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*Research Article*

# Existence Results for Stochastic Semilinear Differential Inclusions with Nonlocal Conditions

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We discuss existence results of mild solutions for stochastic differential inclusions subject to nonlocal conditions. We provide sufficient conditions in order to obtain a priori bounds on possible solutions of a one-parameter family of problems related to the original one. We, then, rely on fixed point theorems for multivalued operators to prove our main results.

## 1. Introduction

We investigate nonlocal stochastic differential inclusions (SDIns) of the form

$$
\begin{gathered}
dx(t) \in \left[Ax(t) + f(t, x_t)\right]dt + G(t, x_t)\,dw(t), \quad t \in J = [0, T], \\
x(0) = \sum_{i=1}^{m} \gamma_i x(t_i), \\
x(t) = \varphi(t), \quad t \in J_1 = (-\infty, 0],
\end{gathered}
\tag{1.1}
$$

where $T > 0$, $0 < t_1 < t_2 < \cdots < t_m < T$, $\gamma_i$ are real numbers, $f$ is a single-valued function, and $G$ is multivalued map.

The importance of nonlocal conditions and their applications in different field have been discussed in [1–3]. Existence results for semilinear evolution equations with nonlocal conditions were investigated in [4–7], and the case of semilinear evolution inclusions with nonlocal conditions and a nonconvex right-hand side was discussed in [8].

Stochastic differential equations (SDEs) play a very important role in formulation and analysis in mechanical, electrical, control engineering and physical sciences, and economic and social sciences. See for instance [9–12] and the references therein. So far, very few articles have been devoted to the study of stochastic differential inclusions with nonlocal conditions, see [13–15] and the references therein. Our objective is to contribute to the study of SDIns with nonlocal conditions. Motivated by the above-mentioned works and using the technique developed in [11, 16, 17], we study the SDIns of the form (1.1). The paper is organized as follows: some preliminaries are presented in Section 2. In Section 3, we investigate the existence of mild solutions for SDIns by using fixed point theorems for Kakutani maps. Finally in Section 4, we give an application to our abstract result.

## 2. Preliminaries

Let $X, Y$ be real separable Hilbert spaces and $L(Y, X)$ be the space of bounded linear operators mapping $Y$ into $X$. For convenience, we will use $\langle \cdot, \cdot \rangle$ to denote inner product of $X$ and $Y$ and $\| \cdot \|$ to denote norms in $X$, $Y$, and $L(Y, X)$ without any confusion.

Let $(\Omega, \mathcal{F}, P; \mathbb{F})(\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0})$ be a complete filtered probability space such that $\mathcal{F}_0$ contains all $P$-null sets of $\mathcal{F}$. An $X$-valued random variable is an $\mathcal{F}$-measurable function $x(t) : \Omega \to X$ and the collection of random variables $\mathcal{H} = \{x(t, \omega) : \Omega \to X : t \in J\}$ is called a stochastic process. Generally, we just write $x(t)$ instead of $x(t, \omega)$ and $x(t) : J \to X$ is the space of $\mathcal{H}$. Let $\{e_i\}_{i\geq 1}$ be a complete orthonormal basis of $Y$. Suppose that $\{w(t) : t \geq 0\}$ is a cylindrical $Y$-valued Wiener process with finite trace nuclear covariance operator $Q \geq 0$, denote $\mathrm{Tr}(Q) = \sum_{i=1}^{\infty}\lambda_i = \lambda < \infty$, which satisfies $Qe_i = \lambda_i e_i$. Actually, $w(t) = \sum_{i=1}^{\infty}\sqrt{\lambda_i}w_i(t)e_i$, where $\{w_i(t)\}_{i=1}^{\infty}$ are mutually independent one-dimensional standard Wiener processes. We assume that $\mathcal{F}_t = \sigma\{w(s) : 0 \leq s \leq t\}$ is the $\sigma$-algebra generated by $w$ and $\mathcal{F}_t = \mathcal{F}$. Let $\mu \in L(Y, X)$ and define

$$\|\mu\|_Q^2 = \mathrm{Tr}(\mu Q \mu^*) = \sum_{n=1}^{\infty}\left\|\sqrt{\lambda_n}\mu e_n\right\|^2. \tag{2.1}$$

If $\|\mu\|_Q < \infty$, then $\mu$ is called a $Q$-Hilbert-Schmidt operator. Let $L_Q(Y, X)$ denote the space of all $Q$-Hilbert-Schmidt operators $\mu : Y \to X$. The completion $L_Q(Y, X)$ of $L(Y, X)$ with respect to the topology induced by the norm $\| \cdot \|_Q$, where $\|\mu\|_Q^2 = \langle \mu, \mu \rangle$ is a Hilbert space with the above norm topology.

We now make the system (1.1) precise. Let $A : X \to X$ be the infinitesimal generator of a compact analytic semigroup $\{S(t), t \geq 0\}$ defined on X. Let $D_\tau = D((-\infty, 0], X)$ denote the family of all right continuous functions with left-hand limit $\varphi$ from $(-\infty, 0]$ to $X$ and $\mathcal{P}(\mathbb{E})$ is the family of all nonempty measurable subsets of $\mathbb{E}$. The functions $f : [0, T] \times D_\tau \to X; G : [0, T] \times D_\tau \to \mathcal{P}(L_Q(Y, X))$ are Borel measurable. The phase space $D((-\infty, 0], X)$ is equipped with the norm $\|\phi\| = \sup_{-\infty<\theta\leq 0}\|\phi(\theta)\|$. We denote by $D^b_{\mathcal{F}_0}((-\infty, 0], X)$ the family of all almost surely bounded, $\mathcal{F}_0$-measurable, $D_\tau$-valued random variables. Further, let $\mathcal{B}_\mathcal{T}$ be the Banach space of all $\mathcal{F}_t$-adapted process $\phi(t, w)$ which is almost surely continuous in $t$ for fixed $w \in \Omega$, with norm

$$\|\phi\|_{\mathcal{B}_\mathcal{T}} = \left(\sup_{0\leq t\leq T} E\|\phi\|^2\right)^{1/2}, \tag{2.2}$$

for any $\phi \in \mathcal{B}_{\mathcal{T}}$. Here the expectation $E$ is defined by

$$E\chi = \int_{\Omega} \chi(w)dP. \tag{2.3}$$

We shall assume throughout the remainder of the paper that the initial function $\varphi \in D^b_{\mathcal{F}_0}((-\infty,0],X)$.

