) := \#\left\{x \le M : \zeta_x(s,x) = kx + u; \alpha_1 \le \frac{\zeta_{xt}}{(\zeta_{xx} - k)} \le \alpha_2\right\} \quad \text{for } 0 \le s \le t. \tag{4.4}$$

Now, define the latter number per unit time as

$$N_{sp}(u,\alpha_1,\alpha_2,t) := \frac{1}{t}\int_0^t \widetilde{N}_{sp}(s,u,\alpha_1,\alpha_2)ds. \tag{4.5}$$

Notice that the process $\mathcal{Z}(t) = \widetilde{N}_{sp}(t,u,\alpha_1,\alpha_2)$ is stationary, has finite mean, and it is Riemann integrable, as a function of t. Define $\mathcal{A}_t = \sigma\{\zeta(\tau,x,0) :\ \tau > t, x \in [0,M]\}$ and the σ-algebra of invariant events $\mathscr{A} = \cap \mathscr{A}_t$. Under the hypothesis that for each $x \in [0, M]$, $K(t, x, 0) \to 0$ whenever $t \to 0$ the σ-algebra $\mathscr{A}$ is trivial. By the Birkoff-Khintchine Ergodic Theorem, we have

$$\frac{\int_0^t \mathscr{Z}(s)ds}{t} \longrightarrow \mathbb{E}_{\mathcal{B}}[\mathscr{Z}(0)], \tag{4.6}$$

where $\mathcal{B}$ is the σ-algebra of t-invariants associated to $\mathscr{Z}$. Since for each t, $\mathcal{B}_t = \sigma\{\mathscr{Z}(\tau) : \tau > t\} \subset \mathscr{A}_t$, it follows that $\mathcal{B} \subset \mathscr{A}$, so that $\mathbb{E}_{\mathcal{B}}[\mathscr{Z}(0)] = \mathbb{E}[\mathscr{Z}(0)] = \mathbb{E}[\widetilde{N}_{sp}(0, u, \alpha_1, \alpha_2)]$ (for references, see [19, page 151]).

Our interest here is to compute the Palm distribution of the number of specular points having speed between $[\alpha_1, \alpha_2]$ defined as

$$F(\alpha_2) - F(\alpha_1) = \lim_{t \to \infty} \frac{N_{sp}(0, \alpha_1, \alpha_2, t)}{\mathrm{N}_{sp}(0, -\infty, \infty, t)} = \frac{\mathbb{E}\left[\widetilde{N}_{sp}(0, 0, \alpha_1, \alpha_2)\right]}{\mathbb{E}\left[\widetilde{N}_{sp}(0, 0, -\infty, \infty)\right]}. \tag{4.7}$$

The last equality, as we have seem, is a consequence of the Ergodic Theorem. We will show the following result.

Proposition 4.1. *Let $\zeta(t, x, y)$ be a Gaussian random field* (3.2) *and assume that it belongs to $C^3(\mathbb{R}^3, \mathbb{R})$ and for each pair (x, y), $K(t, x, y) \to 0$ whenever $t \to 0$. The Palm distribution F defined above satisfies*

$$F(\alpha_2) - F(\alpha_1) = \frac{\mathbb{E}\left[\mathbf{1}_{[\alpha_1, \alpha_2]}(\zeta_{xt}(0,0)/(\zeta_{xx}(0,0) - k))|\zeta_{xx}(0,0) - k|\right]}{\sqrt{m_{400}}\,\mathfrak{R}(k)}, \tag{4.8}$$

where $\mathfrak{R}(k) := (\sqrt{2/\pi})(e^{-k^2/2m_{400}} + (k/\sqrt{m_{400}})\int_0^{k/\sqrt{m_{400}}} e^{-v^2/2}dv)$.

