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Research Article

A q-Weibull Counting Process through a Fractional Differential Operator

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We use the q-Weibull distribution and define a new counting process using the fractional order. As a consequence, we introduce a q-process with q-Weibull interarrival times. Some interesting special cases are also discussed which leads to a Mittag-Leffler form.

1. Introduction

The concept of renewal process has been developed for describing a counting process with the assumption that the times between successive events are independent and identically distributed nonnegative random variables. Count models are used in a wide range of disciplines. For an early application and survey in economics see Cameron and Trivedi [1]; for more recent developments, see Winkelmann [2]; for a comprehensive survey of the literature, see Gurmu and Trivedi [3]. The Poisson process plays a fundamental role in renewal theory (see Muralidharan [4] for more details and the references contained therein).

The data of Jaggia and Thosar [5] on the number of takeover bids received by a target firm after an initial bid illustrate the use of small counts in a Poisson model. For a Poisson count model, there is a general consensus that the inter arrival time follows an exponential distribution. The Poisson count models are valid only when the mean and variance are equal, which gives the condition of equidispersion. But in general, in real life situations, the models are overdispersed which satisfies the heavy-tailedness property. The data of Greene [6] on the number of major derogatory reports in the credit history of individual credit card applicants illustrate over-dispersion, that is, the sample variance is considerably greater than the sample mean, compared to the Poisson which imposes equality of population mean and variance, and excess zeros are present since the observed proportion of zero counts is considerably greater than the predicted probability. An indication of the likely magnitude of underdispersion and over-dispersion can be obtained by comparing the sample mean and variance of the dependent count variable, as subsequent Poisson regression will decrease the conditional variance of the dependent variable. If the sample variance is less than the sample mean, the data will be even more under-dispersed once regressors are included, while if the sample variance is more than twice the sample mean, the data will almost certainly be overdispersed upon inclusion of regressors.

Poisson processes with exponential, gamma, and Weibull, distributions as inter arrival times were developed and their limitations were studied by many authors including Winkelmann [2, 7]. The simplest model for duration data is the exponential, the duration distribution being implied by the pure Poisson process, with density $\lambda e^{-\lambda t}$ and constant hazard rate λ. The restriction of a constant hazard rate is generally not appropriate for econometric data, and we move immediately to the analysis of the Weibull model, which nests the exponential as a special case. Here we consider such a problem with q-Weibull inter arrival times, which incorporates the Weibull distribution as the pathway parameter tends to unity. This model generalizes the Poisson model with the generalized Weibull as inter arrival times since the q-Weibull distribution allows a transition to the original Weibull distribution. Also the q-Weibull distribution nests the exponential, Weibull and q-exponential distributions as special cases.

In 2005, Mathai introduced a pathway model connecting matrix variate Gamma and normal densities. For the pathway parameter $q > 1$, $-\infty < x < \infty$, $a > 0$, $\alpha > 0$, $\beta > 0$, $\delta > 0$, the following is the scalar version of the pathway model:

$$f_1(x) = c_1|x|^{\alpha-1}\left[1 + a(q-1)|x|^{\delta}\right]^{-\beta/(q-1)}, \tag{1.1}$$

where the normalizing constant c_1 is given by

$$c_1 = \frac{\delta\left[a(q-1)\right]^{\alpha/\delta}\Gamma(\beta/(q-1))}{2\Gamma(\alpha/\delta)\Gamma(\beta/(q-1)-\alpha/\delta)}, \qquad \frac{\beta}{q-1} > \frac{\alpha}{\delta}. \tag{1.2}$$

Observe that, for $q < 1$, on writing $q-1 = -(1-q)$, the density in (1.1) reduces to the following form:

$$f_2(x) = c_2|x|^{\alpha-1}\left[1 - a(1-q)|x|^{\delta}\right]^{\beta/(q-1)}, \tag{1.3}$$

for $q < 1$, $a > 0$, $\alpha > 0$, $\beta > 0$, $\delta > 0$, $1 - a(1-q)|x|^{\delta} > 0$ and the normalizing constant c_2 being given by

$$c_2 = \frac{\delta\left[a(1-q)\right]^{\alpha/\delta}\Gamma(\beta/(1-q)+\alpha/\delta+1)}{2\Gamma(\alpha/\delta)\Gamma(\beta/(1-q)+1)}. \tag{1.4}$$

As $q \to 1$, $f_1(x)$ and $f_2(x)$ tend to $f_3(x)$, which is referred to as the "extended symmetric Weibull distribution", where $f_3(x)$ is given by

$$f_3(x) = \frac{\delta(\alpha\beta)^{\alpha/\delta}}{2\Gamma(\alpha/\delta)}|x|^{\alpha-1}\exp\left(-\alpha\beta|x|^{\delta}\right), \quad -\infty < x < \infty, \quad a, \ \alpha, \ \beta, \ \delta > 0. \tag{1.5}$$

