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Research Article

Probabilistic Solution of the General Robin Boundary Value Problem on Arbitrary Domains

Khalid Akhlil

Institut für Angewandte Analysis, Universität Ulm, 89069 Ulm, Germany

Correspondence should be addressed to Khalid Akhlil, akhlil.khalid@gmail.com

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Using a capacity approach and the theory of the measure's perturbation of the Dirichlet forms, we give the probabilistic representation of the general Robin boundary value problems on an arbitrary domain Ω, involving smooth measures, which give rise to a new process obtained by killing the general reflecting Brownian motion at a random time. We obtain some properties of the semigroup directly from its probabilistic representation, some convergence theorems, and also a probabilistic interpretation of the phenomena occurring on the boundary.

1. Introduction

The classical Robin boundary conditions on a smooth domain Ω of $\mathbb{R}^N$ $(N \geq 0)$ is giving by

$$\frac{\partial u}{\partial \nu} + \beta u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where ν is the outward normal vector field on the boundary $\partial\Omega$ and β a positive bounded Borel measurable function defined on $\partial\Omega$.

The probabilistic treatment of Robin boundary value problems has been considered by many authors [1–4]. The first two authors considered bounded C^3-domains since the third considered bounded domains with Lipschitz boundary, and the study of [4] was concerned with C^3-domains but with smooth measures instead of β. If one wants to generalize the probabilistic treatment to a general domain, a difficulty arise when we try to get a diffusion process representing Neumann's boundary conditions.

In fact, the Robin boundary conditions (1.1) are nothing but a perturbation of $\partial/\partial\nu$, which represent Neumann's boundary conditions, by the measure $\mu = \beta \cdot \sigma$, where σ is the surface measure. Consequently, the associated diffusion process is the reflecting Brownian motion killed by a certain additive functional, and the semigroup generated by the Laplacian with classical Robin boundary conditions is then giving by

$$\mathcal{P}_t^{\mu} f(x) = E_x\left[f(X_t)e^{-\int_0^t \beta(X_s)dL_s}\right], \tag{1.2}$$

where $(X_t)_{t\geq 0}$ is a reflecting Brownian motion (RBM) and L_t is the boundary local time, which corresponds to σ by Revuz correspondence. It is clear that the smoothness of the domain Ω in classical Robin boundary value problem follows the smoothness of the domains where RBM is constructed (see [5–10] and references therein for more details about RBM).

In [6], the RBM is defined to be the Hunt process associated with the form $(\mathcal{E}, \mathcal{F})$ defined on $L^2(\Omega)$ by

$$\mathcal{E}(u,v) = \int_{\Omega} \nabla u \nabla v dx, \quad \forall u, v \in \mathcal{F} = H^1(\Omega), \tag{1.3}$$

where Ω is assumed to be bounded with Lipschitz boundary so that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is regular. If Ω is an arbitrary domain, then the Dirichlet form needs not to be regular, and to not to lose the generality we consider $\mathcal{F} = \widetilde{H}^1(\Omega)$, the closure of $H^1(\Omega) \cap C_c(\overline{\Omega})$ in $H^1(\Omega)$. The domain $\widetilde{H}^1(\Omega)$ is so defined to insure the Dirichlet form $(\mathcal{E}, \mathcal{F})$ to be regular.

Now, if we perturb the Neumann boundary conditions by Borel's positive measure [11–13], we get the Dirichlet form $(\mathcal{E}^{\mu}, \mathcal{F}^{\mu})$ defined on $L^2(\Omega)$ by

$$\mathcal{E}^{\mu}(u,v) = \int_{\Omega} \nabla u \nabla v dx + \int_{\partial\Omega} \widetilde{u}\widetilde{v} d\mu, \quad \forall u, v \in \mathcal{F}^{\mu} = \widetilde{H}^1(\Omega) \cap L^2(\partial\Omega, d\mu). \tag{1.4}$$

In the case of $\mu = \beta \cdot \sigma$ (Ω bounded with Lipschitz boundary), (1.4) is the form associated with Laplacian with classical Robin boundary conditions and (1.2) gives the associated semigroup. In the case of an arbitrary domain Ω, we make use of the theory of the measure's perturbation of the Dirichlet forms, see, for example, [14–23].

More specifically, we adapt the potential theory and associated stochastic analysis to our context, which is the subject of Section 2. In Section 3, we focus on the diffusion process $(X_t)_{t\geq 0}$ associated with the regular Dirichlet form $(\mathcal{E}, \widetilde{H}^1(\Omega))$. We apply a decomposition theorem of additive functionals to write X_t in the form $X_t = x + B_t + N_t$, we prove that the additive functional N_t is supported by $\partial\Omega$, and we investigate when it is of bounded variations.

In Section 3, we get the probabilistic representation of the semigroup associated with (1.4), and we prove that it is sandwiched between the semigroup generated by the Laplacian with the Dirichlet boundary conditions and that of the Neumann ones. In addition, we prove some convergence theorems, and we give a probabilistic interpretation of the phenomena occurring on the boundary.

2. Preliminaries and Notations

The aim of this section is to adapt the potential theory and the stochastic analysis for application to our problem. More precisely, it concerns the notion of relative capacity, smooth measures, and its corresponding additive functionals. This section relies heavily on the book of Fukushima [17], particularly, chapter 2 and 5, and the paper in [11]. Throughout [17], the form $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on $L^2(X,m)$, where X is a locally compact separable metric space, and m a positive Radon measure on X with $\mathrm{supp}[m]=X$.

For our purposes, we take $X=\overline{\Omega}$, where Ω is an Euclidean domain of $\mathbb{R}^N$, and the measure m on the σ-algebra $\mathcal{B}(X)$ is given by $m(A)=\lambda(A\cap\Omega)$ for all $A\in\mathcal{B}(X)$ with λ the Lebesgue measure; it follows that $L^2(\Omega)=L^2(X,\mathcal{B}(X),m)$, and we define a regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(\Omega)$ by

$$\mathcal{E}(u,v)=\int_\Omega \nabla u\nabla v\,dx, \qquad \mathcal{F}=\widetilde{H^1}(\Omega), \tag{2.1}$$

where $\widetilde{H}^1(\Omega)=\overline{H^1(\Omega)\cap C_c(\overline{\Omega})}^{H^1(\Omega)}$. The domain $\widetilde{H}^1(\Omega)$ is so defined to insure the Dirichlet form $(\mathcal{E},\mathcal{F})$ to be regular, instead of $\mathcal{F}=H^1(\Omega)$ which make the form not regular in general, but if Ω is bounded open set with Lipschiz boundary, then $\widetilde{H}^1(\Omega)=H^1(\Omega)$.