Proof. For a continuous and bounded function h, we have

$$\int_{-\infty}^{\infty} h(u) N_{sp}(u, \alpha_1, \alpha_2, t) du = \frac{1}{t} \int_0^t \int_{-\infty}^{\infty} h(u) \widetilde{N}_{sp}(s, u, \alpha_1, \alpha_2) du\, ds, \tag{4.9}$$

and by the area formula (2.3), we have

$$\begin{aligned} &\int_{-\infty}^{\infty} h(u) N_{sp}(u, \alpha_1, \alpha_2, t) du \\ &\quad = \frac{1}{t} \int_0^t \int_0^M h(\zeta_x(s, x) - kx) \mathbf{1}_{[\alpha_1, \alpha_2]}\left(\frac{\zeta_{xt}(s, x)}{(\zeta_{xx}(s, x) - k)}\right) |(\zeta_{xx}(s, x) - k)| dx\, ds. \end{aligned} \tag{4.10}$$

Taking expectations, by stationarity and duality, for almost all u, it follows that

$$\begin{aligned}\mathbb{E}\left[N_{sp}(u,\alpha_1,\alpha_2,t)\right] \\ = \int_0^M p_{\zeta_x(0,0)}(u+kx)\mathbb{E}\left[\mathbf{1}_{[\alpha_1,\alpha_2]}\left(\frac{\zeta_{xt}(0,0)}{(\zeta_{xx}(0,0)-k)}\right)|\zeta_{xx}(0,0)-k| \mid \zeta_x(0,0)=u+kx\right]dx.\end{aligned} \tag{4.11}$$

This may be written, in the Gaussian case, by independence, as

$$\begin{aligned}\mathbb{E}\left[N_{sp}(u,\alpha_1,\alpha_2,t)\right] &= \int_0^M \frac{\exp\left\{-(u+kx)^2/2m_{200}\right\}}{\sqrt{2\pi m_{200}}}dx \\ &\quad\times \mathbb{E}\left[\mathbf{1}_{[\alpha_1,\alpha_2]}\left(\frac{\zeta_{xt}(0,0)}{(\zeta_{xx}(0,0)-k)}\right)|\zeta_{xx}(0,0)-k|\right].\end{aligned} \tag{4.12}$$

The formula is true for all u as it follows analogously to the result shown by Azaïs and Wschebor [7, page 163].

For the specular points, the interesting level is $u = 0$, thus we obtain that the expectation of the number of specular points having speed between α_1 and α_2 is

$$\int_0^M p_{\zeta_x(0,0)}(kx)\mathbb{E}\left[\mathbf{1}_{[\alpha_1,\alpha_2]}\left(\frac{\zeta_{xt}(0,0)}{(\zeta_{xx}(0,0)-k)}\right)|\zeta_{xx}(0,0)-k| \ \mid \zeta_x(0,0)=kx\right]dx, \tag{4.13}$$

which in the Gaussian case may be written as

$$\begin{aligned}\mathbb{E}\left[N_{sp}(0,\alpha_1,\alpha_2,t)\right] = \mathbb{E}\left[\widetilde{N}_{sp}(0,0,\alpha_1,\alpha_2)\right] &= \int_0^M \frac{\exp\{-k^2x^2/2m_{200}\}}{\sqrt{2\pi m_{200}}}dx \\ &\quad\times \mathbb{E}\left[\mathbf{1}_{[\alpha_1,\alpha_2]}\left(\frac{\zeta_{xt}(0,0)}{(\zeta_{xx}(0,0)-k)}\right)|\zeta_{xx}(0,0)-k|\right].\end{aligned} \tag{4.14}$$

Moreover, the expectation of the number of specular points per unit of time $N_{sp}(0,-\infty,\infty,t)$ is easily computed yielding formula (2.14) of [4, page 846]

$$\begin{aligned}\mathbb{E}\left[N_{sp}(0,-\infty,\infty,t)\right] &= \int_0^M \frac{\exp\{-k^2x^2/2m_{200}\}}{\sqrt{2\pi m_{200}}}dx\,\mathbb{E}|\zeta_{xx}(0,0)-k| \\ &= \Re(k)\sqrt{\frac{m_{400}}{2\pi m_{200}}}\int_0^M \exp\left\{-\frac{k^2x^2}{2m_{200}}\right\}dx,\end{aligned} \tag{4.15}$$

where $\Re(k) := \sqrt{2/\pi}(e^{-k^2/2m_{400}} + (k/\sqrt{m_{400}})\int_0^{k/\sqrt{m_{400}}} e^{-v^2/2}dv)$.