For different values of the parameters in (1.1), (1.3), and (1.5), we get different distributions like the Weibull, gamma, beta type-1, beta type-2, and so forth. More results are available in Mathai and Haubold [8] and Mathai and Provost [9]. The q-Weibull distribution is a generalized model for the q-exponential distribution which facilitates a transition to the Weibull as $q \to 1$ through the pathway parameter q. Some results in this paper are defined in terms of H-function which can be defined as follows:

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\phi(s)z^{-s}ds, \tag{1.6}$$

where

$$\phi(s) = \frac{\left\{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\right\}\left\{\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)\right\}}{\left\{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\right\}\left\{\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)\right\}}. \tag{1.7}$$

Now we define the fractional order derivative in the Caputo sense, which provides a fractional generalization of the first derivative through the following rule in the Laplace domain:

$$L\left\{{}_tD_*^{\beta}f(t);s\right\} = s^{\beta}\widetilde{f(s)} - s^{\beta-1}f(0^+), \quad 0 < \beta \le 1, \quad s > 0. \tag{1.8}$$

The Caputo derivative has been indexed with the subscript $*$ in order to distinguish it from the classical Riemann-Liouville fractional derivative ${}_tD^{\beta}$. It can be noted from the (1.8) that the Caputo derivative provides a sort of regularization of the Riemann-Liouville derivative at $t = 0$.

2. Preliminaries on the q-Weibull Distribution

The q-Weibull distribution gives a wide range of applications in reliability analysis, statistical mechanics, various engineering fields and so forth. (Picoli et al. [10], Jose et al. [11], Jose and Naik [12], etc.] as a consequence of the pathway model introduced by Mathai [13]. The probability density function (pdf) of the q-Weibull random variable is given as follows:

$$f_1(x) = \alpha\lambda^{\alpha}(2-q)x^{\alpha-1}\left[1-(1-q)(\lambda x)^{\alpha}\right]^{1/(1-q)}, \quad 0 \le x \le \frac{1}{\lambda(1-q)^{1/\alpha}}, \quad \alpha, \ \lambda > 0. \tag{2.1}$$

The parameter q is known as the pathway parameter (the entropy index in statistical mechanics), which determines the shape of the curves. We call this p.d.f a q-Weibull distribution of type-1. Observe that, for $q > 1$, writing $1 - q = -(q - 1)$, the density in (2.1) assumes another form which we call it as type-2 q-Weibull model which is defined as follows. For $x \geq 0$ and for $\alpha, \lambda > 0$, $1 < q < 2$ the probability density function is given by

$$f_2(x) = \alpha\lambda^{\alpha}(2-q)x^{\alpha-1}\left[1+(q-1)(\lambda x)^{\alpha}\right]^{-1/(q-1)}, \quad x \geq 0. \tag{2.2}$$

As q approaches to one, we obtain Weibull density.

Figure 1 shows a comparison between the Weibull distribution ($q = 1$) and the q-Weibull distribution for different values of q. As q goes to 2, we get thicker tailed curves, and as q tends to 1 from the right, the curve becomes peaked and slowly moves to the curve corresponding to the Weibull distribution. As q moves from the left, the mode decreases, and the curve slowly approaches the Weibull distribution, that is, the curve for $q = 1$. The main reason for introducing the q-Weibull model is the switching property of the exponential form to corresponding binomial function. That is,

$$\lim_{q\to 1} F_0\left(\frac{1}{q-1};\ -(q-1)z\right) = \lim_{q\to 1}\left[1+(q-1)z\right]^{-1/(q-1)} = e^{-z}. \tag{2.3}$$

The pathway model includes a wide variety of continuous distributions since one can move from one functional form to another through the pathway parameter. For $\alpha = 1$, we get the q-exponential distribution introduced by Tsallis [14]. As $\alpha = 1$ and $q \to 1$, we get the corresponding exponential distribution. The survival function of the q-Weibull distribution is given by

$$\begin{aligned} \overline{F}_1(t) &= \left[1+(q-1)(\lambda t)^{\alpha}\right]^{(q-2)/(q-1)}, \quad \text{for } 1 < q < 2, \\ \overline{F}_2(t) &= \left[1-(1-q)(\lambda t)^{\alpha}\right]^{(2-q)/(1-q)}, \quad \text{for } q < 1. \end{aligned} \tag{2.4}$$

As $q \to 1$ both the survival functions converge to the survival function of the extended Weibull distribution. For both $1 < q < 2$ and $q < 1$, the survival functions are decreasing and tend to 0. The hazard rate function (HRF) of the q-Weibull distribution is given by

$$\begin{aligned} h_1(t) &= \frac{\alpha\lambda^{\alpha}(2-q)t^{\alpha-1}}{1+(q-1)(\lambda t)^{\alpha}}, \quad \text{for } t \geq 0 \quad 1 < q < 2, \\ h_2(t) &= \frac{\alpha\lambda^{\alpha}(2-q)t^{\alpha-1}}{1-(1-q)(\lambda t)^{\alpha}}, \quad \text{for } |t| < \frac{1}{\lambda(1-q)^{1/\alpha}} \quad q < 1. \end{aligned} \tag{2.5}$$

The q-Weibull distribution nests three other distributions as special cases. The hazard rate function is nonmonotonic when $\alpha > 1$ and $\beta > 1$.