Some notions from set-valued analysis are in order. Denote by $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $\mathcal{P}_{bd}(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $\mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ convex}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$, $\mathcal{P}_{cp,cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$. A multivalued map $F : X \to \mathcal{P}(X)$ is convex valued if $F(x) \in \mathcal{P}_{cv}(X)$ for all $x \in X$, closed valued if $F(x) \in \mathcal{P}_{cl}(X)$ for all $x \in X$, $F$ is compact valued if $F(x) \in \mathcal{P}_{cp}(X)$ for all $x \in X$. $F$ is bounded on bounded sets if $F(V) = \cup_{x\in V} F(x)$ is bounded in $X$, for all $V \in \mathcal{P}_{bd}(X)$; that is,

$$\sup_{x\in V}\{\sup\{\|y\| : y \in F(x)\}\} < \infty. \tag{2.4}$$

$F$ is called upper semicontinuous (u.s.c) on $X$, if for each $x_0 \in X$, the set $F(x_0)$ is nonempty, closed subset of $X$, and if for each open set $V$ of $X$ containing $F(x_0)$ there exists an open neighborhood $N$ of $x_0$ such that $F(N) \subseteq V$.

$F$ is said to be completely continuous if $F(V)$ is relatively compact, for every $V \in \mathcal{P}_{bd}(X)$.

If the multivalued map $F$ is completely continuous with nonempty compact values, then $F$ is u.s.c if and only if $F$ has a closed graph (ie., $x_n \to x^*, y_n \to y^*, y_n \in F(x_n)$ imply $y^* \in F(x^*)$).

$F$ has a fixed point if there is $x \in X$ such that $x \in F(x)$. The fixed point set of the multivalued operator $F$ will be denoted by Fix $F$.

The Hausdorff metric on $\mathcal{P}_{bd,cl}(X)$ is the function $H : \mathcal{P}_{bd,cl}(X) \times \mathcal{P}_{bd,cl}(X) \to \Re^+$ defined by

$$H(\mathbb{A},\mathbb{B}) = \max\left\{\sup_{a\in\mathbb{A}} d(a,\mathbb{B}),\ \sup_{a\in\mathbb{B}} d(\mathbb{A},b)\right\}, \tag{2.5}$$

where $d(\mathbb{A},b) = \inf\{\|a-b\|^2, a \in \mathbb{A}\}$, $d(a,\mathbb{B}) = \inf\{\|a-b\|^2, b \in \mathbb{B}\}$.

The multivalued map $M : [0,T]P_{bd,cl}(X)$ is said to be measurable if for each $x \in X$ the function $\zeta : [0,T] \to \Re^+$ defined by

$$\zeta(t) = d(x, M(t)) = \inf\left\{\|x-z\|^2 : z \in M(t)\right\} \text{ is measurable.} \tag{2.6}$$

For more details on multivalued maps see [18–20]. Our existence results are based on the following fixed point theorem (nonlinear alternative) for Kakutani maps [21].

**Theorem 2.1.** *Let $X$ be a Hilbert space, $C$ a closed convex subset of $X$, $Y$ an open subset of $C$ and $0 \in Y$. Suppose that $F : \overline{Y} \to \mathcal{P}_{cl,cv}(C)$ is an upper semicontinuous compact map. Then either (i) $F$ has a fixed point in $\overline{Y}$ or (ii) there are $v \in \partial Y$ and $\lambda \in (0,1)$ with $v \in \lambda F(v)$.*

*Definition 2.2.* The multivalued map $G : J \times D_\tau \to \mathcal{P}(L_Q(Y,X))$ is said to be $L^2$-Carathèodory if

(i) $t \mapsto G(t,u)$ is measurable for each $u \in D_\tau$;

(ii) $u \mapsto G(t,u)$ is upper semicontinuous for almost all $t \in J$;

(iii) for each $q > 0$, there exists $\omega_q \in L^2(J,\mathfrak{R}^+)$ such that

$$\|G(t,u)\|^2 := \sup\left\{\|g\|^2 : g \in G(t,u)\right\} \le \omega_q(t), \tag{2.7}$$

for all $\|u\|^2_{\mathcal{B}_\tau} \le q$ and for a.e. $t \in J$.

For each $x \in L^2(L_Q(Y,X))$ define the set of selections of $G$ by

$$S_{G,x} = \left\{g \in L^2 = L^2\left(L_Q(Y,X)\right) : g(t) \in G(t,x_t) \text{ for a.e, } t \in J\right\}. \tag{2.8}$$

**Lemma 2.3** (see [22]). *Let $I$ be a compact interval and $X$ be a Hilbert space. Let $G$ be an $L^2$-Carathèodory multivalued map with $S_{G,x} \neq \phi$ and let $\Gamma$ be a linear continuous mapping from $L^2(I,X) \to C(I,X)$. Then the operator*

$$\Gamma \circ S_G : C(I,X) \longmapsto \mathcal{P}_{bd,\mathrm{cl},cv}(C(I,X)), \quad x \longmapsto (\Gamma \circ S_G)(x) = \Gamma(S_{G,x}), \tag{2.9}$$

*is a closed graph operator in $C(I,X) \times C(I,X)$.*

*Definition 2.4.* A semigroup $\{S(t), t \ge 0\}$ is said to be uniformly bounded if there exists a constant $M \ge 1$ such that

$$\|S(t)\| \le M, \quad \text{for } t \ge 0. \tag{2.10}$$

Assume that

$$\sum_{i=1}^{m} |\gamma_i| < \frac{1}{M}. \tag{2.11}$$

Then there exists a bounded operator $B$ on $D(B) = X$ given by the formula

$$B = \left(I - \sum_{i=1}^{m} \gamma_i T(t_i)\right)^{-1}. \tag{2.12}$$

*Definition 2.5.* A stochastic process $\{x(t) \in \mathcal{B}_\tau, t \in (-\infty, T]\}$ is called a mild solution of system (1.1) if

(i) $x(t)$ is $\mathcal{F}_t$-adapted with $\int_0^T \|x(t)\|^2\, dt < \infty$ almost surely;

(ii) $x(t)$ satisfies the integral equation

$$x(t) = \begin{cases} \varphi(t), & t \in J_1, \\ \sum_{i=1}^{m} \gamma_i S(t) B \left[ \int_0^{t_i} S(t_i - s) f(s, x_s) ds + \int_0^{t_i} S(t_i - s) g(s) dw(s) \right] & \\ + \int_0^t S(t-s) f(s, x_s) ds + \int_0^t S(t-s) g(s) dw(s), & a.e.\ t \in J, \end{cases} \quad (2.13)$$

where $g \in S_{G,x}$.