We denote for any $\alpha>0:\mathcal{E}_\alpha(u,v)=\mathcal{E}(u,v)+\alpha(u,v)_m$, for all $u,v\in\mathcal{F}$.

2.1. Relative Capacity

The relative capacity is introduced in a first time in [11] to study the Laplacian with general Robin boundary conditions on arbitrary domains. It is a special case of the capacity associated with a regular Dirichlet form as described in chapter 2 of [17]. It seems to be an efficient tool to analyse the phenomena occurring on the boundary $\partial\Omega$ of Ω.

The relative capacity which we denote by $\mathrm{Cap}_{\overline{\Omega}}$ is defined on a subsets of $\overline{\Omega}$ by the following: for $A\subset\overline{\Omega}$ relatively open (i.e., open with respect to the topology of $\overline{\Omega}$) we set

$$\mathrm{Cap}_{\overline{\Omega}}(A):=\inf\left\{\mathcal{E}_1(u,u):u\in\widetilde{H}^1(\Omega):u\geq 1 \text{ a.e. on } A\right\}. \tag{2.2}$$

And for arbitrary $A\subset\overline{\Omega}$, we set

$$\mathrm{Cap}_{\overline{\Omega}}(A):=\inf\left\{\mathrm{Cap}_{\overline{\Omega}}(B):B \text{ relatively open } A\subset B\subset\overline{\Omega}\right\}. \tag{2.3}$$

A set $N\subset\overline{\Omega}$ is called a relatively polar if $\mathrm{Cap}_{\overline{\Omega}}(N)=0$.

The relative capacity (just as a cap) has the properties of a capacity as described in [17]. In particular, $\mathrm{Cap}_{\overline{\Omega}}$ is also an outer measure (but not a Borel measure) and a Choquet capacity.

A statement depending on $x\in A\subset\overline{\Omega}$ is said to hold relatively quasieverywhere (r.q.e.) on A, if there exist a relatively polar set $N\subset A$ such that the statement is true for every $x\in A\setminus N$.

Now we may consider functions in $\widetilde{H}^1(\Omega)$ as defined on $\overline{\Omega}$, and we call a function $u:\overline{\Omega}\to\mathbb{R}$ relatively quasicontinuous (r.q.c.) if for every $\epsilon>0$ there exists a relatively open set $G\subset\overline{\Omega}$ such that $\mathrm{Cap}_{\overline{\Omega}}(G)<\epsilon$ and $u|_{\overline{\Omega}\setminus G}$ is continuous.

It follows [13] that for each $u \in \widetilde{H}^1(\Omega)$ there exists a relatively quasicontinuous function $\tilde{u} : \overline{\Omega} \to \mathbb{R}$ such that $\tilde{u}(x) = u(x)m$—a.e. This function is unique relatively quasieverywhere. We call $\tilde{u}$ the relatively quasicontinuous representative of u.

For more details, we refer the reader to [11, 13], where the relative capacity is investigated, as well as its relation to the classical one. A description of the space $H_0^1(\Omega)$ in term of relative capacity is also given, namely,

$$H_0^1(\Omega) = \left\{ u \in \widetilde{H}^1(\Omega) : \tilde{u}(x) = 0 \text{ r.q.e. on } \partial\Omega \right\}. \tag{2.4}$$

2.2. Smooth Measures

All families of measures on $\partial\Omega$ defined in this subsection were originally defined on X [17], and then in our settings on $X = \overline{\Omega}$, as a special case. We reproduce the same definitions, and most of their properties on $\partial\Omega$, as we deal with measures concentrated on the boundary of Ω for our approach to the Robin boundary conditions involving measures. There is three families of measures, as we will see in the sequel: the familes S_0, S_{00}, and S. We put $\partial\Omega$ between brackets to recall our context, and we keep in mind that the same things are valid if we put Ω or $\overline{\Omega}$ instead of $\partial\Omega$.

Let $\Omega \subset \mathbb{R}^N$ be open. A positive Radon measure μ on $\partial\Omega$ is said to be of finite energy integral if

$$\int_{\partial\Omega} |v(x)|\mu(dx) \leq C\sqrt{\mathcal{E}_1(v,v)}, \quad v \in \mathcal{F} \cap C_c\left(\overline{\Omega}\right) \tag{2.5}$$

for some positive constant C. A positive Radon measure on $\partial\Omega$ is of finite energy integral if and only if there exists, for each $\alpha > 0$, a unique function $U_\alpha\mu \in \mathcal{F}$ such that

$$\mathcal{E}_\alpha(U_\alpha\mu, v) = \int_{\partial\Omega} v(x)\mu(dx). \tag{2.6}$$

We call $U_\alpha\mu$ an α-potential.

We denote by $S_0(\partial\Omega)$ the family of all positive Radon measures of finite energy integral.

Lemma 2.1. *Each measure in $S_0(\partial\Omega)$ charges no set of zero relative capacity.*

Let us consider a subset $S_{00}(\partial\Omega)$ of S_0 defined by

$$S_{00}(\partial\Omega) = \left\{ \mu \in S_0(\partial\Omega) : \mu(\partial\Omega) < \infty, \left\| U_1\mu \right\|_\infty < \infty \right\}. \tag{2.7}$$

Lemma 2.2. *For any $\mu \in S_0(\partial\Omega)$, there exist an increasing sequence $(F_n)_{n\geq 0}$ of compact sets of $\partial\Omega$ such that*

$$\begin{gathered} 1_{F_n} \cdot \mu \in S_{00}(\partial\Omega), \quad n = 1, 2, \ldots, \\ Cap_{\overline{\Omega}}(K \setminus F_n) \longrightarrow 0, \quad n \longrightarrow +\infty \quad \textit{for any compact set } K \subset \partial\Omega. \end{gathered} \tag{2.8}$$

We note that $\mu \in S_0(\partial\Omega)$ vanishes on $\partial\Omega \setminus \cup_n F_n$ for the sets F_n of Lemma 2.2, because of Lemma 2.1.

