Hindawi Publishing Corporation
International Journal of Stochastic Analysis
Volume 2013, Article ID 952628, 4 pages
http://dx.doi.org/10.1155/2013/952628

Research Article

Sharp Large Deviation for the Energy of α-Brownian Bridge

Shoujiang Zhao,[1] Qiaojing Liu,[1] Fuxiang Liu,[1] and Hong Yin[2]

[1] *School of Science, China Three Gorges University, Yichang 443002, China*
[2] *School of Information, Renmin University of China, Beijing 100872, China*

Correspondence should be addressed to Qiaojing Liu; qjliu2002@163.com

Received 26 April 2013; Accepted 23 October 2013

Academic Editor: Yaozhong Hu

We study the sharp large deviation for the energy of α-Brownian bridge. The full expansion of the tail probability for energy is obtained by the change of measure.

1. Introduction

We consider the following α-Brownian bridge:

$$dX_t = -\frac{\alpha}{T-t} X_t dt + dW_t, \qquad X_0 = 0, \tag{1}$$

where W is a standard Brownian motion, $t \in [0, T)$, $T \in (0, \infty)$, and the constant $\alpha > 1/2$. Let P_α denote the probability distribution of the solution $\{X_t, t \in [0, T)\}$ of (1). The α-Brownian bridge is first used to study the arbitrage profit associated with a given future contract in the absence of transaction costs by Brennan and Schwartz [1].

α-Brownian bridge is a time inhomogeneous diffusion process which has been studied by Barczy and Pap [2, 3], Jiang and Zhao [4], and Zhao and Liu [5]. They studied the central limit theorem and the large deviations for parameter estimators and hypothesis testing problem of α-Brownian bridge. While the large deviation is not so helpful in some statistics problems since it only gives a logarithmic equivalent for the deviation probability, Bahadur and Ranga Rao [6] overcame this difficulty by the sharp large deviation principle for the empirical mean. Recently, the sharp large deviation principle is widely used in the study of Gaussian quadratic forms, Ornstein-Uhlenbeck model, and fractional Ornstein-Uhlenbeck (cf. Bercu and Rouault [7], Bercu et al. [8], and Bercu et al. [9, 10]).

In this paper we consider the sharp large deviation principle (SLDP) of energy S_t, where

$$S_t = \int_0^t \frac{X_s^2}{(s-T)^2} ds. \tag{2}$$

Our main results are the following.

Theorem 1. *Let $\{X_t, t \in [0, T)\}$ be the process given by the stochastic differential equation (1). Then $\{S_t/\lambda_t, t \in [0, T)\}$ satisfies the large deviation principle with speed λ_t and good rate function $I(\cdot)$ defined by the following:*

$$I(x) = \begin{cases} \frac{1}{8x}\left((2\alpha_0 - 1)x - 1\right)^2, & \text{if } x > 0; \\ +\infty, & \text{if } x \le 0, \end{cases} \tag{3}$$

where $\lambda_t = \log(T/(T-t))$.

Theorem 2. *$\{S_t/\lambda_t, t \in [0, T)\}$ satisfies SLDP; that is, for any $c > 1/(2\alpha - 1)$, there exists a sequence $b_{c,k}$ such that, for any $p > 0$, when t approaches T enough,*

$$P(S_t \ge c\lambda_t) = \frac{\exp\{-I(c)\lambda_t + H(a_c)\}}{\sqrt{2\pi} a_c \beta_t} \times \left(1 + \sum_{k=1}^{p} \frac{b_{c,k}}{\lambda_t} + O\left(\frac{1}{\lambda_t^{p+1}}\right)\right), \tag{4}$$

where

$$\sigma_c^2 = 4c^2, \qquad \beta_t = \sigma_c \sqrt{\lambda_t},$$
$$a_c = \frac{(1-2\alpha)^2 c^2 - 1}{8c^2}, \tag{5}$$
$$H(a_c) = -\frac{1}{2}\log\left(\frac{1-(1-2\alpha)c}{2}\right).$$