## 3. Existence Results

In this section, we discuss the existence of mild solutions of the system (1.1). We need the following hypotheses.

$(H_1)$: The function $f : J \times D_\tau \to X$ is continuous and there exist two positive constants $C_1, C_2$ such that

$$\|f(t, x_t)\|^2 \leq C_1 \|x\|^2 + C_2, \quad \text{for each } x \in D_\tau,\ t \in J. \quad (3.1)$$

$(H_2)$ : $G : J \times D_\tau \to \mathcal{P}(L_Q(Y, X))$ is an $L^2$-Carathéodory multivalued function with compact and convex values.

$(H_3)$: There exists a continuous nondecreasing function $\psi : \Re^+ \to (0, \infty)$ and $p \in L^1(J, \Re^+)$ such that

$$\|G(t, x_t)\|^2 = \sup\left\{ \|g\|^2 : g \in G(t, x_t) \right\} \leq p(t)\psi\left(\|x\|^2\right), a.\ et \in J,\ \text{all } x \in D_\tau. \quad (3.2)$$

**Theorem 3.1.** *Assume that* $(H_1)$–$(H_3)$ *hold. Then the system* (1.1) *has at least one mild solution on* $(-\infty, T]$, *provided that*

$$3\mathcal{K}_1 C_1 T < 1, \qquad \sup_{\rho \in [0,\infty)} \frac{\{1 - 3T\mathcal{K}_1 C_1\}\rho}{3T\mathcal{K}_2 C_2 + 3\mathcal{K}_1 Tr(Q)\|p\|_{L^1}\psi(\rho)} > 1, \quad (3.3)$$

*where*

$$\mathcal{K}_1 = 3\left(2mM^2 \sum_{i=1}^{m} |\gamma_i|^2 \|B\|^2 + 1\right) M^2, \qquad \mathcal{K}_2 = \left(2mM^2 \sum_{i=1}^{m} |\gamma_i|^2 \|B\|^2 + T\right) M^2. \quad (3.4)$$

*Proof.* Transform the system (1.1) into a fixed point problem. Consider the multivalued operator $\mathcal{M} : \mathcal{B}_{\mathcal{T}} \to \mathcal{P}(\mathcal{B}_{\mathcal{T}})$ defined by

$$\mathcal{M}(x) = h \in \mathcal{B}_{\mathcal{T}} : h(t)$$
$$= \begin{cases} \varphi(t), & t \in J_1 \\ \displaystyle\sum_{i=1}^{m} \gamma_i S(t) B\left[\int_0^{t_i} S(t_i - s) f(s, x_s) ds + \int_0^{t_i} S(t_i - s) g(s) dw(s)\right] & \\ \quad \displaystyle + \int_0^t S(t-s) f(s, x_s) ds + \int_0^t S(t-s) g(s) dw(s), & g \in S_{G,x}, \ a.\ e.\ t \in J. \end{cases} \tag{3.5}$$

It is clear that the fixed points of $\mathcal{M}$ are mild solutions of system (1.1). Hence we have to find solutions of the inclusion $y \in \mathcal{M}(y)$. We show that the multivalued operator $\mathcal{M}$ satisfies all the conditions of Theorem 2.1. The proof will be given in several steps.

*Step 1.* $\mathcal{M}(x)$ is convex for each $x \in \mathcal{B}_{\mathcal{T}}$. Since $G$ has convex values it follows that $S_{G,x}$ is convex; so that if $g_1, g_2 \in S_{G,x}$ then $\alpha g_1 + (1-\alpha) g_2 \in S_{G,x}$, which implies clearly that $\mathcal{M}(x)$ is convex.

*Step 2.* The operator $\mathcal{M}$ is bounded on bounded subsets of $\mathcal{B}_{\mathcal{T}}$. For $q > 0$ let $B_q = \{x \in \mathcal{B}_{\mathcal{T}} : \|x\|_{\mathcal{B}_{\mathcal{T}}} \le q\}$ be a bounded subset of $\mathcal{B}_{\mathcal{T}}$. We show that $\mathcal{M}(B_q)$ is a bounded subset of $\mathcal{B}_{\mathcal{T}}$. For each $x \in B_q$ let $h \in \mathcal{M}(x)$. Then there exists $g \in S_{G,x}$ such that for each $t \in J$ we have

$$\begin{aligned} h(t) = {} & \sum_{i=1}^{m} \gamma_i S(t) B\left[\int_0^{t_i} S(t_i - s) f(s, x_s) ds + \int_0^{t_i} S(t_i - s) g(s) dw(s)\right] \\ & + \int_0^t S(t-s) f(s, x_s) ds + \int_0^t S(t-s) g(s) dw(s), \end{aligned} \tag{3.6}$$

$$\begin{aligned} \|h(t)\|^2 \le {} & 3\left\|\sum_{i=1}^{m} \gamma_i S(t) B\left[\int_0^{t_i} S(t_i - s) f(s, x_s) ds + \int_0^{t_i} S(t_i - s) g(s) dw(s)\right]\right\|^2 \\ & + 3\left\|\int_0^t S(t-s) f(s, x_s) ds\right\|^2 + 3\left\|\int_0^t S(t-s) g(s) dw(s)\right\|^2 \\ \le {} & 6m \sum_{i=1}^{m} |\gamma_i|^2 M^2 \|B\|^2 M^2 \left[\int_0^{t_i} \|f(s, x_s)\|^2 ds + \operatorname{Tr}(Q) \int_0^{t_i} \|g(s)\|^2 ds\right] \\ & + 3M^2 \int_0^t \|f(s, x_s)\|^2 ds + 3M^2 \operatorname{Tr}(Q) \int_0^t \|g(s)\|^2 ds \\ \le {} & 3\left(2mM^2 \sum_{i=1}^{m} |\gamma_i|^2 \|B\|^2 + 1\right) M^2 \int_0^T \|f(s, x_s)\|^2 ds \end{aligned}$$

$$
\begin{aligned}
&+3\left(2mM^2\sum_{i=1}^{m}\left|\gamma_i\right|^2\|B\|^2+1\right)\operatorname{Tr}(Q)M^2\int_0^T\|g(s)\|^2ds\\
&\leq 3M^2\left(2mM^2\sum_{i=1}^{m}\left|\gamma_i\right|^2\|B\|^2+1\right)\left(\int_0^T\|f(s,x_s)\|^2ds+\operatorname{Tr}(Q)\int_0^T\|g(s)\|^2ds\right)\\
&\leq 3M^2\left(2mM^2\sum_{i=1}^{m}\left|\gamma_i\right|^2\|B\|^2+1\right)\\
&\quad\times\left(\int_0^T\left(C_1\|x(s)\|^2+C_2\right)ds+\operatorname{Tr}(Q)\int_0^T p(s)\psi\left(\|x(s)\|^2\right)ds\right)\\
&\leq \mathcal{K}_1\left(TC_1q^2+TC_2+\operatorname{Tr}(Q)\psi\left(q^2\right)\|p\|_{L^1}\right).
\end{aligned}
\tag{3.7}
$$