Remark 2.1. As $q \to 1$ and $\alpha = 1$, we get the hazard rate function of exponential distribution.

Remark 2.2. As $q \to 1$, we get the hazard rate function Weibull model.

Remark 2.3. If $q \neq 1$ and $\alpha = 1$, we get hazard rate function of the q-exponential model.

3. A Generalized q-Counting Process

In this section, we generalize the Poisson process in another direction by introducing a counting process using q-Weibull inter arrival times. For the Poisson process, the mean and variance are equal, which is not realistic. Now, following McShane et al. [15], we introduce a model which gives more flexibility to the mean-variance relationship.

Let Y_n be the time from the measurement origin at which the nth event occurs. Let $X(t)$ be the number of events that have occurred until the time t. Then $Y_n \leq t \Leftrightarrow X(t) \geq n$. In other words the amount of time at which the nth event occurred from the time origin is less than or equal to t if and only if the number of events that have occurred by time t is greater than or equal to n. Now, the count model, $P_n(t)$, is given by

$$\begin{aligned} P_n(t) &= P[X(t) = n] = P[X(t) \geq n] - P[X(t) \geq n+1] \\ &= P[Y_n \leq t] - P[Y_{n+1} \leq t] = F_n(t) - F_{n+1}(t), \end{aligned} \tag{3.1}$$

where $F_n(t)$ is the CDF of Y_n. When the counting process coincide with the occurrence of an event, then $F_n(t)$ is the n-fold convolution of the common inter arrival time distribution. Now we consider the count model for the q-Weibull distribution by assuming that the inter arrival times are i.i.d q-Weibull distributed, with cumulative distribution function (CDF),

$$F(t) = \begin{cases} 1 - \left[1 - (1-q)(\lambda t)^{\alpha}\right]^{(2-q)/(1-q)}, & 0 \leq t \leq \dfrac{1}{\lambda(1-q)^{1/\alpha}}, \\ 1, & t \geq \dfrac{1}{\lambda(1-q)^{1/\alpha}}. \end{cases} \tag{3.2}$$

Here we consider the case $q < 1$. The series expansion for the CDF is given by

$$F(t) = 1 - \sum_{j=0}^{\infty} (-1)^j \frac{((2-q)/(1-q))_j \left[(1-q)(\lambda t)^{\alpha}\right]^j}{\Gamma(j+1)} \tag{3.3}$$

where $(\alpha)_j$ is the Pochammer symbol. Then corresponding $f(t)$ is given by

$$f(t) = \sum_{j=0}^{\infty} (-1)^j \frac{\alpha j((2-q)/(1-q))_j \ (1-q)^j \lambda^{\alpha} t^{\alpha j-1}}{\Gamma(j+1)}. \tag{3.4}$$

We have also the recursive relationship

$$P_n(t) = \int_0^t [F_{n-1}(t-s) - F_n(t-s)] f(s) ds = \int_0^t P_{n-1}(t-s) f(s) ds. \tag{3.5}$$

Now, $F_0(t) = 1$, for all t and $F_1(t) = F(t)$. Also

$$P_0(t) = F_0(t) - F_1(t) = \left[1 - (1-q)(\lambda t)^{\alpha}\right]^{(2-q)/(1-q)} = \sum_{j=0}^{\infty} (-1)^j \frac{((2-q)/(1-q))_j \left[(1-q)(\lambda t)^{\alpha}\right]^j}{\Gamma(j+1)}. \tag{3.6}$$