The coefficients $b_{c,k}$ may be explicitly computed as functions of the derivatives of L and H (defined in Lemma 3) at point a_c. For example, $b_{c,1}$ is given by

$$b_{c,1} = \frac{1}{\sigma_c^2}\left(-\frac{h_2}{2} - \frac{h_1^2}{2} + \frac{l_4}{8\sigma_c^2} + \frac{l_3 h_1}{2\sigma_c^2} - \frac{5l_3^2}{24\sigma_c^4} + \frac{h_1}{a_c} - \frac{l_3}{2a_c\sigma_c^2} - \frac{1}{a_c^2}\right), \tag{6}$$

with $l_k = L^{(k)}(a_c)$, and $h_k = H^{(k)}(a_c)$.

2. Large Deviation for Energy

Given $\alpha > 1/2$, we first consider the following logarithmic moment generating function of S_t; that is,

$$L_t(u) := \log \mathbb{E}_\alpha \exp\left\{u\int_0^t \frac{X_s^2}{(s-T)^2}ds\right\}, \quad \forall\lambda \in \mathbb{R}. \tag{7}$$

And let

$$\mathscr{D}_{L_t} := \{u \in \mathbb{R},\ L_t(u) < +\infty\} \tag{8}$$

be the effective domain of L_t. By the same method as in Zhao and Liu [5], we have the following lemma.

Lemma 3. *Let $\mathscr{D}_L$ be the effective domain of the limit L of L_t; then for all $u \in \mathscr{D}_L$, one has*

$$\frac{L_t(u)}{\lambda_t} = L(u) + \frac{H(u)}{\lambda_t} + \frac{R(u)}{\lambda_t}, \tag{9}$$

with

$$L(u) = -\frac{1-2\alpha-\varphi(u)}{4},$$
$$H(\lambda) = -\frac{1}{2}\log\left\{\frac{1}{2}(1+h(u))\right\}, \tag{10}$$
$$R(u) = -\frac{1}{2}\log\left\{1+\frac{1-h(u)}{1+h(u)}\exp\{2\varphi(u)\lambda_t\}\right\},$$

where $\varphi(u) = -\sqrt{(1-2\alpha)^2 - 8u}$ and $h(u) = (1-2\alpha)/\varphi(u)$. Furthermore, the remainder $R(u)$ satisfies

$$R(u) = O_{t\to T}\left(\exp\{2\varphi(u)\lambda_t\}\right). \tag{11}$$

Proof. By Itô's formula and Girsanov's formula (see Jacob and Shiryaev [11]), for all $u \in \mathscr{D}_L$ and $t \in [0,T)$,

$$\log\frac{dP_\alpha}{dP_\beta}|_{[0,t]} = (\alpha-\beta)\int_0^t \frac{X_s}{s-T}dX_s - \frac{\alpha^2-\beta^2}{2}\int_0^t \frac{X_s^2}{(s-T)^2}ds,$$
$$\int_0^t \frac{X_s}{s-T}dX_s = \frac{1}{2}\left(\frac{X_t^2}{(t-T)} + \int_0^t \frac{X_s^2}{(s-T)^2}ds - \log\left(1-\frac{t}{T}\right)\right). \tag{12}$$

Therefore,

$$\begin{aligned} L_t(u) &= \log\mathbb{E}_\beta\left(\exp\left\{u\int_0^t \frac{X_s^2}{(s-T)^2}ds\right\}\frac{dP_\alpha}{dP_\beta}|_{[0,t]}\right) \\ &= \log\mathbb{E}_\beta\exp\left\{\frac{\alpha-\beta}{2(t-T)}X_t^2 - \frac{\alpha-\beta}{2}\log\left(1-\frac{t}{T}\right) + \frac{1}{2}\left(\beta^2-\alpha^2+\alpha-\beta+2u\right) \times \int_0^t \frac{X_s^2}{(s-T)^2}ds\right\}. \end{aligned} \tag{13}$$