We now turn to a class of measures $S(\partial\Omega)$ larger than $S_0(\partial\Omega)$. Let us call a (positive) Borel measure μ on $\partial\Omega$ smooth if it satisfies the following conditions:

(i) μ charges no set of zero relative capacity;

(ii) there exist an increasing sequence $(F_n)_{n\geq 0}$ of closed sets of $\partial\Omega$ such that

$$\mu(F_n) < \infty, \quad n = 1, 2, \ldots, \tag{2.9}$$

$$\lim_{n\to+\infty} \mathrm{Cap}_{\overline{\Omega}}(K \setminus F_n) = 0 \quad \text{for any compact } K \subset \partial\Omega. \tag{2.10}$$

Let us note that μ then satisfies

$$\mu(\partial\Omega \setminus \cup_n F_n) = 0. \tag{2.11}$$

An increasing sequence (F_n) of closed sets satisfying condition (2.10) will be called a generalized nest; if further each F_n is compact, we call it a generalized compact nest.

We denote by $S(\partial\Omega)$ the family of all smooth measures. The class $S(\partial\Omega)$ is quiet large and it contains all positive Radon measures on $\partial\Omega$ charging no set of zero relative capacity. There exist also, by [15, Theorem 1.1], a smooth measure μ on $\partial\Omega$ (hence singular with respect to m) "nowhere Radon" in the sense that $\mu(G) = \infty$ for all nonempty relatively open subset G of $\partial\Omega$ (see [15, Example 1.6]).

The following Theorem, says that, any measure in $S(\partial\Omega)$ can be approximated by measures in $S_0(\partial\Omega)$ and in $S_{00}(\partial\Omega)$ as well.

Theorem 2.3. *The following conditions are equivalent for a positive Borel measure μ on $\partial\Omega$.*

(i) $\mu \in S(\partial\Omega)$.

(ii) *There exists a generalized nest (F_n) satisfying* (2.11) *and $1_{F_n} \cdot \mu \in S_0(\partial\Omega)$ for each n.*

(iii) *There exists a generalized compact nest (F_n) satisfying* (2.11) *and $1_{F_n} \cdot \mu \in S_{00}(\partial\Omega)$ for each n.*

2.3. Additive Functionals

Now we turn our attention to the correspondence between smooth measures and additive functionals, known as Revuz correspondence. As the support of an additive functional is the quasisupport of its Revuz measure, we restrict our attention, as for smooth measures, to additive functionals supported by $\partial\Omega$. Recall that as the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is regular, then there exists a Hunt process $M = (\Xi, X_t, \xi, P_x)$ on $\overline{\Omega}$ which is m-symmetric and associated with it.

Definition 2.4. A function $A : [0, +\infty[\times \Xi \rightarrow [-\infty, +\infty]$ is said to be an additive functional (AF) if

(1) A_t is $\mathcal{F}_t$-measurable,

(2) there exist a defining set $\Lambda \in \mathcal{F}_\infty$ and an exceptional set $N \subset \partial\Omega$ with $\mathrm{cap}_{\overline{\Omega}}(N) = 0$ such that $P_x(\Lambda) = 1$, for all $x \in \partial\Omega \setminus N$, $\theta_t \Lambda \subset \Lambda$, for all $t > 0$; for all $\omega \in \Lambda$, $A_0(\omega) = 0$; $|A_t(\omega)| < \infty$ for $t < \xi$. $A(\omega)$ is right continuous and has left limit, and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t \omega)s$, $t \geq 0$.

An additive functional is called positive continuous (PCAF) if, in addition, $A_t(\omega)$ is nonnegative and continuous for each $\omega \in \Lambda$. The set of all PCAF's on $\partial\Omega$ is denoted $\mathcal{A}_c^+(\partial\Omega)$.

Two additive functionals A^1 and A^2 are said to be equivalent if for each $t > 0$, $P_x(A_t^1 = A_t^2) = 1$ r.q.e $x \in \overline{\Omega}$.

We say that $A \in \mathcal{A}_c^+(\partial\Omega)$ and $\mu \in S(\partial\Omega)$ are in the Revuz correspondence, if they satisfy, for all γ-excessive function h, and $f \in \mathcal{B}_+(\overline{\Omega})$, the relation

$$\lim_{t \searrow 0} \frac{1}{t} E_{h \cdot m}\left[\int_0^t f(X_s)\, dA_s\right] = \int_{\partial\Omega} h(x)(f \cdot \mu)(dx). \tag{2.12}$$

The family of all equivalence classes of $\mathcal{A}_c^+(\partial\Omega)$ and the family $S(\partial\Omega)$ are in one to one correspondence under the Revuz correspondence. In this case, $\mu \in S(\partial\Omega)$ is called the Revuz measure of A.

Example 2.5. We suppose Ω to be bounded with Lipschitz boundary. We have [2]

$$\lim_{t \searrow 0} \frac{1}{t} E_{h \cdot m}\left[\int_0^t f(X_s)\, dL_s\right] = \frac{1}{2} \int_{\partial\Omega} h(x) f(x) \sigma(dx), \tag{2.13}$$

where L_t is the boundary local time of the reflecting Brownian motion on $\overline{\Omega}$. It follows that $(1/2)\sigma$ is the Revuz measure of L_t.

In the following we give some facts useful in the proofs of our main results. We set

$$\begin{aligned} U_A^\alpha f(x) &= E_x\left[\int_0^\infty e^{-\alpha t} f(X_t)\, dA_t\right], \\ R_\alpha^A f(x) &= E_x\left[\int_0^\infty e^{-\alpha t} e^{-A_t} f(X_t)\, dt\right], \\ R_\alpha f(x) &= E_x\left[\int_0^\infty e^{-\alpha t} f(X_t)\, dt\right]. \end{aligned} \tag{2.14}$$

Proposition 2.6. *Let $\mu \in S_0(\partial\Omega)$ and $A \in \mathcal{A}_c^+(\partial\Omega)$, the corresponding PCAF. For $\alpha > 0$, $f \in \mathcal{B}_b^+$, U_A^α is a relatively quasicontinuous version of $U_\alpha(f \cdot \mu)$.*

Proposition 2.7. *Let $A \in \mathcal{A}_c^+(\partial\Omega)$, and $f \in \mathcal{B}_b^+$, then R_α^A is relatively quasicontinuous and*

$$R_\alpha^A f - R_\alpha f + U_A^\alpha R_\alpha^A f = 0. \tag{2.15}$$

In general, the support of an AF A is defined by

$$\operatorname{supp}[A] = \{x \in X \setminus N : P_x(R = 0) = 1\}, \tag{2.16}$$

where $R(\omega) = \inf\{t > 0 : A_t(\omega) \neq 0\}$.