Using the recursive formula

$$\begin{aligned}
P_1(t) &= \int_0^t P_0(t-s) f(s) ds \\
&= \sum_{j=0}^{\infty} \sum_{i=1}^{\infty} (-1)^{j+i+1} \frac{((2-q)/(1-q))_i ((2-q)/(1-q))_j (1-q)^{i+j} (\lambda t)^{\alpha(i+j)} \alpha i \Gamma(\alpha i+1) \Gamma(\alpha j+1)}{\Gamma(j+1)\Gamma(i+1)\Gamma(\alpha(i+j)+1)} \\
&= \sum_{j=1}^{\infty} (-1)^{j+1} \frac{\left[(1-q)(\lambda t)^{\alpha}\right]^j}{\Gamma(\alpha j+1)} \sum_{i=0}^{j-1} \left(\frac{2-q}{1-q}\right)_{j-i} \left(\frac{2-q}{1-q}\right)_i \frac{\Gamma(\alpha i+1)\Gamma(\alpha(j-i)+1)}{\Gamma(i+1)\Gamma(j-i+1)} \\
&= \sum_{j=1}^{\infty} (-1)^{j+1} \frac{\left[(1-q)(\lambda t)^{\alpha}\right]^j a_i^j}{\Gamma(\alpha j+1)},
\end{aligned} \tag{3.7}$$

where

$$a_i^j = \sum_{i=0}^{j-1} \left(\frac{2-q}{1-q}\right)_i \left(\frac{2-q}{1-q}\right)_{j-i} \frac{\Gamma(\alpha i+1)\Gamma(\alpha(j-i)+1)}{\Gamma(i+1)\Gamma(j-i+1)}, \tag{3.8}$$

which suggest a general form

$$\begin{aligned}
P_{n+1}(t) &= \int_0^t P_n(t-s) f(s) ds \\
&= \sum_{j=n}^{\infty} \sum_{i=1}^{\infty} (-1)^{j+i+n+1} \frac{((2-q)/(1-q))_i (1-q)^{i+j} (\lambda t)^{\alpha(i+j)} a_j^n \Gamma(\alpha i+1)\Gamma(\alpha j+1)}{\Gamma(\alpha j+1)\Gamma(i+1)\Gamma(\alpha(i+j)+1)}.
\end{aligned} \tag{3.9}$$

On simplification, we get

$$\begin{aligned}
P_{n+1}(t) &= \sum_{l=n+1}^{\infty} (-1)^{l+n+1} \frac{\left[(1-q)(\lambda t)^{\alpha}\right]^l}{\Gamma(\alpha l+1)} \sum_{k=n}^{l-1} \frac{((2-q)/(1-q))_{l-k} a_k^n \Gamma(\alpha(l-k)+1)}{\Gamma(\alpha-k+1)} \\
&= \sum_{l=n+1}^{\infty} (-1)^{l+n+1} \frac{\left[(1-q)(\lambda t)^{\alpha}\right]^l a_l^{n+1}}{\Gamma(\alpha l+1)}.
\end{aligned} \tag{3.10}$$

Thus we have the model

$$P[N(t)=n]=\sum_{l=n}^{\infty}(-1)^{l+n}\frac{\left[(1-q)(\lambda t)^{\alpha}\right]^{l}a_{l}^{n}}{\Gamma(\alpha l+1)},\quad n=0,1,2,\ldots, \tag{3.11}$$

where $a_j^0 = ((2-q)/(1-q))_j\Gamma(\alpha j+1)/\Gamma(j+1)$, for $j=0,1,\ldots$ and $a_j^{n+1} = \sum_{i=n}^{j-1}((2-q)/(1-q))_{j-i}\Gamma(\alpha(j-i)+1)/\Gamma(j-i+1)a_i^n$, for $n=0,1,\ldots$, $j=n+1,n+2,\ldots$, which is a Mittag-Leffler type distribution. The mean count of this distribution is given by

$$E(N)=\sum_{n=1}^{\infty}\sum_{l=n}^{\infty}(-1)^{l+n}\frac{n\left[(1-q)(\lambda t)^{\alpha}\right]^{l}a_{l}^{n}}{\Gamma(\alpha l+1)} \tag{3.12}$$

with variance given by

$$\operatorname{Var}(N)=\sum_{n=2}^{\infty}\sum_{l=n}^{\infty}(-1)^{l+n}\frac{n^{2}\left[(1-q)(\lambda t)^{\alpha}\right]^{l}a_{l}^{n}}{\Gamma(\alpha l+1)}-\left(\sum_{n=1}^{\infty}\sum_{l=n}^{\infty}(-1)^{l+n}\frac{n\left[(1-q)(\lambda t)^{\alpha}\right]^{l}a_{l}^{n}}{\Gamma(\alpha l+1)}\right)^{2}. \tag{3.13}$$

Then the moment generating function can be readily obtained as

$$M_i(\theta)=E\left[e^{\theta N}\right]=\sum_{i=0}^{\infty}\sum_{j=i}^{\infty}(-1)^{j+i}\frac{e^{iN}\left[(1-q)(\lambda t)^{\alpha}\right]^{l}a_{l}^{n}}{\Gamma(\alpha j+1)}. \tag{3.14}$$