If $4u \le (1-2\alpha)^2$, we can choose β such that $(\beta-1/2)^2 - (\alpha-1/2)^2 + 2u = 0$. Then

$$\begin{aligned} L_t(u) = &-\frac{1-2\alpha-\varphi(\lambda)}{4}\lambda_t \\ &-\frac{1}{2}\log\left\{\frac{1}{2}(1+h(u))\right\} \\ &-\frac{1}{2}\log\left\{1+\frac{1-h(u)}{1+h(u)}\exp\{2\varphi(u)\lambda_t\}\right\}, \end{aligned} \tag{14}$$

where $\varphi(u) = -\sqrt{(1-2\alpha)^2 - 8u}$, and $h(u) = (1-2\alpha)/\varphi(u)$. Therefore,

$$\begin{aligned} \frac{L_t(u)}{\lambda_t} &= -\frac{1-2\alpha-\varphi(u)}{4} - \frac{1}{2\lambda_t}\log\left\{\frac{1}{2}(1+h(u))\right\} - \frac{1}{2\lambda_t}\log\left\{1+\frac{1-h(u)}{1+h(u)}\exp\{2\varphi(u)\lambda_t\}\right\} \\ &= L(u) + \frac{H(u)}{\lambda_t} + \frac{R(u)}{\lambda_t}. \end{aligned} \tag{15}$$

□

Proof of Theorem 1. From Lemma 3, we have

$$L(u) = \lim_{t\to T}\frac{L_t(u)}{\lambda_t} = \frac{1-2\alpha-\varphi(u)}{4}, \tag{16}$$

and $L(\cdot)$ is steep; by the Gärtner-Ellis theorem (Dembo and Zeitouni [12]), S_t/λ_t satisfies the large deviation principle with speed λ_t and good rate function $I(\cdot)$ defined by the following:

$$I(x) = \begin{cases} \frac{1}{8x}((2\alpha-1)x-1)^2, & \text{if } x>0; \\ +\infty, & \text{if } +x \le 0. \end{cases} \tag{17}$$

□

Remark 4. Theorem 1 can also be obtained by using Theorem 1 in Zhao and Liu [5].

3. Sharp Large Deviation for Energy

For $c > 1/(2\alpha-1)$, let

$$a_c = \frac{(1-2\alpha)^2c^2-1}{8c^2}, \qquad \sigma_c^2 = L''(a_c) = 4c^3,$$
$$H(a_c) = -\frac{1}{2}\log(1-(1-2\alpha)c). \tag{18}$$

Then

$$\begin{aligned} P(S_t \ge c\lambda_t) &= \int_{S_t\ge c\lambda_t} \exp\{L(a_c) - ca_c\lambda_t + ca_c\lambda_t - a_cS_t\}\,dQ_t \\ &= \exp\{L(a_c) - ca_c\lambda_t\}\mathbb{E}_Q \exp\{-a_c\beta_tU_tI_{\{U_t\ge 0\}}\} = A_tB_t, \end{aligned} \tag{19}$$

where $\mathbb{E}_Q$ is the expectation after the change of measure

$$\frac{dQ_t}{dP} = \exp\{a_cS_t - L_t(a_c)\},$$
$$U_t = \frac{S_t - c\lambda_t}{\beta_t}, \qquad \beta_t = \sigma_c\sqrt{\lambda_t}. \tag{20}$$

By Lemma 3, we have the following expression of A_t.

Lemma 5. *For all $c > 1/(2\alpha-1)$, when t approaches T enough,*

$$A_t = \exp\{-I(c)\lambda_t + H(a_c)\}(1+O((T-t)^c)). \tag{21}$$

For B_t, one gets the following.

Lemma 6. *For all $c > 1/(2\alpha-1)$, the distribution of U_t under Q_t converges to $N(0,1)$ distribution. Furthermore, there exists a sequence ψ_k such that, for any $p > 0$ when t approaches T enough,*

$$B_t = \frac{1}{a_c\sigma_c\sqrt{2\pi\lambda_t}}\left(1+\sum_{k=1}^{p}\frac{\psi_k}{\lambda_t^k} + O\left(\lambda_t^{-(p+1)}\right)\right). \tag{22}$$

Proof of Theorem 2. The theorem follows from Lemma 5 and Lemma 6. □

It only remains to prove Lemma 6. Let $\Phi_t(\cdot)$ be the characteristic function of U_t under Q_t; then we have the following.