As the process ζ satisfies that $K(t,x,0) \to 0$, we obtain (4.8) by simple division.

Let us now define $p_{x,t}(\xi_1, \xi_2)$ the Gaussian density of the random vector $(\zeta_{xt}(0,0), \zeta_{xx}(0,0))$. We may write (4.8) as

$$F(\alpha_2) - F(\alpha_1) = \frac{1}{\sqrt{m_{400}}\Re(k)} \int_{\alpha_1}^{\alpha_2} \int_{-\infty}^{\infty} (\xi_2 - k)^2 p_{x,t}(c(\xi_2 - k), \xi_2) dc\, d\xi_2. \tag{4.16}$$

□

Remark 4.2. Differentiating the previous expression one obtains the density of the velocity of the specular points:

$$\widehat{p}_k(c) = \frac{1}{\sqrt{m_{400}}\Re(k)} \int_{-\infty}^{\infty} (\xi_2 - k)^2 p_{x,t}(c(\xi_2 - k), \xi_2) d\xi_2. \tag{4.17}$$

If $k = 0$ we recover formula (2-5-19) of [2] (modified in order to consider the case of specular points)

$$\widehat{p}_0(c) = \frac{\Delta_2}{2m_{004}^2} \left((c - \overline{c})^2 + \Delta_2 m_{400}^{-2} \right)^{-3/2}, \tag{4.18}$$

where

$$\begin{aligned} \Delta_2 &= \left[\det \begin{pmatrix} m_{200} & m_{110} \\ m_{110} & m_{020} \end{pmatrix} \right]^{-1}, \\ \overline{c} &= -\frac{m_{301}}{m_{400}}. \end{aligned} \tag{4.19}$$

5. Number of Specular Points in Two Dimensions

The specular points in two dimensions are described, as we have seen in the last section in the one-dimensional case, by the condition $(\zeta_x(t,x,y), \zeta_y(t,x,y)) = (kx, ky)$ at point (x,y) and for a fixed time t.

Defining the vectorial process

$$Z(t,x,y) = (\zeta_x(t,x,y) - kx, \zeta_y(t,x,y) - ky), \tag{5.1}$$

we say that we have a specular point if $Z = 0$ and the number of such points in a fixed time t and in a region $\Omega \subset \mathbb{R}^2$ will be

$$N_\Omega^Z(0,0) = \#\{(x,y) \in \Omega \ : Z(t,x,y) = (0,0)\}. \tag{5.2}$$

We denote as in formula (2.5) $\mathbf{x} = (x, y)$ and $\mathbf{w} = (w_1, w_2)$. Then applying this formula to the process Z, we get for almost all (w_1, w_2)

$$\mathbb{E}\left[N_\Omega^Z(w_1, w_2)\right] = \int_\Omega p_{Z(0,x,y)}(w_1, w_2)\mathbb{E}\left[\left|\Delta(t, x, y)\right| / Z(0, x, y) = (w_1, w_2)\right] dx\, dy, \tag{5.3}$$

where $\Delta(t, x, y) = (\zeta_{xx}(t, x, y) - k)(\zeta_{yy}(t, x, y) - k) - \zeta_{xy}^2(t, x, y)$.

This formula turns out to be valid for all (w_1, w_2) under the hypotheses of Theorem 6.2 in [7] (in our case $\zeta \in C^3(\mathbb{R}^3)$ and $\operatorname{Var} Z > 0$ will be enough) and in particular for the specular points, that is when $(w_1, w_2) = (0, 0)$,

$$\mathbb{E}\left[N_\Omega^Z(0, 0)\right] = \int_\Omega p_{Z(0,x,y)}(0, 0)\mathbb{E}\left[|\Delta(0, 0, 0)| / Z(x, y) = (0, 0)\right] dx\, dy. \tag{5.4}$$

The independence property allows writing

$$\mathbb{E}\left[N_\Omega^Z(0, 0)\right] = \mathbb{E}[|\Delta(0, 0, 0)|] \int_\Omega p_{Z(0,x,y)}(0, 0) dx\, dy, \tag{5.5}$$

obtaining finally the following result.