Now this model generalizes the commonly used Poisson and Weibull count models; we call this new model a q-Weibull count model. We have already seen the probability density curves for the Weibull, Poisson, and the q-Weibull count models that are shown in Figure 1. Here we consider the under-dispersed case. In a similar manner, we can construct a heterogeneous q-Weibull count model with λ replaced by λ_i where λ_i follows gamma distribution with parameters m and p. The p.m.f. of such a process is given by

$$\begin{aligned}P[N(t)=n]&=\int_{0}^{\infty}\sum_{l=n}^{\infty}(-1)^{l+n}\frac{\left[(1-q)(\lambda t)^{\alpha}\right]^{l}a_{l}^{n}}{\Gamma(\alpha l+1)}f(\lambda_i\mid m,p)\,d\lambda_i\\&=\sum_{l=n}^{\infty}(-1)^{l+n}\frac{\left[(1-q)(t)^{\alpha}\right]^{l}a_{l}^{n}}{\Gamma(\alpha l+1)}\int_{0}^{\infty}\frac{1}{\Gamma(p)}m^{p}\lambda_i^{p+\alpha l-1}e^{-m\lambda_i}\,d\lambda_i\\&=\sum_{l=n}^{\infty}(-1)^{l+n}\frac{e^{im}\left[(1-q)(t)^{\alpha}\right]^{l}a_{l}^{n}\Gamma(p+\alpha l)}{\Gamma(\alpha l+1)m^{\alpha l}\Gamma(p)}.\end{aligned} \tag{3.15}$$

This probability model provides an entirely new class of counting processes derived using q-Weibull inter arrival times and is an improvement over the traditional Poisson process.

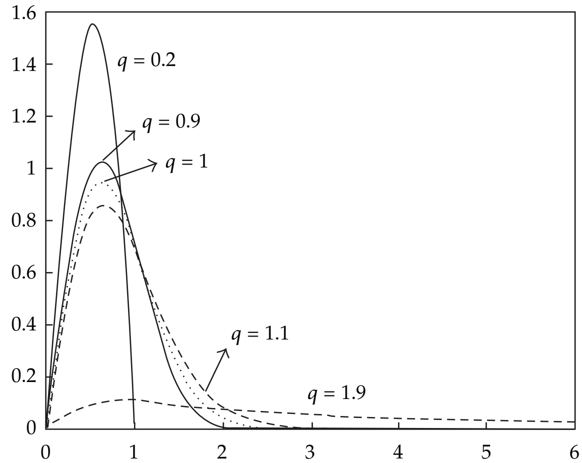

Figure 1: Weibull and q-Weibull pdf for different values of q.

4. Fractional Order Process with q-Weibull InterArrival Time

Let us consider the q-counting process with q-Weibull interarrival times. Here we consider a fractional generalization of the process using the fractional order differential operator instead of the ordinary differentiation. As the fractional order β is unity, one can obtain the corresponding process with q-Weibull interarrival time. If the survival probability for the generalized q-counting process is

$$\Psi(t) = \left[1 + (q-1)(\lambda t)^{\alpha}\right]^{(q-2)/(q-1)}, \tag{4.1}$$

then it obeys the following ordinary differential equation:

$$\frac{d}{dt}\Psi(t) = K t^{\alpha-1}[\Psi(t)]^{1/(2-q)}, \quad \text{where } K = \alpha(q-2)\lambda^{\alpha}. \tag{4.2}$$

Now, the generalization is obtained by replacing the first derivative by a fractional derivative (Caputo derivative of order $\beta \in (0,\ 1]$) in (4.2). Thus we have

$$\begin{aligned} {}_tD_*^{\beta}\Psi(t) &= K t^{\alpha-1}[\Psi(t)]^{1/(2-q)} \\ L\left\{{}_tD_*^{\beta}\Psi(t)\right\} &= KL\left\{t^{\alpha-1}[\Psi(t)]^{1/(2-q)}\right\} \\ p^{\beta}L\{[\Psi(t)]\} - p^{\beta-1} &= KL\left\{t^{\alpha-1}[\Psi(t)]^{1/(2-q)}\right\}. \end{aligned} \tag{4.3}$$

To find the Laplace transform involved in the right hand side of (4.3), we have to evaluate the integral

$$I_1(\alpha, \lambda, q : p) = \int_0^{\infty} e^{-px} x^{\alpha-1}\left[1 + (q-1)(\lambda x)^{\alpha}\right]^{-1/(q-1)} dx. \tag{4.4}$$