Lemma 7. *When t approaches T, Φ_t belongs to $L^2(\mathbb{R})$ and, for all $u \in \mathbb{R}$,*

$$\Phi_t(u) = \exp\left\{-\frac{iu\sqrt{\lambda_t}c}{\sigma_c}\right\} \times \exp\left\{\left(L_t\left(a_c + \frac{iu}{\beta_t}\right) - L_t(a_c)\right)\right\}. \tag{23}$$

Moreover,

$$B_t = \mathbb{E}_Q \exp\{-a_c\beta_tU_tI_{\{U_t\ge 0\}}\} = C_t + D_t, \tag{24}$$

with

$$C_t = \frac{1}{2\pi a_c\beta_t}\int_{|u|\le s_t}\left(1+\frac{iu}{a_c\beta_t}\right)^{-1}\Phi_t(u)\,du,$$
$$D_t = \frac{1}{2\pi a_c\beta_t}\int_{|u|> s_t}\left(1+\frac{iu}{a_c\beta_t}\right)^{-1}\Phi_t(u)\,du, \tag{25}$$
$$|D_t| = O\left(\exp\{-D\lambda_t^{1/3}\}\right),$$

where

$$s_t = s\left(\log\left(\frac{T}{T-t}\right)\right)^{1/6}, \tag{26}$$

for some positive constant s, and D is some positive constant.

Proof. For any $u \in \mathbb{R}$,

$$\begin{aligned} \Phi_t(u) &= \mathbb{E}(\exp\{iuU_t\}\exp\{a_cS_t - L_t(a_c)\}) \\ &= \exp\left\{-\frac{iu\sqrt{\lambda_t}c}{\sigma_c}\right\} \times \exp\left\{\left(L_t\left(a_c+\frac{iu}{\beta_t}\right) - L_t(a_c)\right)\right\}. \end{aligned} \tag{27}$$

By the same method as in the proof of Lemma 2.2 in [7] by Bercu and Rouault, there exist two positive constants τ and κ such that

$$|\Phi_t(u)|^2 \le \left(1+\frac{\tau u^2}{\lambda_t}\right)^{-(\kappa/2)\lambda_t}; \tag{28}$$

therefore, $\Phi_t(\cdot)$ belongs to $L^2(\mathbb{R})$, and by Parseval's formula, for some positive constant s, let

$$s_t = s\left(\log\left(\frac{T}{T-t}\right)\right)^{1/6}; \tag{29}$$

we get

$$B_t = \frac{1}{2\pi a_c\beta_t}\int_{|u|\le s_t}\left(1+\frac{iu}{a_c\beta_t}\right)^{-1}\Phi_t(u)\,du + \frac{1}{2\pi a_c\beta_t} \times \int_{|u|>s_t}\left(1+\frac{iu}{a_c\beta_t}\right)^{-1}\Phi_t(u)\,du \tag{30}$$

$$= : C_t + D_t, \tag{31}$$

$$|D_t| = O\left(\exp\{-D\lambda_t^{1/3}\}\right), \tag{32}$$

where D is some positive constant. □

Proof of Lemma 6. By Lemma 3, we have

$$\frac{L_t^{(k)}(a_c)}{\lambda_t} = L^{(k)}(a_c) + \frac{H^{(k)}(a_c)}{\lambda_t} + \frac{O\left(\lambda_t^k (T-t)^{-2c}\right)}{\lambda_t}. \quad (33)$$