Proposition 5.1. *Let the stationary mean zero Gaussian random field* $\zeta(t, x, y) \in C^3(\mathbb{R}^3)$ *a.s. and* $\operatorname{Var} Z > 0$. *Then,*

$$\mathbb{E}\left[N_\Omega^Z(0, 0)\right] = \mathbb{E}\left[\left|(\zeta_{xx}(0, 0, 0) - k)(\zeta_{yy}(0, 0, 0) - k) - (\zeta_{xy}(0, 0, 0))^2\right|\right] \int_\Omega p_{Z(0,x,y)}(0, 0) dx\, dy. \tag{5.6}$$

Remark 5.2. The Li and Wei formula (cf. [20]) provides a way to compute the expectation of the absolute value of the determinant in the above formula, see Azaïs et al. [10], which one will not pursue. Instead one will apply a Monte Carlo method. Let us consider the regression model

$$\zeta_{yy}(0, 0, 0) = \alpha\zeta_{xx}(0, 0, 0) + \beta\zeta_{xy}(0, 0, 0) + \sigma_1\varepsilon_1, \tag{5.7}$$

where

$$\alpha = \frac{m_{220}^2 - m_{310}m_{130}}{m_{400}m_{220} - m_{310}^2} \qquad \beta = \frac{m_{400}m_{130} - m_{310}m_{220}}{m_{400}m_{220} - m_{310}^2}, \tag{5.8}$$

$\varepsilon_1 = N(0, 1) \perp (\zeta_{xx}(0, 0, 0), \zeta_{xy}(0, 0, 0))$ and $\sigma^2 = -m_{400} + \alpha^2 m_{400} + \beta^2 m_{220} + 2\alpha\beta m_{220}$. Therefore it yields

$$\mathbb{E}[|\Delta(0, 0, 0)|] = \mathbb{E}\left[\left|(\zeta_{xx}(0, 0, 0) - k)\left((\zeta_{xx}(0, 0, 0) + \beta\zeta_{xy}(0, 0, 0) + \sigma\varepsilon_1) - k\right) - (\zeta_{xy}(0, 0, 0))^2\right|\right]. \tag{5.9}$$

This last expression can be evaluated readily by using Monte Carlo. Indeed let $\vec{\varepsilon}^{i} = (\varepsilon_1^i, \varepsilon_2^i, \varepsilon_3^i)^t$, $i = 1, \ldots, N$, be a sample of standard Gaussian vectors in $\mathbb{R}^3$. We have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N}\Bigg| \left(\sqrt{m_{400}}\varepsilon_3^i - k\right) \times \left(\left(\left(\alpha\sqrt{m_{400}}\varepsilon_3^i + \beta\left(\left(m_{220} - \frac{m_{310}^2}{m_{400}}\right)^{1/2}\varepsilon_2^i + \frac{m_{310}}{\sqrt{m_{400}}}\varepsilon_3^i\right) + \sigma\varepsilon_1^i\right)\right) - k\right) - \left(\left(m_{220} - \frac{m_{310}^2}{m_{400}}\right)^{1/2}\varepsilon_2^i + \frac{m_{310}}{\sqrt{m_{400}}}\varepsilon_3^i\right)^2 \Bigg| = \mathbb{E}[|\Delta(0,0,0)|]. \tag{5.10}$$

6. Movement and Velocity of the Specular Points

In this section we will compute the density of the velocity of the specular points in two spatial dimensions. Let us consider the random field $Z(t,x,y) = (\zeta_x(t,x,y) - kx, \zeta_y(t,x,y) - ky)$; the number of specular points of the field $\zeta(t,x,y)$, in a fixed time t and in a region $\Omega \subset \mathbb{R}^2$, was defined in (5.2) and denoted as $N_\Omega^Z(0,0)$. We have already computed the expectation of the number of specular points

$$\mathbb{E}\left[N_\Omega^Z(0,0)\right] = \mathbb{E}[|\Delta(0,0,0)|]\int_\Omega p_{Z(0,x,y)}(0,0)dx\,dy. \tag{6.1}$$