The integrand can be taken as a product of two integrable functions. Let x_1 and x_2 be two independent scalar random variables with probability density functions $f_1(x)$ and $f_2(x)$, respectively. Consider the transformation $u = x_1/x_2$ and $v = x_2 \Rightarrow dx_1 \wedge dx_2 = v du \wedge dv$, where $\wedge$ is the wedge product discussed in Mathai [13]. Then the joint density of u and v is $g(u,v) = v f_1(uv) f_2(v)$. Now, $g_1(u)$ is obtained by integrating the joint probability density function $g(u,v)$ with respect to v. That is, $g_1(u) = \int_v v f_1(uv) f_2(v) dv$. Let $f_1(x) = c_1[1 + (q-1)\lambda^\alpha {x_1}^\alpha]^{-1/(q-1)}$, $x_1 \geq 0$, and $f_2(x_2) = c_2 {x_2}^{\alpha-2} e^{-px_2}$, $x_2 \geq 0$, $\alpha, \lambda > 0$, where c_1 and c_2 are normalizing constants. These constants can be obtained by integrating $f_1(x)$ and $f_2(x)$ with respect to x. Thus we have

$$g_1(u) = c_1 c_2 \int_0^\infty e^{-pv} v^{\alpha-1} \left[1 + (q-1)(\lambda u v)^\alpha\right]^{-1/(q-1)} dv. \tag{4.5}$$

Now

$$\begin{aligned} E\left(x_1^{s-1}\right) &= c_1 \int_0^\infty {x_1}^{s-1} \left[1 + (q-1)(\lambda x_1)^\alpha\right]^{-1/(q-1)} dx_1 \\ &= \frac{c_1}{\alpha \lambda^s (q-1)^{s/\alpha}} \int_0^\infty w^{(s/\alpha)-1} \, (1+w)^{-1/(q-1)} dw \\ &= \frac{c_1}{\alpha \lambda^s (q-1)^{s/\alpha}} \frac{\Gamma(s/\alpha)\Gamma(1/(q-1) - s/\alpha)}{\Gamma(1/(q-1))}, \quad \Re\left(\frac{1}{q-1} - \frac{s}{\alpha}\right) > 0, \end{aligned} \tag{4.6}$$

where $\Re(\cdot)$ denote the real part of $(\cdot)$. Similarly

$$\begin{aligned} E\left(x_2^{1-s}\right) &= c_2 \int_0^\infty {x_2}^{\alpha-s-1} e^{-px_2} dx_2 \\ &= \frac{c_2 \Gamma(\alpha - s)}{p^{\alpha-s}}. \end{aligned} \tag{4.7}$$

Then

$$\begin{aligned} E\left(u^{s-1}\right) &= E\left(x_1^{s-1}\right) E\left(x_2^{1-s}\right) \\ &= c_1 c_2 \frac{p^{s-\alpha}}{\alpha \lambda^s (q-1)^{s/\alpha}} \frac{\Gamma(s/\alpha)\Gamma(1/(q-1) - s/\alpha)\Gamma(\alpha - s)}{\Gamma(1/(q-1))}. \end{aligned} \tag{4.8}$$

Now, the density of u is obtained by taking the inverse Mellin transform. The detailed existence conditions for the sMellin and inverse Mellin transforms are available in Mathai [13]. Thus

$$g_1(u) = \frac{c_1 c_2}{\alpha p^\alpha \Gamma(1/(q-1))} \frac{1}{2\pi_i} \int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{s}{\alpha}\right) \Gamma\left(\frac{1}{q-1} - \frac{s}{\alpha}\right) \Gamma(\alpha - s) \left(\frac{u(q-1)^{1/\alpha} \lambda}{p}\right)^{-s} ds, \tag{4.9}$$

and the quantity in (4.4) becomes

$$I_1(\alpha,\lambda,q:p) = \frac{c_1c_2}{\alpha p^{\alpha}\Gamma(1/(q-1))} H_{2,1}^{1,2}\left[\frac{(q-1)^{1/\alpha}\lambda}{p}\middle| \begin{matrix}(1-1/(q-1),1/\alpha),\ (1-\alpha,1)\\ (0,1/\alpha)\end{matrix}\right],$$

$$\Re\left(\frac{1}{q-1}-\frac{s}{\alpha}\right) > 0. \tag{4.10}$$

Hence (4.3) simplifies to

$$p^{\beta}\left(L\{[\Psi(t)]\}-p^{-1}\right) = \frac{K}{\alpha p^{\alpha}\Gamma(1/(q-1))} H_{2,1}^{1,2}\left[\frac{(q-1)^{1/\alpha}\lambda}{p}\middle| \begin{matrix}(1-1/(q-1),1/\alpha),\ (1-\alpha,1)\\ (0,1/\alpha)\end{matrix}\right],$$

$$\Re\left(\frac{1}{q-1}-\frac{s}{\alpha}\right) > 0. \tag{4.11}$$

Now

$$\Psi(t) = 1+\frac{\lambda^{\alpha}(q-2)t^{\alpha+\beta-1}}{\Gamma(1/(q-1))}\frac{1}{2\pi_i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s/\alpha)\Gamma(1/(q-1)-s/\alpha)\Gamma(\alpha-s)\left((q-1)^{1/\alpha}\lambda t\right)^{-s}}{\Gamma(\alpha+\beta-s)}ds$$

$$= 1+\frac{\lambda^{\alpha}(q-2)t^{\alpha+\beta-1}}{\Gamma(1/(q-1))} H_{2,2}^{1,2}\left[(q-1)^{1/\alpha}\lambda t\middle| \begin{matrix}(1-1/(q-1),1/\alpha),(1-\alpha,1)\\ (0,1/\alpha),(1-\alpha-\beta,1)\end{matrix}\right]. \tag{4.12}$$