Noting that $L'(a_c) = 0, L''(a_c) = \sigma_c^2$ and

$$\frac{L''(a_c)}{2}\left(\frac{iu}{\beta_t}\right)^2 \lambda_t = -\frac{u^2}{2}, \quad (34)$$

for any $p > 0$, by Taylor expansion, we obtain

$$\begin{aligned}\log \Phi_t(u) = & -\frac{u^2}{2} + \lambda_t \sum_{k=3}^{2p+3} \left(\frac{iu}{\beta_t}\right)^k \frac{L^{(k)}(a_c)}{k!} \\ & + \sum_{k=1}^{2p+1} \left(\frac{iu}{\beta_t}\right)^k \frac{H^{(k)}(a_c)}{k!} \\ & + O\left(\frac{\max\left(1, |u|^{2p+4}\right)}{\lambda_t^{p+1}}\right);\end{aligned} \quad (35)$$

therefore, there exist integers $q(p), r(p)$ and a sequence $\varphi_{k,l}$ independent of$p$; when t approaches T, we get

$$\begin{aligned}\Phi_t(u) = & \exp\left\{-\frac{u^2}{2}\right\}\left(1 + \frac{1}{\sqrt{\lambda_t}} \sum_{k=0}^{2p} \sum_{l=k+1}^{q(p)} \frac{\varphi_{k,l} u^l}{\lambda_t^{k/2}} \right. \\ & \left. + O\left(\frac{\max\left(1, |u|^{r(p)}\right)}{\lambda_t^{p+1}}\right)\right),\end{aligned} \quad (36)$$

where O is uniform as soon as $|u| \le s_t$.

Finally, we get the proof of Lemma 6 by Lemma 7 together with standard calculations on the $N(0,1)$ distribution. □

Acknowledgment

This research was supported by the National Natural Science of Tianyuan Foundation under Grant 11226202.

References

[1] M. J. Brennan and E. S. Schwartz, "Arbitrage in stock index futures," *The Journal of Business*, vol. 63, pp. 7–31, 1990.

[2] M. Barczy and G. Pap, "Asymptotic behavior of maximum likelihood estimator for time inhomogeneous diffusion processes," *Journal of Statistical Planning and Inference*, vol. 140, no. 6, pp. 1576–1593, 2010.

[3] M. Barczy and G. Pap, "Explicit formulas for Laplace transforms of certain functionals of some time inhomogeneous diffusions," *Journal of Mathematical Analysis and Applications*, vol. 380, no. 2, pp. 405–424, 2011.

[4] H. Jiang and S. Zhao, "Large and moderate deviations in testing time inhomogeneous diffusions," *Journal of Statistical Planning and Inference*, vol. 141, no. 9, pp. 3160–3169, 2011.

[5] S. Zhao and Q. Liu, "Large deviations for parameter estimators of α-Brownian bridge," *Journal of Statistical Planning and Inference*, vol. 142, no. 3, pp. 695–707, 2012.

[6] R. R. Bahadur and R. Ranga Rao, "On deviations of the sample mean," *Annals of Mathematical Statistics*, vol. 31, pp. 1015–1027, 1960.

[7] B. Bercu and A. Rouault, "Sharp large deviations for the Ornstein-Uhlenbeck process," *Rossiĭskaya Akademiya Nauk. Teoriya Veroyatnosteĭ i ee Primeneniya*, vol. 46, no. 1, pp. 74–93, 2001.