Theorem 2.8. *The support of $A \in \mathcal{A}_c^+(\partial\Omega)$ is the relative quasisupport of its Revuz measure.*

In the following, we give a well-known theorem of decomposition of additive functionals of finite energy. We will apply it to get a decomposition of the diffusion process associated with $(\mathcal{E}, \mathcal{F})$.

Theorem 2.9. *For any $u \in \mathcal{F}$, the AF $A^{[u]} = \tilde{u}(X_t) - \tilde{u}(X_0)$ can be expressed uniquely as*

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M^{[u]} + N^{[u]}, \tag{2.17}$$

where $M_t^{[u]}$ is a martingale additive functional of finite energy and $N_t^{[u]}$ is a continuous additive functional of zero energy.

A set $\sigma(u)$ is called the (0)-spectrum of $u \in \mathcal{F}$, if $\sigma(u)$ is the complement of the largest open set G such that $\mathcal{E}(u, v)$ vanishes for any $v \in \mathcal{F} \cap C_0(X)$ with $\operatorname{supp}[v] \subset G$. The following Theorem means that $\operatorname{supp}[N^{[u]}] \subset \sigma(u)$, for all $u \in \mathcal{F}$.

Theorem 2.10. *For any $u \in \mathcal{F}$, the CAF $N^{[u]}$ vanishes on the complement of the spectrum $F = \sigma(u)$ of u in the following sense:*

$$P_x\left(N_t^{[u]} = 0 : \forall t < \sigma_F\right) = 1 \quad r.q.e\ x \in X. \tag{2.18}$$

3. General Reflecting Brownian Motion

Now we turn our attention to the process associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\Omega)$ defined by

$$\mathcal{E}(u, v) = \int_\Omega \nabla u \nabla v dx, \qquad \mathcal{F} = \widetilde{H^1}(\Omega). \tag{3.1}$$

Due to the theorem of Fukushima (1975), there is a Hunt process $(X_t)_{t \geq 0}$ associated with it. In addition, $(\mathcal{E}, \mathcal{F})$ is local, thus the Hunt process is in fact a diffusion process (i.e., A strong Markov process with continuous sample paths). The diffusion process $M = (X_t, P_x)$ on $\overline{\Omega}$ is associated with the form $\mathcal{E}$ in the sense that the transition semigroup $p_t f(x) = E_x[f(X_t)]$, $x \in \overline{\Omega}$ is a version of the L^2-semigroup $\mathcal{P}_t f$ generated by $\mathcal{E}$ for any nonnegative L^2-function f.

M is unique up to a set of zero relative capacity.

Definition 3.1. We call the diffusion process on $\overline{\Omega}$ associated with $(\mathcal{E}, \mathcal{F})$ the general reflecting Brownian motion.

The process X_t is so named to recall the standard reflecting Brownian motion in the case of bounded smooth Ω, as the process associated with $(\mathcal{E}, H^1(\Omega))$. Indeed, when Ω is bounded with Lipschitz boundary, we have that $\widetilde{H}^1(\Omega) = H^1(\Omega)$ and by [6] the reflecting Brownian motion X_t admits the following Skorohod representation:

$$X_t = x + W_t + \frac{1}{2}\int_0^t \nu(X_s)\, dL_s, \tag{3.2}$$

where W is a standard N-dimensional Brownian motion, L is the boundary local (continuous additive functional) associated with surface measure σ on $\partial\Omega$, and ν is the inward unit normal vector field on the boundary.

For a general domain, the form $(\mathcal{E}, H^1(\Omega))$ needs not to be regular. Fukushima [9] constructed the reflecting Brownian motion on a special compactification of Ω, the so-called Kuramuchi compactification. In [6], it is shown that if Ω is a bounded Lipschitz domain, then the Kuramochi compactification of Ω is the same as the Euclidean compactification. Thus for such domains, the reflecting Brownian motion is a continuous process who does live on the set $\overline{\Omega}$.

Now, we apply a general decomposition theorem of additive functionals to our process M, in the same way as in [6]. According to Theorem 2.9, the continuous additive functional $\widetilde{u}(X_t) - \widetilde{u}(X_0)$ can be decomposed as follows:

$$\widetilde{u}(X_t) - \widetilde{u}(X_0) = M_t^{[u]} + N_t^{[u]}, \tag{3.3}$$

where $M_t^{[u]}$ is a martingale additive functional of finite energy and $N_t^{[u]}$ is a continuous additive functional of zero energy.

Since $(X_t)_{t\geq 0}$ has continuous sample paths, $M_t^{[u]}$ is a continuous martingale whose quadratic variation process is

$$\left\langle M^{[u]}, M^{[u]} \right\rangle_t = \int_0^t |\nabla u|^2 (X_s)\, ds. \tag{3.4}$$

Instead of u, we take coordinate function $\phi_i(x) = x_i$. We have

$$X_t = X_0 + M_t + N_t. \tag{3.5}$$

We claim that M_t is a Brownian motion with respect to the filtration of X_t. To see that, we use Lévys criterion. This follows immediately from (3.2), which became in the case of coordinate function

$$\left\langle M^{[\phi_i]}, M^{[\phi_i]} \right\rangle = \delta_{ij} t. \tag{3.6}$$

Now we turn our attention to the additive functional N_t. Two natural questions need to be answered. The first is, where is the support of N_t located and the second concern the boundedness of its total variation.

For the first question we claim the following.