The condition satisfied for the specular points (i.e., $(\zeta_x(t,x,y), \zeta_y(t,x,y)) = (kx, ky)$) and the implicit function theorem entails

$$\begin{aligned} \zeta_{xt} &= -(\zeta_{xx} - k)\frac{dx}{dt} - \zeta_{xy}\frac{dy}{dt}, \\ \zeta_{yt} &= -\zeta_{xy}\frac{dx}{dt} - (\zeta_{yy} - k)\frac{dy}{dt}. \end{aligned} \tag{6.2}$$

Let us define as Longuet-Higgins $c_x = dx/dt$ and $c_y = dy/dt$. The objective is to find the Palm distribution associated to the velocity field $(c_x, c_y) = (dx/dt), (dy/dt)$. The following computations, in the case $k = 0$, are essentially contained in the very original and seminal work of Longuet-Higgins [2].

Now define for $\mathbf{u} = (u_1, u_2)$:

$$\widetilde{N}_{sp}(s, \mathbf{u}, v_1, v_2, v_3, v_4) = \#\left\{(x,y) \in \Omega : Z(s,x,y) = \mathbf{u}; v_1 \le c_x \le v_2; v_3 \le c_y \le v_4\right\} \tag{6.3}$$

for $0 \le s \le t$ and Ω a compact set in $\mathbb{R}^2$.

Consider

$$N_{sp}(\mathbf{u}, v_1, v_2, v_3, v_4, t) := \frac{1}{t}\int_0^t \widetilde{N}_{sp}(s, \mathbf{u}, v_1, v_2, v_3, v_4)ds. \tag{6.4}$$

Let g be a continuous bounded function. On the one hand, using (2.3), we have

$$\begin{aligned}\int_{\mathbb{R}^2} g(\mathbf{u})N_{sp}(\mathbf{u}, v_1, v_2, v_3, v_4, t)d\mathbf{u} &= \frac{1}{t}\int_0^t\int_{\mathbb{R}^2} g(\mathbf{u})\widetilde{N}_{sp}(s, \mathbf{u}, v_1, v_2, v_3, v_4)d\mathbf{u}\, ds \\ &= \frac{1}{t}\int_0^t\int_{\Omega} g(Z(s,x,y)-\mathbf{u})\mathbf{1}_{[v_1,v_2]}(c_x)\mathbf{1}_{[v_3,v_4]}(c_y)\left|\Delta(t,x,y)\right|dx\, dy\, ds.\end{aligned} \tag{6.5}$$

Let us denote by $p(\xi_4, \xi_5, \xi_6, \xi_7, \xi_8)$ the density of the Gaussian random vector

$$\left(\zeta_{xx}(\mathbf{0}), \zeta_{xy}(\mathbf{0}), \zeta_{yy}(\mathbf{0}), \zeta_{xt}(\mathbf{0}), \zeta_{yt}(\mathbf{0})\right). \tag{6.6}$$

It follows that

$$p\left(\xi_4, \xi_5, \xi_6, c_x, c_y\right) := p\left(\xi_4, \xi_5, \xi_6, -(\xi_4 - k)c_x - \xi_5 c_y, -\xi_5 c_x - (\xi_6 - k)c_y\right) \tag{6.7}$$

is the density function of the random vector $(\zeta_{xx}(\mathbf{0}), \zeta_{xy}(\mathbf{0})\zeta_{yy}(\mathbf{0}), c_x(\mathbf{0}), c_y(\mathbf{0}))$. Taking expectations in (6.5), using duality and putting $\mathbf{u} = \mathbf{0}$ (under the hypothesis $\zeta \in C^3(\mathbb{R}^3, \mathbb{R})$ and $\operatorname{Var}\zeta > 0$ the formula holds for all levels $\mathbf{u}$ (cf. [7, page 163]) we obtain