For $t \geq 0$, $\alpha,\beta > 0$ and for $1 < q < 2$, the pdf is obtained as

$$f(t) = \frac{(2-q)\lambda^{\alpha}t^{\alpha+\beta-2}}{\Gamma(1/(q-1))} H_{2,2}^{1,2}\left[(q-1)^{1/\alpha}\lambda t\middle| \begin{matrix}(1-1/(q-1),1/\alpha),(1-\alpha,1)\\ (0,1/\alpha),(2-\alpha-\beta,1)\end{matrix}\right] \tag{4.13}$$

and 0 otherwise. Similar expressions for survival and density functions exist for $q < 1$.

4.1. Series Expansion for the H-Function

The H-function defined in (4.12) can be expanded in a series form by evaluating the contour integral by means of the residue calculus. The poles of the integrand $\Gamma(s/\alpha)\Gamma(1/(q-1)-$

$s/\alpha)\Gamma(\alpha-s)((q-1)^{1/\alpha}\lambda t)^{-s}/\Gamma(\alpha+\beta-s)$ coming from $\Gamma(s/\alpha)$, which are $s/\alpha=-\nu$, $\nu=0,1,\ldots$ The residue at $s/\alpha=-\nu$, denoted by R_ν, is given by

$$\begin{aligned}R_\nu &= \lim_{s/\alpha\to-\nu}\frac{((s/\alpha)+\nu)\Gamma(s/\alpha)\Gamma(1/(q-1)-s/\alpha)\Gamma(\alpha-s)}{\Gamma(\alpha+\beta-s)}\left((q-1)^{1/\alpha}\lambda t\right)^{-s}\\ &= \lim_{s/\alpha\to-\nu}\frac{((s/\alpha)+\nu)((s/\alpha)+\nu-1)((s/\alpha)+\nu-2)\ldots(s/\alpha)\Gamma(s/\alpha)\Gamma(1/(q-1)-s/\alpha)}{(s/\alpha+\nu-1)(s/\alpha+\nu-2)\ldots(s/\alpha)\Gamma(\alpha+\beta-s)}\\ &\quad\times\Gamma(\alpha-s)\left((q-1)^{1/\alpha}\lambda t\right)^{-s}.\end{aligned}\tag{4.14}$$

Hence

$$\begin{aligned}&\frac{1}{2\pi_i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s/\alpha)\Gamma(1/(q-1)-s/\alpha)\Gamma(\alpha-s)}{\Gamma(\alpha+\beta-s)}\left((q-1)^{1/\alpha}\lambda t\right)^{-s}ds\\ &\quad=\sum_{\nu=0}^{\infty}R_\nu=\sum_{\nu=0}^{\infty}\frac{\Gamma(1/(q-1)+\nu)\Gamma(\alpha+\alpha\nu)}{\Gamma(\alpha+\beta+\nu)\nu!}\left((q-1)(\lambda t)^{\alpha}\right)^{\nu}.\end{aligned}\tag{4.15}$$

Therefore,

$$\begin{aligned}\Psi(t)&=1+\frac{\lambda^{\alpha}(q-2)t^{\alpha+\beta-1}}{\Gamma(1/(q-1))}\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{\Gamma(1/(q-1)+\nu)\Gamma(\alpha+\alpha\nu)}{\Gamma(\alpha+\beta+\alpha\nu)\nu!}\left((q-1)(\lambda t)^{\alpha}\right)^{\nu},\\ \Psi(t)&=1+\frac{\lambda^{\alpha}(q-2)t^{\alpha+\beta-1}}{\Gamma(1/(q-1))}\,{}_2\Psi_1\left[-(q-1)(\lambda t)^{\alpha}\left|\begin{array}{c}(\alpha,\alpha),(1/(q-1),1)\\(\alpha+\beta,\alpha)\end{array}\right.\right].\end{aligned}\tag{4.16}$$

${}_2\Psi_1(\cdot)$ denotes the Wright function introduced by Wright in 1935 which is defined by

$${}_p\Psi_q\left[z\left|\begin{array}{c}(a_p,A_p)\\(b_q,B_q)\end{array}\right.\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+nA_i)}{\prod_{j=1}^{q}\Gamma(b_i+nB_j)}\frac{z^n}{n!},\tag{4.17}$$

where $a_i,b_j\in C$ and $A_i,B_j\in\Re=(-\infty,\infty)$, $A_i,B_j\neq0$, $i=1,\ 2,\ldots,\ p$, $j=1,2,\ldots,q$ with the convergence condition that $\sum_{j=1}^{q}B_j-\sum_{i=1}^{p}A_i>-1$. The plot of $\Psi(t)$ for various values of parameters is given in Figure 2. It is seen from the graph that $\Psi(t)$ behaves like a survival function.