Proposition 3.2. *The additive functional N_t is supported by $\partial\Omega$.*

Proof. Following Theorem 2.10, we have that $\mathrm{supp}[N_t] \subset \sigma(\phi)$, where $\sigma(\phi)$ is the (0)-spectrum of ϕ, which means the complement of the largest open set G such that $\mathcal{E}(\phi_i, v) = 0$ for all $v \in \mathcal{F} \cap C_c(\overline{\Omega})$ with $\mathrm{supp}[v] \subset G$.

Step 1. If Ω is smooth (Bounded with Lipschitz boundary, e.g.), then we have

$$\mathcal{E}(\phi_i, v) = -\int_{\partial\Omega} v \cdot n_i d\sigma. \tag{3.7}$$

Then, $\mathcal{E}(\phi_i, v) = 0$ for all $v \in \mathcal{F} \cap C_c(\overline{\Omega})$ with $\mathrm{supp}[v] \subset \Omega$. We can then see that the largest G is Ω. Consequently $\sigma(\phi) = \overline{\Omega} \setminus \Omega$ and then $\sigma(\phi) = \partial\Omega$.

Step 2. If Ω is arbitrary, then we take an increasing sequence of subset of Ω such that $\bigcup_{n=0}^{\infty} \Omega_n = \Omega$. Define the family of Dirichlet forms $(\mathcal{E}_{\Omega_n}, \mathcal{F}_{\Omega_n})$ to be the parts of the form $(\mathcal{E}, \mathcal{F})$ on each Ω_n as defined in Section 4.4 of [17]. By Theorem 4.4.5 in the same section, we have that $\mathcal{F}_{\Omega_n} \subset \mathcal{F}$ and $\mathcal{E}_{\Omega_n} = \mathcal{E}$ on $\mathcal{F}_{\Omega_n} \times \mathcal{F}_{\Omega_n}$. We have that Ω_n is the largest open set such that $\mathcal{E}_{\Omega_n}(\phi_i, v) = 0$ for all $v \in \mathcal{F}_{\Omega_n} \cap C_c(\overline{\Omega_n})$. By limit, we get the result. □

The interest of the question of boundedness of total variation of N_t appears when one needs to study the semimartingale property and the Skorohod equation of the process X_t of type 3.2. Let $|N|$ be the total variation of N_t, that is,

$$|N|_t = \sup \sum_{i=1}^{n-1} |N_{t_i} - N_{t_{i-1}}|, \tag{3.8}$$

where the supremum is taken over all finite partition $0 = t_0 < t_1 < \cdots < t_n = t$, and $|\cdot|$ denote the Euclidian distance. If $|N|$ is bounded, then we have the following expression:

$$N_t = \int_0^t \nu_s d|N|_s, \tag{3.9}$$

where ν is a process such that $|\nu|_s = 1$ for $|N|$-almost all s.

According to § 5.4. in [17], we have the following result.

Theorem 3.3. *Assume that Ω is bounded and that the following inequality is satisfied:*

$$\left| \int_{\Omega} \frac{\partial v}{\partial x_i} dx \right| \leq C \|v\|_{\infty}, \quad \forall v \in \widetilde{H}^1(\Omega) \cap C_b\left(\overline{\Omega}\right), \tag{3.10}$$

for some constant C. Then, N_t is of bounded variation.

A bounded set verifying (2.10) is called *strong Caccioppoli set*. This notion is introduced in [8], and is a purely measured theoretic notion. An example of this type of sets is bounded sets with Lipschitz boundary.

Theorem 3.4. *If Ω is a Caccioppoli set, then there exist a finite signed smooth measure ν such that*

$$\int_{\Omega} \frac{\partial v}{\partial x_i} dx = -\int_{\partial\Omega} v d\mu, \quad \forall v \in \widetilde{H}^1(\Omega) \cap C_b\left(\overline{\Omega}\right). \tag{3.11}$$

And $\nu = \nu^1 - \nu^2$ is associated with the CAF $-N_t = -A_t^1 + A_t^2$ with the Revuz correspondence. Consequently ν charges no set of zero relative capacity.

To get a Skorohod type representation, we set

$$\begin{gathered} \nu = \sum_{i=1}^{N} |\mu_i|, \\ \phi_i = \frac{d\mu_i}{d\nu} \quad i = 1, \dots, N. \end{gathered} \tag{3.12}$$

We define the measure σ on $\partial\Omega$ by

$$\sigma(dx) = 2\left(\sum_{i=1}^{N} |\phi_i(x)|^2\right)^{1/2} \nu(dx) \tag{3.13}$$

and the vector of length 1 at $x \in \partial\Omega$ by

$$n_i(x) = \begin{cases} \dfrac{\phi_i(x)}{\left(\sum_{i=1}^{N} |\phi_i(x)|^2\right)^{1/2}} & \text{if } \displaystyle\sum_{i=1}^{N} |\phi_i(x)|^2 > 0; \\ 0 & \text{if } \displaystyle\sum_{i=1}^{N} |\phi_i(x)|^2 = 0. \end{cases} \tag{3.14}$$

Thus, $\mu_i(dx) = (1/2) n_i(x)\sigma(dx)$, $i = 1, \dots, N$.

Then

$$N_t = \int_0^t n(X_s) dL_s, \tag{3.15}$$

where L is the PCAF associated with $(1/2)\sigma$.

Theorem 3.5. *If Ω is a Caccioppoli set, then for r.q.e $x \in \overline{\Omega}$, one has*

$$X_t = x + B_t + \int_0^t n(X_s) dL_s, \tag{3.16}$$

where B is an N-dimensional Brownian motion, and L is a PCAF associated by the Revuz correspondence to the measure $(1/2)\sigma$.

Remark 3.6. The above theorem can be found in [9, 24]. In particular, Fukushima proves an equivalence between the property of Caccioppoli sets and the Skorohod representation.

4. Probabilistic Solution to General Robin Boundary Value Problem

This section is concerned with the probabilistic representation to the semigroup generated by the Laplacian with general Robin boundary conditions, which is, actually, obtained by perturbing the Neumann boundary conditions by a measure. We start with the regular Dirichlet form defined by (3.1), which we call always as the Dirichlet form associated with the Laplacian with Neumann boundary conditions.