$$\begin{aligned}&\mathbb{E}\left[N_{sp}(\mathbf{0}, v_1, v_2, v_3, v_4, t)\right] \\ &\quad = \int_{\Omega} p_{Z(0,0,0)}(-kx, -ky)dx\, dy \\ &\quad\quad \times \int_{v_1}^{v_2}\int_{v_3}^{v_4}\int_{\mathbf{R}^3} p\left(\xi_4, \xi_5, \xi_6, c_x, c_y\right)\left|(\xi_4 - k)(\xi_6 - k) - \xi_5^2\right| d\xi_4 d\xi_5 d\xi_6 dc_x dc_y.\end{aligned} \tag{6.8}$$

Hence, analogously as in Section 4 and by using the same arguments that lead to apply the Ergodic Theorem, the Palm distribution of a specular point having the components of its velocity $c_x \in [v_1, v_2]$ and $c_y \in [v_3, v_4]$ is

$$\begin{aligned}&\lim_{t\to\infty}\frac{N_{sp}(\mathbf{0}, v_1, v_2, v_3, v_4, t)}{N_{sp}(\mathbf{0}, -\infty, \infty, -\infty, \infty, t)} \\ &\quad = \frac{\int_{v_1}^{v_2}\int_{v_3}^{v_4}\int_{\mathbf{R}^3} p\left(\xi_4, \xi_5, \xi_6, c_x, c_y\right)\left|(\xi_4 - k)(\xi_6 - k) - \xi_5^2\right| d\xi_4 d\xi_5 d\xi_6 dc_x dc_y}{\mathbb{E}|\Delta(0,0,0)|}.\end{aligned} \tag{6.9}$$

We can summarize the above computations in the following result.

Proposition 6.1. *Let $\zeta(t,x,y)$ be a mean stationary Gaussian field which is a.s. three times continuously differentiable and* $\operatorname{Var}\zeta > 0$. *Assume also that its covariance function satisfies $K(t,x,y) \to 0$ for each (x,y) whenever $t \to 0$. Hence the Palm distribution of a specular point having the components of its velocity $c_x \in [v_1, v_2]$ and $c_y \in [v_3, v_4]$ is*

$$F(v_3,v_4) - F(v_1,v_2) = \frac{\int_{v_1}^{v_2}\int_{v_3}^{v_4}\int_{\mathbf{R}^3} p(\xi_4,\xi_5,\xi_6,c_x,c_y)\left|(\xi_4-k)(\xi_6-k)-\xi_5^2\right| d\xi_4 d\xi_5 d\xi_6 dc_x dc_y}{\mathbb{E}|\Delta(0,0,0)|}. \tag{6.10}$$

Remark 6.2. Taking derivatives one gets the density of the speed of the two-dimensional specular points

$$\widehat{p}_{(x,y,k)}(c_x,c_y) = \frac{\int_{\mathbf{R}^3} p(\xi_4,\xi_5,\xi_6,c_x,c_y)\left|(\xi_4-k)(\xi_6-k)-\xi_5^2\right| d\xi_4 d\xi_5 d\xi_6}{\mathbb{E}|\Delta(0,0,0)|}. \tag{6.11}$$

In the particular case (infinite distance), where $k = 0$, we obtain the Longuet-Higgins formula (see [2, pages 362–365]). Nevertheless, formula (6.11) is well suited for numerical computations, for $k \neq 0$.

7. Another Application of Rice Formula

7.1. Angle between the Normal and the Level Curves Defining a Crest in Direction θ

Let $\zeta(t,\mathbf{x}) := \zeta(t,x,y)$ be again a stationary zero mean Gaussian random field modeling the height of the sea waves, here $t \in \mathbb{R}^+$ and $\mathbf{x} = (x,y) \in \mathbb{R}^2$. Let us recall that such a field has the spectral representation given in (3.1). Also in (3.5), we give an expression for its covariance function. In this expression, the function $\widehat{S}(\omega,\Theta)$ is known as the directional spectral function and if it does not depend on Θ the random field ζ is called isotropic.