4.2. Some Interesting Special Cases

Note that, for $\alpha=\beta=1$, $\Psi(t)$ reduces to the following form:

$$\Psi(t)=1+\frac{(q-2)\lambda t}{\Gamma(1/(q-1))}\frac{1}{2\pi_i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)\Gamma(1/(q-1)-s)\Gamma(1-s)}{\Gamma(2-s)}\left((q-1)\lambda t\right)^{-s}ds.\tag{4.18}$$

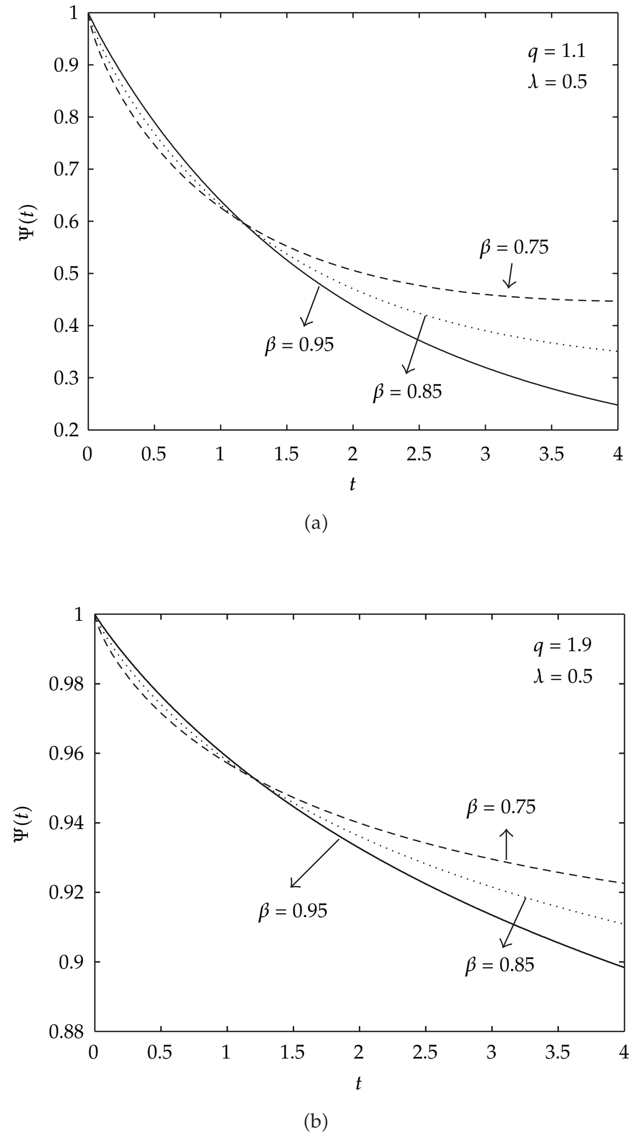

Figure 2: The plot of $\Psi(t)$ for various values of parameters.

Now evaluating the contour integral by the residue calculus, the poles of the integrand $\Gamma(s)\Gamma(1/(q-1)-s)\Gamma(1-s)/\Gamma(2-s)((q-1)\lambda t)^{-s}$ coming from $\Gamma(s)$ are $s=-\nu$, $\nu=0,1,\ldots$. The residue at $s=-\nu$ denoted by R_ν is given by

$$R_\nu = \lim_{s\to-\nu}\frac{(s+\nu)\Gamma(s)\Gamma\left(1/(q-1)-s\right)\Gamma(1-s)}{\Gamma(2-s)}\left((q-1)\lambda t\right)^{-s}. \tag{4.19}$$

Therefore the corresponding survival function is

$$\begin{aligned}\Psi(t) &= 1+\frac{(q-2)\lambda t}{\Gamma(1/q-1)}\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{\Gamma(1/(q-1)+\nu)}{\Gamma(\nu+1)}((q-1)(\lambda t))^{\nu}\\ &= \left[1+(q-1)\lambda t\right]^{(q-2)/(q-1)}\end{aligned} \tag{4.20}$$

which is the survival function of q-exponential distribution. For $\alpha = \beta = 1$ and as $q \to 1$, then (4.20) will become the survival function of the exponential distribution (interarrival time of the Poisson process). For $\alpha = 1$ and as $q \to 1$, we get the "Mittag-Leffler type renewal process" introduced by Minardi [16].

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