Let μ be a positive Radon measure on $\partial\Omega$ charging no set of zeo relative capacity. Consider the perturbed Dirichlet form $(\mathcal{E}_\mu, \mathcal{F}_\mu)$ on $L^2(\Omega)$ defined by

$$\begin{gathered}\mathcal{F}_\mu = \mathcal{F} \cap L^2(\partial\Omega, \mu), \\ \mathcal{E}_\mu(u,v) = \mathcal{E}(u,v) + \int_{\partial\Omega} uv\,d\mu \quad u, v \in \mathcal{F}_\mu.\end{gathered} \tag{4.1}$$

We will see in the following theorem that the transition function

$$\mathcal{P}_t^\mu f(x) = E_x\left[f(X_t)e^{-A_t^\mu}\right] \tag{4.2}$$

is associated with $(\mathcal{E}_\mu, \mathcal{F}_\mu)$, where A_t^μ is a positive additive functional whose Revuz measure is μ; note that the support of the AF is the same as the relative quasisupport of its Revuz measure.

Proposition 4.1. *$\mathcal{P}_t^\mu$ is a strongly continuous semigroup on $L^2(\Omega)$.*

Proof. The proof of the above proposition can be found in [14]. □

Theorem 4.2. *Let μ be a positive Radon measure on $\partial\Omega$ charging no set of zero relative capacity and $(A_t^\mu)_{t\geq 0}$ be its associated PCAF of $(X_t)_{t\geq 0}$. Then $\mathcal{P}_t^\mu$ is the strongly continuous semigroup associated with the Dirichlet form $(\mathcal{E}_\mu, \mathcal{F}_\mu)$ on $L^2(\Omega)$.*

Proof. To prove that $\mathcal{P}_t^\mu$ is associated with the Dirichlet form $(\mathcal{E}_\mu, \mathcal{F}_\mu)$ on $L^2(\Omega)$, it suffices to prove the assertion

$$R_\alpha^A f \in \mathcal{F}^\mu, \quad \mathcal{E}_\alpha^\mu\left(R_\alpha^A, u\right) = (f, u), \quad f \in L^2(\Omega, m),\ u \in \mathcal{F}^\mu. \tag{4.3}$$

Since $\|R_\alpha^A f\|_{L^2(\Omega)} \leq \|R_\alpha f\|_{L^2(\Omega)} \leq (1/\alpha)\|f\|_{L^2(\Omega)}$, we need to prove (4.7) only for bounded $f \in L^2(\Omega)$. We first prove that (4.7) is valid when $\mu \in S_{00}(\partial\Omega)$. According to Proposition 2.7 we have

$$R_\alpha^A f - R_\alpha f + U_A^\alpha R_\alpha^A f = 0, \quad \alpha > 0,\ f \in \mathcal{B}^+\left(\overline{\Omega}\right). \tag{4.4}$$

If $\mu \in S_{00}(\partial\Omega)$, and if f is bounded function in $L^2(\Omega)$, then $\|R_\alpha f\| < \infty$, and $U_A^\alpha R_\alpha^A f$ is a relative quasicontinuous version of the α-potential $U_\alpha(R_\alpha^A f \cdot \mu) \in \mathcal{F}$ by Proposition 2.6. Since $\|U_\alpha(R_\alpha^A f \cdot \mu)\|_\infty \leq \|R_\alpha^A f\|_\infty \|U_\alpha \mu\|_\infty < \infty$ and $\mu(\partial\Omega) < \infty$, we have that

$$R_\alpha^A f = R_\alpha f - U_A^\alpha R_\alpha^A f \in \mathcal{F}^\mu \tag{4.5}$$

and that

$$\begin{aligned} \mathcal{E}_\alpha\left(R_\alpha^A f, u\right) &= \mathcal{E}_\alpha(R_\alpha f, u) - \mathcal{E}_\alpha\left(U_A^\alpha R_\alpha^A f, u\right) \\ &= (f, u) - \left(R_\alpha^A f, u\right)_\mu, \quad u \in \mathcal{F}^\mu, \end{aligned} \tag{4.6}$$

then (4.7) follows.

For general positive measure μ charging no set of zero relative capacity, we can take by virtue of Theorem 2.3 and Lemma 2.2 an increasing sequence (F_n) of generalized nest of $\partial\Omega$, and $\mu_n = 1_{F_n} \cdot \mu \in S_{00}(\partial\Omega)$. Since μ charges no set of zero relative capacity, $\mu_n(B)$ increases to $\mu(B)$ for any $B \in \mathcal{B}(\partial\Omega)$.

Let $A_n = 1_{F_n} \cdot A$. Then A_n is a PCAF of X_t with Revuz measure μ_n. Since $\mu_n \in S_{00}(\partial\Omega)$ we have for $f \in L^2(\Omega)$:

$$R_\alpha^{A_n} f \in \mathcal{F}^{\mu_n}, \quad \mathcal{E}_\alpha^{\mu_n}\left(R_\alpha^{A_n}, u\right) = (f, u), \quad f \in L^2(\Omega, m),\ u \in \mathcal{F}^{\mu_n}. \tag{4.7}$$

Clearly $|R_\alpha^{A_n} f| \leq R_\alpha|f| < \infty$ r.q.e, and hence $\lim_{n\to+\infty} R_\alpha^{A_n} f(x) = R_\alpha^A f(x)$ for r.q.e $x \in \overline{\Omega}$. For $n < m$, we get from (4.8)

$$\mathcal{E}_\alpha^{\mu_n}\left(R_\alpha^{A_n} f - R_\alpha^{A_m} f, R_\alpha^{A_n} f - R_\alpha^{A_m} f\right) \leq \left(f, R_\alpha^{A_n} f - R_\alpha^{A_m} f\right), \tag{4.8}$$

which converges to zero as $n, m \to +\infty$. Therefore, $(R_\alpha^{A_n} f)_n$ is $\mathcal{E}_1$-convergent in $\mathcal{F}$ and the limit function $R_\alpha^A f$ is in $\widetilde{\mathcal{F}}$. On the other hand, we also get from (4.8)

$$\left\|R_\alpha^{A_n} f\right\|_{L^2(\partial\Omega,\mu)} \leq \left(f, R_\alpha^{A_n} f\right)_{L^2(\Omega)} \leq \left(\frac{1}{\alpha}\right)\|f\|_{L^2(\Omega)}. \tag{4.9}$$