Hence for each $h \in \mathcal{M}(B_q)$, we get

$$
\|h\|_{\mathcal{B}_\tau}^2=\sup_{t\in[0,T]}E\|h\|^2\leq \mathcal{K}_1T\left(TC_1q^2+TC_2+\operatorname{Tr}(Q)\psi\left(q^2\right)\|p\|_{L^1}\right).
\tag{3.8}
$$

Then, for each $h \in \mathcal{M}(x)$, we have $\|h\|_{\mathcal{B}_\tau}^2 \leq \widehat{\wedge}$, where $\widehat{\wedge} := \mathcal{K}_1T(TC_1q^2+TC_2+\operatorname{Tr}(Q)\psi(q^2)\|p\|_{L^1})$.

*Step 3.* $\mathcal{M}$ sends bounded sets into equicontinuous sets in $\mathcal{B}_\tau$. For each $x \in B_q$ let $h \in \mathcal{M}(x)$ be given by (3.6). Let $\tau_1, \tau_2 \in J$ with $0 < \tau_1 < \tau_2 \leq T$. Then

$$
\begin{aligned}
h(\tau_2)-h(\tau_1)=&\sum_{i=1}^{m}\gamma_i[S(\tau_2)-S(\tau_1)]B\left[\int_0^{t_i}S(t_i-s)f(s,x_s)ds+\int_0^{t_i}S(t_i-s)g(s)dw(s)\right]\\
&+\int_0^{\tau_2}S(\tau_2-s)f(s,x_s)ds+\int_0^{\tau_2}S(\tau_2-s)g(s)dw(s)\\
&-\int_0^{\tau_1}S(\tau_1-s)f(s,x_s)ds-\int_0^{\tau_1}S(\tau_1-s)g(s)dw(s).
\end{aligned}
\tag{3.9}
$$

This implies that

$$
\begin{aligned}
h(\tau_2)-h(\tau_1)=&\sum_{i=1}^{m}\gamma_i[S(\tau_2)-S(\tau_1)]B\left[\int_0^{t_i}S(t_i-s)f(s,x_s)ds+\int_0^{t_i}S(t_i-s)g(s)dw(s)\right]\\
&+\int_0^{\tau_1}[S(\tau_2-s)-S(\tau_1-s)]f(s,x_s)ds+\int_0^{\tau_1}[S(\tau_2-s)-S(\tau_1-s)]g(s)dw(s)\\
&+\int_{\tau_1}^{\tau_2}S(\tau_2-s)f(s,x_s)ds+\int_{\tau_1}^{\tau_2}S(\tau_2-s)g(s)dw(s).
\end{aligned}
\tag{3.10}
$$

It follows that

$$\begin{aligned}
\|h(\tau_2)-h(\tau_1)\|^2 \le{}& 5m\|B\|^2\|S(\tau_2)-S(\tau_1)\|^2\sum_{i=1}^{m}\left|\gamma_i\right|^2\int_0^{t_i}\|S(t_i-s)\|^2\left\|f(s,x_s)\right\|^2ds \\
&+5m\|B\|^2\|S(\tau_2)-S(\tau_1)\|^2\sum_{i=1}^{m}\left|\gamma_i\right|^2\operatorname{Tr}(Q)\int_0^{t_i}\|S(t_i-s)\|^2\left\|g(s)\right\|^2ds \\
&+5\int_0^{\tau_1}\|S(\tau_2-s)-S(\tau_1-s)\|^2\left\|f(s,x_s)\right\|^2ds \\
&+5\int_{\tau_1}^{\tau_2}\|S(\tau_2-s)\|^2\left\|f(s,x_s)\right\|^2ds \\
&+5\operatorname{Tr}(Q)\int_0^{\tau_1}\|S(\tau_2-s)-S(\tau_1-s)\|^2\left\|g(s)\right\|^2ds \\
&+5\operatorname{Tr}(Q)\int_{\tau_1}^{\tau_2}\|S(\tau_1-s)\|^2\left\|g(s)\right\|^2ds \\
\le{}& 5mM^2\|B\|^2\|S(\tau_2)-S(\tau_1)\|^2\sum_{i=1}^{m}\left|\gamma_i\right|^2\left\{\left(C_1q^2+C_2\right)+\operatorname{Tr}(Q)\psi\left(q^2\right)\|p\|_{L^1}\right\} \\
&+5\left(C_1q^2+C_2\right)\int_0^{\tau_1}\|S(\tau_2-s)-S(\tau_1-s)\|^2ds \\
&+5M^2\left(C_1q^2+C_2\right)(\tau_2-\tau_1) \\
&+5\operatorname{Tr}(Q)\psi\left(q^2\right)\int_0^{\tau_1}\|S(\tau_2-s)-S(\tau_1-s)\|^2p(s)ds \\
&+5M^2\operatorname{Tr}(Q)\psi\left(q^2\right)\|p\|_{L^1}(\tau_2-\tau_1).
\end{aligned} \tag{3.11}$$

Since there is $\delta>0$ such that

$$\|S(\tau_2)-S(\tau_1)\|\le\frac{\delta}{\sqrt{\tau_1}}\sqrt{\tau_2-\tau_1}, \tag{3.12}$$

(see [23, proposition 1]) and the compactness of $S(t)$ for $t>0$ implies the continuity in the uniform operator topology, we have

$$\|S(\tau_2)-S(\tau_1)\|^2\longrightarrow 0,\quad \|S(\tau_2-s)-S(\tau_1-s)\|^2\longrightarrow 0\quad\text{as }\tau_2\longrightarrow\tau_1. \tag{3.13}$$

Therefore

$$E\|h(\tau_2)-h(\tau_1)\|^2\longrightarrow 0\quad\text{as }\tau_2\longrightarrow\tau_1. \tag{3.14}$$