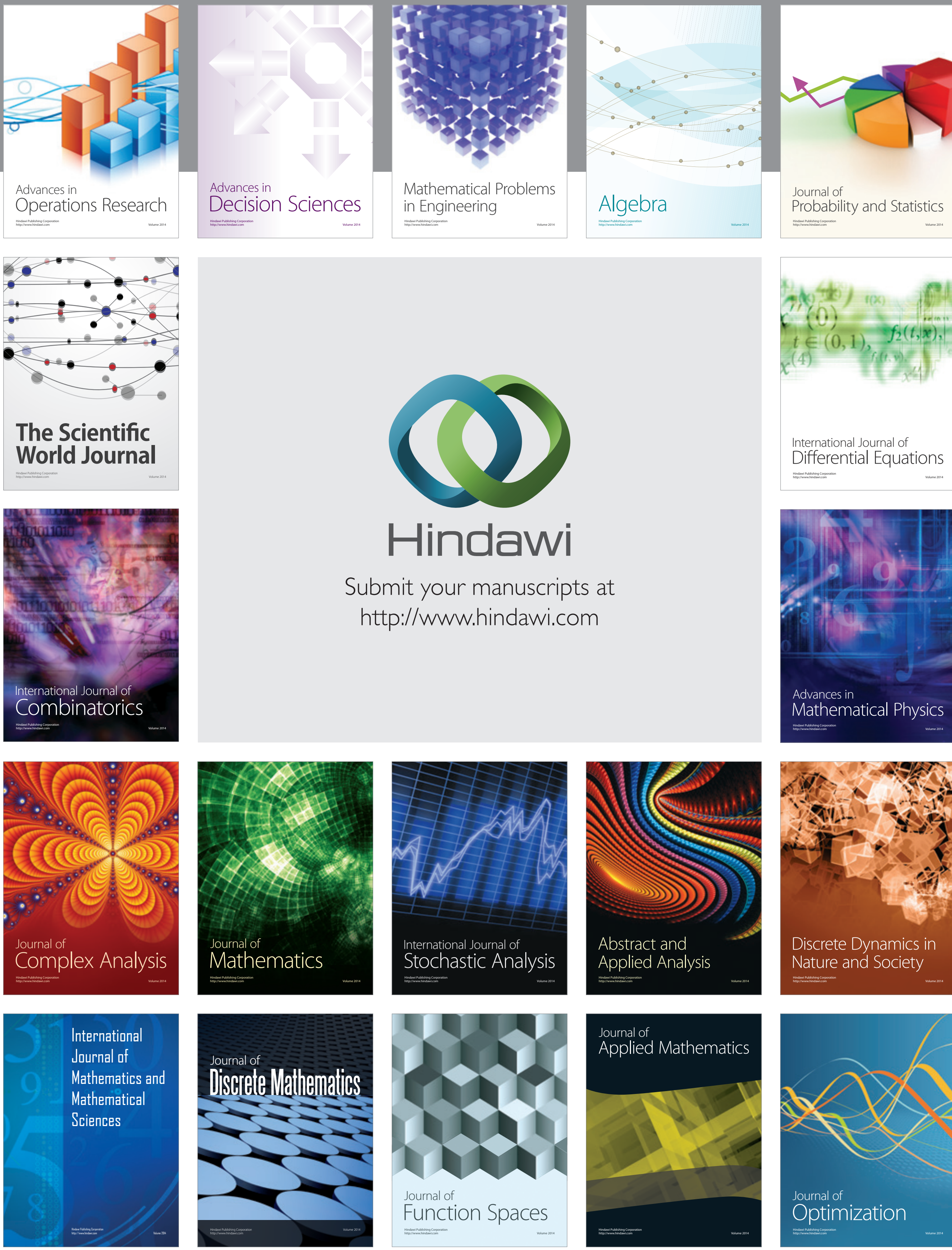
[8] B. Bercu, F. Gamboa, and M. Lavielle, "Sharp large deviations for Gaussian quadratic forms with applications," *European Series in Applied and Industrial Mathematics. Probability and Statistics*, vol. 4, pp. 1–24, 2000.

[9] B. Bercu, L. Coutin, and N. Savy, "Sharp large deviations for the fractional Ornstein-Uhlenbeck process," *SIAM Theory of Probability and Its Applications*, vol. 55, pp. 575–610, 2011.

[10] B. Bercu, L. Coutin, and N. Savy, "Sharp large deviations for the non-stationary Ornstein-Uhlenbeck process," *Stochastic Processes and their Applications*, vol. 122, no. 10, pp. 3393–3424, 2012.

[11] J. Jacod and A. N. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer Press, Berlin, Germany, 1987.

[12] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*, vol. 38, Springe, Berlin, Germany, 2nd edition, 1998.


Advances in Operations Research
Advances in Decision Sciences
Mathematical Problems in Engineering
Algebra
Journal of Probability and Statistics
The Scientific World Journal
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Differential Equations
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
International Journal of Stochastic Analysis
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
Journal of Applied Mathematics
Journal of Optimization