And by Fatou's lemma: $\|R_\alpha^A f\|_{L^2(\Omega)} \leq (1/\sqrt{\alpha})\|f\|_{L^2(\Omega)}$, getting $R_\alpha^A f \in \mathcal{F}^\mu$. Finally, observe the estimate

$$\left|\left(R_\alpha^{A_n} f, u\right)_{\mu_n} - \left(R_\alpha^A f, u\right)_\mu\right| \leq \left\|R_\alpha^{A_n} f - R_\alpha^A f\right\|_{L^2(\partial\Omega,\mu_n)} \|u\|_{L^2(\partial\Omega,\mu)} + \left|(R_\alpha f, u)_{\mu-\mu_n}\right| \tag{4.10}$$

holding for $u \in L^2(\partial\Omega, \mu)$. The second term of the right-hand side tends to zero as $n \to +\infty$. The first term also tends to zero because we have from (4.8) $\|R_\alpha^{A_n} f - R_\alpha^{A_m} f\|_{L^2(\partial\Omega,\mu_n)} \leq (f, R_\alpha^{A_n} f - R_\alpha^{A_m} f)$, and it suffices to let first $m \to +\infty$ and then $n \to +\infty$. By letting $n \to +\infty$ in (4.7), we arrive to desired equation (4.3). □

The proof of Theorem 4.2 is similar to [17, Theorem 6.1.1] which was formulated in the first time by Albeverio and Ma [14] for general smooth measures in the context of general (X, m). In the case of $X = \overline{\Omega}$, and working just with measures on $S_0(\partial\Omega)$, the proof still the same and works also for any smooth measure concentrated on $\partial\Omega$. Consequently, the theorem is still verified for smooth measures "nowhere Radon," that is, measures locally infinite on $\partial\Omega$.

Example 4.3. We give some particular examples of $\mathcal{P}_t^\mu$.

(1) If $\mu = 0$, then

$$\mathcal{P}_t^0 f(x) = E_x\left[f(X_t)\right], \tag{4.11}$$

the semigroup generated by Laplacian with, Neumann boundary conditions.

(2) If μ is locally infinite (nowhere Radon) on $\partial\Omega$, then

$$\mathcal{P}_t^\infty f(x) = E_x\left[f(B_t)1_{\{t<\tau\}}\right], \tag{4.12}$$

the semigroup generated by the Laplacian with Dirichlet boundary conditions (see [13, Proposition 3.2.1]).

(3) Let Ω be a bounded and enough smooth to insure the existence of the surface measure σ, and $\mu = \beta \cdot \sigma$, with β is a measurable bounded function on $\partial\Omega$, then $A_t^\mu = \int_0^t \beta(X_s)dL_s$, where L_t is a boundary local time. Consequently

$$\mathcal{P}_t^\mu f(x) = E_x\left[f(X_t)\exp\left(-\int_0^t \beta(X_s)dL_s\right)\right] \tag{4.13}$$

is the semigroup generated by the Laplacian with (classical) Robin boundary conditions given by (1.1).

The setting of the problem from the stochastic point of view and the stochastic representation of the solution of the problem studied are important on themselves and are new. In fact before there was always additional hypothesis on the domain or on the class of measures. Even if our approach is inspired by the works [14, 15] and Chapter 6 of [17], the link is not obvious and give as rise to a new approach to the Robin boundary conditions. As a consequence, the proof of many propositions and properties become obvious and direct. The advantage of the stochastic approach is, then, to give explicitly the representation of the semigroup and an easy access of it.

Proposition 4.4. *$\mathcal{P}_t^\mu$ is sub-Markovian, that is, $\mathcal{P}_t^\mu \geq 0$ for all $t \geq 0$, and*

$$\left\| \mathcal{P}_t^\mu f \right\|_\infty \leq \| f \|_\infty \quad (t \geq 0). \tag{4.14}$$

Proof. It is clear that if $f \in L^2(\Omega)_+$, then $\mathcal{P}_t^\mu f \geq 0$ for all $t \geq 0$. In addition we have $|\mathcal{P}_t^\mu f(x)| \leq E_x[|f|(X_t)]$, and then $\|\mathcal{P}_t^\mu f\|_\infty \leq \|f\|_\infty \quad (t \geq 0)$. □

Remark 4.5. The analytic proof needs the first and the second BeurlingDeny criterion [11, Proposition 3.10] while our proof is obvious and direct.

Let Δ_μ be the self-adjoint operator on $L^2(\Omega)$ generator of the semigroup $\mathcal{P}_t^\mu$, we write

$$\mathcal{P}_t^\mu f(x) = e^{-t\Delta_\mu} f(x). \tag{4.15}$$

Following [13], we know that Δ_μ is a realization of the Laplacian. Then we call Δ_μ the Laplacian with General Robin boundary conditions.

Theorem 4.6. *Let $\mu \in S(\partial\Omega)$, then the semigroup $\mathcal{P}_t^\mu$ is sandwiched between the semigroup of Neumann Laplacian, and the semigroup of Dirichlet Laplacian. That is*

$$0 \leq e^{-t\Delta_D} \leq \mathcal{P}_t^\mu \leq e^{-t\Delta_N} \tag{4.16}$$

for all $t \geq 0$, in the sense of positive operators.

Proof. Let $f \in L^2(\Omega)_+$. Since $A_t^\mu \geq 0$ we get easily the following: $\mathcal{P}_t^\mu f(x) \leq E_x[f(X_t)]$ for any $x \in \overline{\Omega}$. On the other hand, we have $\mathcal{P}_t^\mu f(x) \geq E_x[f(X_t)e^{-A_t^\mu} 1_{\{t<\sigma_{\partial\Omega}\}}]$, where $\sigma_{\partial\Omega}$ is the first hitting time of $\partial\Omega$. Since the relative quasisupport of A_t^μ and N_t are in $\partial\Omega$, then in $\{t < \sigma_{\partial\Omega}\}$, N_t and A_t^μ vanishes. Consequently, $X_t = B_t$ in $\{t < \sigma_{\partial\Omega}\}$ and $\mathcal{P}_t^\mu f(x) \geq E_x[f(B_t)1_{\{t<\sigma_{\partial\Omega}\}}]$. The theorem follows. □

Remark 4.7. The fact that the semigroup $\mathcal{P}_t^\mu$ is sandwiched between the Neumann semigroup and the Dirichlet one as proved in [13, Theorem 3.4.1] is not obvious and needs a result characterizing the domination of positive semigroups due to Ouhabaz, while our proof is simple and direct.