When $\tau_1 = 0$ we have

$$\|h(\tau_2) - h(0)\|^2 \le 5mM^2\|B\|^2\|S(\tau_2) - S(0)\|^2 \sum_{i=1}^{m} |\gamma_i|^2 \left\{ t_i\left(C_1 q^2 + C_2\right) + \operatorname{Tr}(Q)\psi\left(q^2\right)\|p\|_{L^1}\right\}$$
$$+ 5M^2\left(C_1 q^2 + C_2\right)\tau_2 + 5M^2 \operatorname{Tr}(Q)\psi\left(q^2\right)\|p\|_{L^1}\tau_2, \tag{3.15}$$

so that, similar to the previous situation, we have

$$E\|h(\tau_2) - h(0)\|^2 \longrightarrow 0 \quad \text{as } \tau_2 \longrightarrow 0. \tag{3.16}$$

*Step 4.* $\mathcal{M}$ sends bounded sets into relatively compact sets in $\mathcal{B}_{\mathcal{T}}$. Let $0 < \epsilon < t$, for $t \in J$. For $x \in B_q$ define a function $h_\epsilon$ by

$$h_\epsilon(t) = \sum_{i=1}^{m} \gamma_i S(t) B\left[\int_0^{t_i} S(t_i - s) f(s, x_s) ds + \int_0^{t_i} S(t_i - s) g(s) dw(s)\right]$$
$$+ \int_0^{t-\epsilon} S(t-s) f(s, x_s) ds + \int_0^{t-\epsilon} S(t-s) g(s) dw(s), \tag{3.17}$$

where $g \in S_{G,x}$. Since $S(t)$ is a compact operator, the set $V_\epsilon(t) = \{h_\epsilon(t) : h_\epsilon \in \mathcal{M}(x)\}$ is relatively compact in $\mathcal{B}_{\mathcal{T}}$ for every $\epsilon$ in $(0, t)$. Moreover, for every $h \in \mathcal{M}(x)$ we have

$$E\|h - h_\epsilon\|^2 \le 2\epsilon M^2 \int_{t-\epsilon}^{t} \left[C_1 E\|x(s)\|^2 + C_2\right] ds + 2M^2 \operatorname{Tr}(Q) \int_{t-\epsilon}^{t} \omega_q(s) ds$$
$$\le 2\epsilon^2 M^2 (C_1 q + C_2) + 2M^2 \operatorname{Tr}(Q) \int_{t-\epsilon}^{t} \omega_q(s) ds. \tag{3.18}$$

Since $\omega_q \in L^1(J)$ and $\operatorname{meas}([t-\epsilon, t]) = \epsilon$ it follows that

$$\|h - h_\epsilon\|_{\mathcal{B}_{\mathcal{T}}} \longrightarrow 0 \quad \text{as } \epsilon \longrightarrow 0. \tag{3.19}$$

As a consequence of Step 1 through Step 4, together with Ascoli-Arzela theorem, we can conclude that the multivalued operator $\mathcal{M}$ is compact.

*Step 5.* $\mathcal{M}$ has a closed graph. Let $x_n \to x^*$ and $h_n \in \mathcal{M}(x_n)$ with $h_n \to h^*$. We shall show that $h^* \in \mathcal{M}(x^*)$.

There exists $g_n \in S_{G,x_n}$ such that

$$h_n(t) = \sum_{i=1}^{m} \gamma_i S(t) B\left[\int_0^{t_i} S(t_i - s) f(s, x_{n,s}) ds + \int_0^{t_i} S(t_i - s) g_n(s) dw(s)\right]$$
$$+ \int_0^{t} S(t-s) f(s, x_{n,s}) ds + \int_0^{t} S(t-s) g_n(s) dw(s). \tag{3.20}$$

We must prove that there exists $g^* \in S_{G,x^*}$ such that

$$h^*(t) = \sum_{i=1}^{m}\gamma_i S(t)B\left[\int_0^{t_i} S(t_i - s)f(s, x_s^*)ds + \int_0^{t_i} S(t_i - s)g^*(s)dw(s)\right] + \int_0^{t} S(t-s)f(s, x_s^*)ds + \int_0^{t} S(t-s)g^*(s)dw(s). \tag{3.21}$$

Consider the linear continuous operator $\Gamma : L^2(L_Q(Y, X)) \to \mathcal{B}_{\mathcal{T}}$ defined by

$$\Gamma(g)(t) = \sum_{i=1}^{m}\gamma_i S(t)B\int_0^{t_i} S(t_i - s)g(s)dw(s) + \int_0^{t} S(t-s)g(s)dw(s). \tag{3.22}$$

Clearly, $\Gamma$ is linear and continuous. Indeed, one has

$$\begin{aligned} \|\Gamma(g)(t)\|^2 &\leq \left(2mM^2\sum_{i=1}^{m}|\gamma_i|^2\|B\|^2 + 1\right)\mathrm{Tr}(Q)M^2\int_0^t \|g(s)\|^2 ds \\ E\|\Gamma(g)\|^2 &\leq \left(2mM^2\sum_{i=1}^{m}|\gamma_i|^2\|B\|^2 + 1\right)\mathrm{Tr}(Q)M^2\|\omega_q\|_{L^1}. \end{aligned} \tag{3.23}$$

Let

$$\begin{aligned} \Theta_n(t) &= h_n(t) - \sum_{i=1}^{m}\gamma_i S(t)B\int_0^{t_i} S(t_i - s)f(s, x_{n,s})ds - \int_0^{t} S(t-s)f(s, x_{n,s})ds, \\ \Theta^*(t) &= h^*(t) - \sum_{i=1}^{m}\gamma_i S(t)B\int_0^{t_i} S(t_i - s)f(s, x_s^*)ds - \int_0^{t} S(t-s)f(s, x_s^*)ds. \end{aligned} \tag{3.24}$$

We have

$$\Theta_n(t) \in \Gamma \circ S_{G,x_n}. \tag{3.25}$$

Since $f$ is continuous (see $(H_1)$)

$$\|\Theta_n(t) - \Theta^*(t)\|^2 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.26}$$

Lemma 2.3 implies that $\Gamma \circ S_G$ has a closed graph. Hence there exists $g^* \in S_{G,x^*}$ such that

$$\Theta^*(t) = \sum_{i=1}^{m}\gamma_i S(t)B\int_0^{t_i} S(t_i - s)g^*(s)dw(s) + \int_0^{t} S(t-s)g^*(s)dw(s). \tag{3.27}$$

Hence $h^* \in \mathcal{M}(x^*)$, which shows that graph $\mathcal{M}$ is closed.