In what follows, we will get information about the crest of the waves in a direction θ. Let us define, as in [8], the crest of the waves ζ in direction θ at time s as the level set

$$\mathfrak{C}_Q(s,\theta) = \left\{(x,y) \in Q : \ \zeta'_\theta(s,x,y) = 0;\ \zeta''_{\theta\theta}(s,x,y) < 0\right\}, \tag{7.1}$$

where ζ'_θ and $\zeta''_{\theta\theta}$ denote the first and second derivatives in the direction θ, respectively. This set is the zero level set $\mathcal{C}_Q^{Z_\theta}(s,0)$ of the field

$$Z_\theta(s,x,y) = \zeta_x(s,x,y)\cos\theta + \zeta_y(s,x,y)\sin\theta, \tag{7.2}$$

under the additional condition that $Z''_{\theta\theta}(s,x,y) < 0$. If $\widehat{\theta}$ is the direction orthogonal to θ, we can express the gradient of $Z''_\theta(s,x,y)$ with respect to θ and its orthogonal, denoted as ∇_θ, as

$$\begin{aligned}\nabla_\theta Z_\theta(s,x,y) &= \left(\partial_\theta Z_\theta(s,x,y), \partial_{\widehat{\theta}} Z_\theta(s,x,y)\right)\\ &= \left\|\nabla_\theta Z_\theta(s,x,y)\right\|\left(\cos\Phi(s,x,y), \sin\Phi(s,x,y)\right),\end{aligned} \tag{7.3}$$

where $\Phi(s,x,y) = \arctan \partial_{\widehat{\theta}} Z_\theta(s,x,y)/\partial_\theta Z_\theta(s,x,y)$. Thus taking into account that in the crest $\Phi \in [\pi/2, 3\pi/2]$, if $H : [\pi/2, 3\pi/2] \to \mathbb{R}$ is a continuous function by using Theorem 2.8 we get

$$\frac{1}{t}\int_0^t \int_{\mathfrak{C}_Q(s,\theta)} H(\Phi(s,\mathbf{x}))d\sigma_1(\mathbf{x})ds = \frac{1}{t}\int_0^t \int_{\mathcal{C}_Q^{Z_\theta}(s,0)} H(\Phi(s,\mathbf{x}))1_{[-1,0)}(\cos\Phi(s,\mathbf{x}))d\sigma_1(\mathbf{x})ds \longrightarrow \mathbb{E}\left[\int_{\mathcal{C}_Q^{Z_\theta}(0,0)} H(\Phi(0,\mathbf{x}))1_{[-1,0)}(\cos\Phi(0,\mathbf{x}))d\sigma_1(\mathbf{x})\right]. \tag{7.4}$$

To obtain the above result we assume, as in the precedent sections, that the covariance $K(t,x,y)$ of the stationary field ζ satisfies that for each (x,y)

$$K(t,x,y) \longrightarrow 0 \quad \text{whenever } t \longrightarrow \infty. \tag{7.5}$$

This hypothesis allows us to place ourselves in the framework of the Ergodic Theorem. Defining the Palm distribution ν_θ of the normal angle at the crest in the direction θ as the following integral:

$$\int_{\pi/2}^{3\pi/2} H(\varphi)d\nu_\theta(\varphi) := \lim_{t\to\infty} \frac{\int_0^t \int_{\mathfrak{C}_Q(s,\theta)} H(\Phi(s,\mathbf{x}))d\sigma_1(\mathbf{x})}{\int_0^t \sigma_1(\mathfrak{C}_Q(s,\theta))ds}. \tag{7.6}$$

Let us denote $\mathbf{E}(k)$ as the elliptic integral of the first kind. Also let us define $\gamma^2(\theta) = \lambda_-(\theta)/\lambda_+(\theta)$, where $\lambda_-(\theta) \le \lambda_+(\theta)$ are the eigenvalues of the covariance matrix of the Gaussian vector $(\partial_\theta Z_\theta(0,0,0), \partial_{\widehat{\theta}} Z_\theta(0,0,0))$ and $\kappa(\theta)$ the angle that turn diagonal this matrix. We get the following result.