Proposition 4.8. *Let $\mu, \nu \in S(\partial\Omega)$ such that $\nu \leq \mu$ (i.e., $\nu(A) \leq \mu(A)$, for all $A \in \mathcal{B}(\partial\Omega)$), then*

$$0 \leq e^{-t\Delta_D} \leq \mathcal{P}_t^\mu \leq \mathcal{P}_t^\nu \leq e^{-t\Delta_N} \tag{4.17}$$

for all $t \geq 0$, in the sense of positive operators.

Proof. It follows from the remark that if $\nu \leq \mu$, then $A_t^\nu \leq A_t^\mu$, which means that $(A_t^\mu)_\mu$ is increasing, and then $(\mathcal{P}_t^\mu)_\mu$ is decreasing. □

There exist a canonical Hunt process X_t^A possessing the transition function $\mathcal{P}_t^\mu$ which is directly constructed from X_t by killing the paths with rate $-dL_t$, where $L_t = e^{-A_t}$.

To construct the process associated with $\mathcal{P}_t^\mu$, we follow A.2 of [17], so we need a nonnegative random variable $Z(\omega)$ on $(\Xi, \mathcal{M}, P_x)$ which is of an exponential distribution with mean 1, independent of $(X_t)_{t\geq 0}$ under P_x for every $x \in \overline{\Omega}$ satisfying $Z(\theta_s(\omega)) = (Z(\omega) - s) \vee 0$. Introducing now a Random time ξ^A defined by

$$\xi^A = \inf\{t \geq 0 : A_t \geq Z\}. \tag{4.18}$$

We define the process $(X_t^A)_{t\geq 0}$ by

$$X_t^A = \begin{cases} X_t & \text{if } t < \xi^A; \\ \Delta & \text{if } t \geq \xi^A, \end{cases} \tag{4.19}$$

where Δ is a one-point compactification.

And, the admissible filtration of the process $(X_t^A)_{t\geq 0}$ is defined by

$$\mathcal{F}_t^A = \{\Lambda \in \mathcal{F}_\infty : \Lambda \cap \{A_t < Z\} = \Lambda_t \cap \{A_t < Z\},\ \exists \Lambda_t \in \mathcal{F}_t\}. \tag{4.20}$$

Since $\{A_t < Z\} \cap \{A_t = \infty\} = \emptyset$, we may and will assume that $\Lambda_t \supset \{A_t = \infty\}$.

Now, we can write

$$\begin{aligned} E_x\left[f\left(X_t^A\right)\right] &= E_x\left[f(X_t) : t < \xi^A\right] \\ &= E_x\left[f(X_t) : A_t < Z\right] \\ &= E_x\left[f(X_t)e^{-A_t}\right] \\ &= \mathcal{P}_t^\mu f(x). \end{aligned} \tag{4.21}$$

The Hunt process $(X_t^A)_{f\geq 0}$ is called the canonical subprocess of $(X_t)_{t\geq 0}$ relative to the multiplicative functional L_t. In fact, $(X_t^A)_{t\geq 0}$ is a Diffusion process as $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ is local.

In the literature, the Diffusion process X_t^A is called a partially reflected Brownian motion [25], in the sense that, the paths of X_t are reflected on the boundary since they will be killed (absorbed) at the random time ξ^A with rate $-dL_t$.

Theorem 4.9. *Let $\mu, \mu_n \in S(\partial\Omega)$ such that μ_n is monotone and converges setwise to μ, that is, $\mu_n(B)$ converges to $\mu(B)$ for any $B \in \mathcal{B}(\partial\Omega)$, then Δ_{μ_n} converges to Δ_μ in strongly resolvent sense.*

Proof. We prove the theorem for μ_n increasing, the proof of the decreasing case is similar. Let A_n (resp. A) be the additive functional associated to μ_n (resp. μ) by the Revuz correspondence. Similarly to the second part of the proof of Theorem 4.2, we have $\lim R_\alpha^{A_n} f(x) = R_\alpha^A f(x)$ for r.q.e $x \in \overline{\Omega}$. Consequently $\lim_{n\to+\infty} \|R_\alpha^{A_n} f - R_\alpha^A f\|_{L^2(\Omega)} = 0$. For $n < m$, we have $\mathcal{F}^{\mu_m} \subset \mathcal{F}^{\mu_n}$, and then

$$\mathcal{E}_\alpha^{\mu_n}\left(R_\alpha^{A_n} f - R_\alpha^{A_m} f, R_\alpha^{A_n} f - R_\alpha^{A_m} f\right) \leq \left(f, R_\alpha^{A_n} f - R_\alpha^{A_m} f\right), \tag{4.22}$$

which converges to zero as $n, m \to +\infty$. Therefore, $(R_\alpha^{A_n} f)_n$ is $\mathcal{E}_1$-convergent in $\mathcal{F}$ and the limit function $R_\alpha^A f$ is in $\widetilde{\mathcal{F}}$. The result follows. □

Corollary 4.10. *Let $\mu \in S(\partial\Omega)$ finite and let $k \in \mathbb{N}^*$. We defined for $u, v \in \mathcal{F}^\mu$:*

$$\mathcal{E}^{\mu_k}(u, v) = \int_\Omega \nabla u \nabla v dx + \frac{1}{k} \int_{\partial\Omega} \tilde{u}\tilde{v} d\mu, \tag{4.23}$$

then $\Delta_{\mu_k} \to \Delta_N$ in the strong resolvent sense.

Intuitively speaking, when the measure μ is infinity (locally infinite on the boundary), the semigroup $\mathcal{P}_t^\mu$ is the Dirichlet semigroup as said in the example 2 in section 4, which means that the boundary became "completely absorbing," and any other additive functional in the boundary cannot influence this phenomena, which explain why N_t does not appear yet in the decomposition of X_t, which means that the reflecting phenomena disappear, and so any path of X_t is immediately killed when it arrives to the boundary.

When μ is null on the boundary, then the semigroup $\mathcal{P}_t^\mu$ is the Neumann one, and then the boundary became completely reflecting, but for a general measure μ the paths are reflected many times before they are absorbed at a random time.

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