*Step 6.* Let $\lambda \in (0,1)$ and let $x \in \lambda \mathcal{M}(x)$. Then there exists $g \in S_{G,x}$ such that

$$x(t) = \lambda \sum_{i=1}^{m} \gamma_i S(t) B \left[ \int_0^{t_i} S(t_i - s) f(s, x_s) ds + \int_0^{t_i} S(t_i - s) g(s) dw(s) \right] + \lambda \int_0^t S(t-s) f(s, x_s) ds + \lambda \int_0^t S(t-s) g(s) dw(s). \tag{3.28}$$

Thus

$$\|x(t)\|^2 \le 3\left( 2mM^2 \sum_{i=1}^{m} |\gamma_i|^2 \|B\|^2 + T \right) M^2 \int_0^t \|f(s, x_s)\|^2 ds + 3\left( 2mM^2 \sum_{i=1}^{m} |\gamma_i|^2 \|B\|^2 + 1 \right) M^2 \operatorname{Tr}(Q) \int_0^t \|g(s)\|^2 ds. \tag{3.29}$$

Conditions $(H_1)$–$(H_3)$ imply that for each $t \in J$

$$E\|x\|^2 \le 3\left( 2mM^2 \sum_{i=1}^{m} |\gamma_i|^2 \|B\|^2 + T \right) M^2 \int_0^t \left[ C_1 E\|x(s)\|^2 + C_2 \right] ds + 3\left( 2mM^2 \sum_{i=1}^{m} |\gamma_i|^2 \|B\|^2 + 1 \right) M^2 \operatorname{Tr}(Q) \int_0^t p(s) \psi\left( E\|x(s)\|^2 \right) ds. \tag{3.30}$$

The function $\varrho$ defined on $[0, T]$ by

$$\varrho(t) = \sup\left\{ E\|x(s)\|^2 : 0 \le s \le t \right\} \tag{3.31}$$

satisfies

$$\varrho(t) \le 3T\left( 2mM^2 \sum_{i=1}^{m} |\gamma_i|^2 \|B\|^2 + T \right) M^2 C_2 + 3T\left( 2mM^2 \sum_{i=1}^{m} |\gamma_i|^2 \|B\|^2 + T \right) M^2 C_1 \varrho(t) + 3\left( 2mM^2 \sum_{i=1}^{m} |\gamma_i|^2 \|B\|^2 + 1 \right) M^2 \operatorname{Tr}(Q) \|p\|_{L^1} \psi(\varrho(t)). \tag{3.32}$$

This yields

$$\varrho(t) \le \frac{3T\mathcal{K}_2 C_2 + 3\mathcal{K}_1 \operatorname{Tr}(Q) \|p\|_{L^1} \psi(\varrho(t))}{1 - 3T\mathcal{K}_1 C_1}. \tag{3.33}$$

Since

$$\|x\|_{\mathcal{B}_{\mathcal{T}}} = \sup_{0 \le t \le T} \varrho(t), \tag{3.34}$$

it follows that

$$\|x\|_{\mathcal{B}_\tau} \le \frac{3T\mathcal{K}_2 C_2 + 3\mathcal{K}_1 \operatorname{Tr}(Q)\|p\|_{L^1}\psi(\|x\|_{\mathcal{B}_\tau})}{1 - 3T\mathcal{K}_1 C_1}. \tag{3.35}$$

Therefore

$$\frac{(1 - 3T\mathcal{K}_1 C_1)\|x\|_{\mathcal{B}_\tau}}{3T\mathcal{K}_2 C_2 + 3\mathcal{K}_1 \operatorname{Tr}(Q)\|p\|_{L^1}\psi(\|x\|_{\mathcal{B}_\tau})} \le 1. \tag{3.36}$$

Now, by (3.3) there exists $\rho_0 > 0$ such that

$$\frac{\{1 - 3T\mathcal{K}_1 C_1\}\rho_0}{3T\mathcal{K}_2 C_2 + 3\mathcal{K}_1 \operatorname{Tr}(Q)\|p\|_{L^1}\psi(\rho_0)} > 1. \tag{3.37}$$

Let $\mathfrak{Y} = \{v \in \mathcal{B}_\tau : \|v\|_{\mathcal{B}_\tau} < \rho_0\}$. Suppose that there is $v \in \partial\mathfrak{Y}$ such that $v \in \lambda\mathcal{M}(v)$ for $\lambda \in (0,1)$. Then $\|x\|_{\mathcal{B}_\tau} = \varrho_0$ satisfies (3.36), which contradicts (3.37). So, alternative (ii) in Theorem 2.1. does not hold, and consequently, the multivalued operator $\mathcal{M}$ has a fixed point, which is a solution of (1.1). □

We now present another existence result for system (1.1). We shall assume that the single-valued $f$ and the multivalued $G$ satisfy a Wintner-type growth condition with respect to their second variable.

**Theorem 3.2.** *Assume that* $(H_2)$ *and the following condition hold.*

$(H_{fG})$: *There exists* $\ell \in L^1([0,T],\mathfrak{R}^+)$ *such that*

$$\begin{gathered} H(f(t,x_t), f(t,y_t)) \vee H(G(t,x_t), G(t,y_t)) \le \ell(t)\|x-y\|^2, \quad \forall t \in J, x, y \in D_\tau, \\ H(0, f(t,0)) \vee H(0, G(t,0)) \le \ell(t), \quad a.e.\ t \in J, \end{gathered} \tag{3.38}$$

*then the system* (1.1) *has at least one mild solution on* $(-\infty, T]$.

*Remark 3.3.* $H(f(t,x_t), f(t,y_t)) = \|f(t,x_t) - f(t,y_t)\|^2$.

*Proof.* The multivalued operator $\mathcal{M}$ defined in the proof of the previous theorem is completely continuous and upper semicontinuous. Now, we prove that

$$\mathfrak{Y} = \{x \in \mathcal{B}_\tau : x \in \lambda\mathcal{M}(x) \text{ for some } \lambda \in (0,1)\} \tag{3.39}$$

is bounded. Let $x \in \mathfrak{Y}$. Then there exists $g \in S_{G,x}$ such that for each $t \in J$

$$\begin{aligned} x(t) = \lambda \sum_{i=1}^{m} \gamma_i S(t) B \left[ \int_0^{t_i} S(t_i - s) f(s, x_s) ds + \int_0^{t_i} S(t_i - s) g(s) dw(s) \right] \\ + \lambda \int_0^t S(t-s) f(s, x_s) ds + \lambda \int_0^t S(t-s) g(s) dw(s), \end{aligned} \tag{3.40}$$