Proposition 7.1. *If the mean zero and stationary random field ζ is three times continuously differentiable and hypothesis* (7.5) *holds, the Palm distribution ν_θ of the normal angle at the crest in the direction θ satisfies*

$$\int_{\pi/2}^{3\pi/2} H(\varphi)d\nu_\theta(\varphi) = \frac{\gamma^2(\theta)}{2\mathbf{E}\left(\sqrt{1-\gamma^2(\theta)}\right)} \int_{\pi/2}^{3\pi/2} \frac{H(\varphi)}{\left(1-(1-\gamma^2(\theta))\sin^2(\varphi-\kappa(\theta))\right)^{3/2}} d\varphi. \tag{7.7}$$

Proof. By using the Ergodic Theorem, we get

$$\begin{aligned}\int_{\pi/2}^{3\pi/2} H(\varphi)d\nu_\theta(\varphi) &= \frac{\mathbb{E}\left[\int_{\mathcal{C}_Q^{Z_\theta}(0,0)} H(\Phi(0,\mathbf{x}))1_{[-1,0)}(\cos\Phi(0,\mathbf{x}))d\sigma_1(\mathbf{x})\right]}{\mathbb{E}\sigma_1(\mathfrak{C}_Q(0,\theta))} \\ &= \frac{\mathbb{E}\left[\int_{\mathcal{C}_Q^{Z_\theta}(0,0)} H(\Phi(0,\mathbf{x}))1_{[-1,0)}(\cos\Phi(0,\mathbf{x}))d\sigma_1(\mathbf{x})\right]}{(1/2)\mathbb{E}\sigma_1\left(\mathcal{C}_Q^{Z_\theta}(0,0)\right)}.\end{aligned} \tag{7.8}$$

To get (7.7) it is enough to compute the two expectations in the last equality. Let us define $a(\theta) = \mathbb{E}(Z_\theta^2(0,0,0))$. First we consider the numerator. We have changing to polar coordinates

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathcal{C}_Q^{Z_\theta}(0,0)} H(\Phi(0,\mathbf{x}))1_{[-1,0)}(\cos\Phi(0,\mathbf{x}))d\sigma_1(\mathbf{x})\right] \\
&\quad= \frac{\lambda_-(\theta)}{(2\pi)^{2/3}\sqrt{a(\theta)}(\lambda_+(\theta)\lambda_-(\theta))^{1/2}} \\
&\qquad\times\int_0^\infty\int_{\pi/2}^{3\pi/2}\rho^2 H(\varphi)e^{-(\rho^2/2\lambda_+(\theta)\lambda_-(\theta))(\lambda_+\cos^2(\varphi-\kappa(\theta))+\lambda_-\sin^2(\varphi-\kappa(\theta))}d\rho\, d\varphi \\
&\quad= \frac{\lambda_-(\theta)}{\sqrt{a(\theta)\lambda_+(\theta)}4\pi}\int_{\pi/2}^{3\pi/2}H(\varphi)\frac{1}{\left(1-(1-\gamma^2(\theta))\sin^2(\varphi-\kappa(\theta))^{3/2}\right)}d\varphi.
\end{aligned} \tag{7.9}$$

For the denominator, we have $H = 1$ obtaining, $(1/2)\mathbb{E}\sigma_1(\mathcal{C}_Q^{Z_\theta}(0,0)) = (1/2\pi)(\sqrt{\lambda_+(\theta)}/\sqrt{a(\theta)})\mathbf{E}(\sqrt{1-\gamma^2(\theta)})$. Thus (7.7) follows by a simple division. □

Remark 7.2. Formula (7.7) can be developed further for the model where the directional spectrum has the following representation $\widehat{S}(\omega,\Theta) = f(\omega)\Omega(\Theta)$, function Ω is usually called the spreading function. This matter needs further research and we will not pursue this study in this work.

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