for some $\lambda \in (0,1)$. Then

$$\begin{aligned} \|x(t)\|^2 &\le 3\left( 2mM^2 \sum_{i=1}^{m} t_i \|\gamma_i\|^2 \|B\|^2 + T \right) M^2 \int_0^t \|f(s, x_s)\|^2 ds \\ &\quad + 3\left( 2mM^2 \sum_{i=1}^{m} \|\gamma_i\|^2 \|B\|^2 + 1 \right) M^2 \operatorname{Tr}(Q) \int_0^t \|g(s)\|^2 ds \\ &\le 6\left( 2mM^2 \sum_{i=1}^{m} t_i \|\gamma_i\|^2 \|B\|^2 + T \right) M^2 \int_0^t \ell(s) \left( 1 + \|x(s)\|^2 \right) ds \\ &\quad + 6\left( 2mM^2 \sum_{i=1}^{m} \|\gamma_i\|^2 \|B\|^2 + 1 \right) M^2 \operatorname{Tr}(Q) \int_0^t \ell(s) \left( 1 + \|x(s)\|^2 \right) ds. \end{aligned} \tag{3.41}$$

Thus

$$E\|x(t)\|^2 \le Q_1 + Q_2 \int_0^t \ell(s) E\|x(s)\|^2 ds, \tag{3.42}$$

where

$$\begin{aligned} Q_1 &= 6\left( 2mM^2 \sum_{i=1}^{m} t_i \|\gamma_i\|^2 \|B\|^2 + T \right) M^2 (T + \operatorname{Tr}(Q)) \|\ell\|_{L^1}, \\ Q_2 &= 6\left( 2mM^2 \sum_{i=1}^{m} \|\gamma_i\|^2 \|B\|^2 + 1 \right) M^2 (T + \operatorname{Tr}(\mathrm{Q})). \end{aligned} \tag{3.43}$$

Using the function $\varrho(t)$, defined by (3.31), we obtain

$$\varrho(t) \le Q_1 + Q_2 \int_0^t \ell(s) \varrho(s) ds. \tag{3.44}$$

Gronwall's inequality gives

$$\varrho(t) \le Q_1 \exp(Q_2 \|\ell\|_{L^1}), \quad \forall t \in J. \tag{3.45}$$

Therefore there exists $\beta > 0$ such that

$$\varrho(t) \leq \beta, \quad \forall t \in J, \tag{3.46}$$

which implies that

$$\|x\|_{\mathcal{B}_\tau}^2 \leq \beta. \tag{3.47}$$

This shows that $\mathfrak{Y}$ is bounded. Theorem 2.1. shows that $\mathcal{M}$ has a fixed point, which is a solution of (1.1), and this completes the proof. □

## 4. Example

Consider the following stochastic partial differential inclusion with infinite delay

$$\begin{aligned}
&\frac{\partial}{\partial t}v(t,x) \in \sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}v(t,x)\right) - a_0 v(t,x) + \epsilon\sum_{i=1}^{n}\frac{\partial}{\partial x_i}v(t-r,x) \\
&\qquad + \int_{-\infty}^{0}\beta_1(\theta)v(t+\theta,x)d\theta + \int_{-\infty}^{0}\beta_2(t,x,\theta)G_1(v(t+\theta,x))d\theta d\beta(t) \\
&v(t,x) = 0, \quad t \in J,\ x \in \partial\Delta, \\
&v(0,x) = \sum_{i=1}^{n}\widehat{\beta}_k(x)v(x,t_k), \quad x \in \Delta,\ t_k \in [0,T], \\
&v(\theta,x) = \varphi(0,x), \quad -\infty < \theta \leq 0,\ x \in \Delta,
\end{aligned} \tag{4.1}$$

where $a_0$, $r$, and $\epsilon$ are positive constants, $J = [0,T]$, $\Delta$ is an open bounded set in $\mathfrak{R}^n$ with a smooth boundary $\partial\Delta$, $\beta_1 : (-\infty,0] \to \mathfrak{R}$ is a positive function, $\beta(t)$ stands for a standard cylindrical Wiener process in $L^2(\Delta)$ defined on a stochastic basis $(\Omega, \mathcal{F}, P)$, and $\varphi \in D^b_{\mathcal{F}_0}((-\infty,0],\ L^2(\Delta))$.

The coefficients $a_{ij} \in L^\infty(\Delta)$ are symmetric and satisfy the ellipticity condition

$$\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j \geq \varkappa|\xi|^2, \quad x \in \Delta,\ \xi \in \mathfrak{R}^n, \tag{4.2}$$

for a positive constant $\varkappa$.

In order to rewrite (4.1) in the abstract form, we introduce $X = L^2(\Delta)$ and we define the linear operator $A : D(A) \subset X \to X$ by

$$D(A) = H^2(\Delta) \cap H_0^1(\Delta); \qquad A = -\sum_{i,j}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right). \tag{4.3}$$

Here $H^1(\Delta)$ is the Sobolev space of functions $u \in L^2(\Delta)$ with distributional derivative $u' \in L^2(\Delta)$, $H_0^1(\Delta) = \{u \in H^1(\Delta);\ u = 0 \text{ on } \partial\Delta\}$ and $H^2(\Delta) = \{u \in L^2(\Delta);\ u',\ u'' \in L^2(\Delta)\}$.

Then $A$ generates a symmetric compact analytic semigroup $e^{-tA}$ in $X$, and there exists a constant $M_1 > 0$ such that $\|e^{-tA}\| \le M_1$. Also, note that there exists a complete orthonormal set $\{\xi_n\}$, $(n = 1, 2, \ldots)$ of eigenvectors of $A$ with $\xi_n(x) = \sqrt{(2/n)}\sin(nx)$.

We assume the following conditions hold.

(i) The function $\beta_1(\cdot)$ is continuous in $J$ with

$$\int_{-\infty}^{0} \beta_1(\theta)^2 d\theta < \infty. \tag{4.4}$$

(ii) The function $\beta_2(\cdot) \ge 0$ is continuous in $J \times \Delta \times (-\infty, 0)$ with

$$\int_{-\infty}^{0} \beta_2(t, x, \theta) d\theta = p_1(t, x) < \infty, \qquad \left(\int_{\Delta} p_1^2(t, x) dx\right)^{1/2} < \infty. \tag{4.5}$$

(iii) The multifunction $G_1(\cdot)$ is an $L^2$-Carathèodory multivalued function with compact and convex values and

$$0 \le \|G_1(v(\theta, x))\| \le \varphi_0(\|v(\theta, \cdot)\|_{L^2}), \quad (\theta, x) \in J \times \Delta, \tag{4.6}$$

where $\varphi_0(\cdot) : [0, \infty) \longrightarrow (0, \infty)$ is continuous and nondecreasing.

Assuming that conditions (i)–(iii) are verified, then the problem (4.1) can be modeled as the abstract stochastic partial functional differential inclusions of the form (1.1), with

$$\begin{gathered} f(t, v_t) = \int_{-\infty}^{0} \beta_1(\theta) v(t + \theta, x) d\theta \\ G(t, v_t) = \int_{-\infty}^{0} \beta_2(t, x, \theta) G_1(v(t + \theta, x)) d\theta, \qquad \gamma_i = \widehat{\beta}_k(x). \end{gathered} \tag{4.7}$$

The next result is a consequence of Theorem 3.1.

**Proposition 4.1.** *Assume that the conditions* (*i*)–(*iii*) *hold. Then there exists at least one mild solution* $v$ *for the system* (4.1) *provided that*

$$\sup_{\rho \in [0, \infty)} \frac{\{1 - 3T\mathbb{K}_2 C_1\}\rho}{3\mathbb{K}_1 Tr(Q)\|p\|_{L^1}\varphi_0(\rho)} > 1, \tag{4.8}$$

*where* $\mathbb{K}_1 = (2mM_1^2\sum_{i=1}^{m}\|\gamma_i\|^2\|B\|^2 + 1)M_1^2$ *and* $\mathbb{K}_2 = (2mM_1^2\sum_{i=1}^{m} t_i\|\gamma_i\|^2\|B\|^2 + T)M_1^2$.

*Proof.* Condition (i) implies that $(H_1)$ holds with $C_1 = \int_{-\infty}^{0} \beta_1^2(\theta) d\theta$ and $C_2 = 0$. $(H_2)$ and $(H_3)$ follow from conditions (ii) and (iii) with $p(t) = (\int_{\Delta} p_1^2(t, x) dx)^{1/2}$ and $\varphi = \varphi_0$. □

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## References


[1] K. Balachandran and K. Uchiyama, "Existence of solutions of nonlinear integrodifferential equations of Sobolev type with nonlocal condition in Banach spaces," *Proceedings of the Indian Academy of Sciences (Mathematical Sciences)*, vol. 110, no. 2, pp. 225–232, 2000.

[2] L. Byszewski, "Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem," *Journal of Mathematical Analysis and Applications*, vol. 162, no. 2, pp. 494–505, 1991.

[3] K. Deng, "Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions," *Journal of Mathematical Analysis and Applications*, vol. 179, no. 2, pp. 630–637, 1993.

[4] M. Benchohra, E. P. Gatsori, J. Henderson, and S. K. Ntouyas, "Nondensely defined evolution impulsive differential inclusions with nonlocal conditions," *Journal of Mathematical Analysis and Applications*, vol. 286, no. 1, pp. 307–325, 2003.

[5] A. Boucherif and R. Precup, "Semilinear evolution equations with nonlocal initial conditions," *Dynamic Systems and Applications*, vol. 16, no. 3, pp. 507–516, 2007.

[6] X. Fu and K. Ezzinbi, "Existence of solutions for neutral functional differential evolution equations with nonlocal conditions," *Nonlinear Analysis*, vol. 54, no. 2, pp. 215–227, 2003.

[7] J. Liang and Ti.-J. Xiao, "Semilinear integrodifferential equations with nonlocal initial conditions," *Computers & Mathematics with Applications*, vol. 47, no. 6-7, pp. 863–875, 2004.

[8] A. Boucherif, "Semilinear evolution inclusions with nonlocal conditions," *Applied Mathematics Letters*, vol. 22, no. 8, pp. 1145–1149, 2009.

[9] N. U. Ahmed, "Nonlinear stochastic differential inclusions on Banach space," *Stochastic Analysis and Applications*, vol. 12, no. 1, pp. 1–10, 1994.

[10] P. Balasubramaniam and S. K. Ntouyas, "Controllability for neutral stochastic functional differential inclusions with infinite delay in abstract space," *Journal of Mathematical Analysis and Applications*, vol. 324, no. 1, pp. 161–176, 2006.

[11] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, vol. 44, Cambridge University Press, Cambridge, UK, 1992.

[12] P. Krée, "Diffusion equation for multivalued stochastic differential equations," *Journal of Functional Analysis*, vol. 49, no. 1, pp. 73–90, 1982.

[13] S. Aizicovici and V. Staicu, "Multivalued evolution equations with nonlocal initial conditions in Banach spaces," *Nonlinear Differential Equations and Applications*, vol. 14, no. 3-4, pp. 361–376, 2007.

[14] D. N. Keck and M. A. McKibben, "Functional integro-differential stochastic evolution equations in Hilbert space," *Journal of Applied Mathematics and Stochastic Analysis*, vol. 16, no. 2, pp. 141–161, 2003.

[15] A. Lin and L. Hu, "Existence results for impulsive neutral stochastic functional integro-differential inclusions with nonlocal initial conditions," *Computers & Mathematics with Applications*, vol. 59, no. 1, pp. 64–73, 2010.

[16] A. Elazzouzi and A. Ouhinou, "Optimal regularity and stability analysis in the $\alpha$-norm for a class of partial functional differential equations with infinite delay," *Discrete and Continuous Dynamical Systems*, vol. 30, no. 1, pp. 115–135, 2011.

[17] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, vol. 44, Springer, New York, NY, USA, 1983.

[18] J.-P. Aubin and A. Cellina, *Differential Inclusions*, vol. 264, Springer, Berlin, Germany, 1984.

[19] K. Deimling, *Multivalued Differential Equations*, vol. 1, Walter de Gruyter, New York, NY, USA, 1992.

[20] S. Hu and N. S. Papageorgiou, "On the existence of periodic solutions for nonconvex-valued differential inclusions in $\mathbb{R}^n$," *Proceedings of the American Mathematical Society*, vol. 123, no. 10, pp. 3043–3050, 1995.

[21] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer, New York, NY, USA, 2003.

[22] A. Lasota and Z. Opial, “An application of the Kakutani-Ky-Fan theorem in the theory of ordinary differential equations,” *Bulletin de l’Académie Polonaise des Sciences. Série des Sciences Mathématiques, Astronomiques et Physiques*, vol. 13, pp. 781–786, 1965.

[23] J. Hofbauer and P. L. Simon, “An existence theorem for parabolic equations on $\mathbb{R}^N$ with discontinuous nonlinearity,” *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 8, pp. 1–9, 2001.