

STABILITY OF SOLUTIONS OF A NONSTANDARD ORDINARY DIFFERENTIAL SYSTEM BY LYAPUNOV'S SECOND METHOD¹

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ABSTRACT

Differential equations of the form $y' = f(t, y, y')$ where f is not necessarily linear in its arguments represent certain physical phenomena and are known for quite some time. The well known Clairut's and Chrystal's equations fall into this category. Earlier we established the existence of a (unique) solution of the nonstandard initial value problem $y' = f(t, y, y'), y(t_0) = y_0$ under certain natural hypotheses on f . In this paper, we studied the stability of solutions of a nonstandard first order ordinary differential system.

Key words: Nonstandard, initial, value, problem, existence, unique, solution, continuous, differentiable, contraction, system, stable, uniform, asymptotic.

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1. INTRODUCTION

Differential equations of the form $y' = f(t, y, y')$ where f is not necessarily linear in its arguments represent certain physical phenomena and are known for quite some time. The well known Clairut's and Chrystal's equations fall into this category [4]. A few authors, notably E.L. Ince [5], H.T. Davis [4] et. al. have given some methods for finding solutions of equations of the above type. Apart from these, to the authors knowledge, there does not seem to exist any systematic study of these equations.

In our earlier paper [7], we have studied the solutions of initial value problems (IVPs) associated with the above type of equations. The next fundamental aspect of these equations is the study of stability of solutions. In the present paper, we shall study the stability of solutions of a system of the first order equations of the above type.

Before proceeding to the stability theorems we shall introduce a few notations, definitions and present a few existence results under certain usual assumptions. we shall use the following notation:

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$I = [0, \infty)$,

R^n denotes the n -dimensional real space equipped with the box norm $|\cdot|$ given by

$$|x| = \sum_{i=1}^n |x_i|, \text{ where } x = \text{column}(x_1, x_2, \dots, x_n) \in R^n,$$

$S = \{z \in R^n \mid |z| \leq c\}$, where c is a positive constant,

D is a connected subset of R^n with the non empty interior,

A^0 denotes the interior of a subset A of R^n ,

$C(J, R^n)$ is the set of all continuous vector valued functions defined on a subinterval J of I with values in R^n ,

y' and y'' denote the first and second order derivatives of y , respectively, if they exist.

$V_1 \times V_2$ is the Cartesian product of two non-empty sets V_1 and V_2 , taken in that order,

$f(t, y, z) = \text{column}(f_1(t, y, z), f_2(t, y, z), \dots, f_n(t, y, z))$ is a vector valued not necessarily linear function defined for $(t, y, z) \in I \times D \times S$ with values in R^n ,

g_j denotes j^{th} ($j = 1, 2, \dots, n$) component of a function g defined on $I \times D \times S$ with values in R^n ,

B^{-1} is the inverse matrix of the matrix B , if it exists.

Consider the nonstandard initial value problem

$$y' = f(t, y, y'), \tag{1}$$

$$y(t_0) = y_0 \tag{2}$$

where $(t_0, y_0) \in I \times D^0$. Problem (1),(2) will be denoted by NSTD IVP.

Definition 1: By a solution of NSTD IVP (1),(2) we mean a continuously differentiable function $y \in C(J, R^n)$, where J is some subinterval of I containing t_0 , such that

(i) $y(t_0) = y_0$,

(ii) $(y(t), y'(t)) \in D \times S$ for all $t \in J$

and

(iii) $y'(t) = f(t, y(t), y'(t))$ holds for all $t \in J$.

We shall denote a solution of problem (1),(2) by $y(t, t_0, y_0)$ instead of $y(t)$.

Also, consider the (not necessarily linear) functional equation

$$z(\cdot) = f(\cdot, y_0 + \int_{t_0}^{\cdot} z(s) ds, z(\cdot)). \quad (3)$$

Definition 2: By a solution of equation (3) we mean a function $z \in C(J, R^n)$, where J is some subinterval of I containing t_0 , such that

$$(ii') \quad (y_0 + \int_{t_0}^t z(s) ds, z(t)) \in D \times S \text{ for all } t \in J$$

and

$$(iii') \quad z(t) = f(t, y_0 + \int_{t_0}^t z(s) ds, z(t)) \text{ holds for all } t \in J.$$

The following result can be easily verified (see [7]):

Result 1: IVP (1), (2) has a [unique] solution if equation (3) has a [unique] solution.

Definition 3: By a solution of system (1) we mean a continuously differentiable function $y \in C(J, R^n)$, where J is some subinterval of I , such that conditions (ii) and (iii) of Definition 1 are true for $t \in J$.

Definition 4: Let $\phi(t)$ be a solution of system (1) existing on the interval I . We say that $\phi(t)$ is a stable solution of system (1) if for every $\epsilon > 0$ and $t_0 \in I$ there exists a $\delta(t_0, \epsilon) > 0$ such that whenever $|y_0 - \phi(t_0)| < \delta$, where $y_0 \in D^0$, then the solution $y(t, t_0, y_0)$ exists for all $t \geq t_0$ and $|y(t, t_0, y_0) - \phi(t)| < \epsilon$ for $t \geq t_0$.

Let $y(t)$ be an arbitrary solution of (1) existing on the interval $[0, +\infty)$. Substituting the transformation

$$y = w + \phi$$

into (1) we get

$$w' = g(t, w, w') \quad (4)$$

$$\text{where } g(t, w, w') = f(t, w + \phi, w' + \phi') - f(t, \phi, \phi'). \quad (5)$$

We note that $g(t, 0, 0) = 0$, and that the stability of $\phi(t)$ is equivalent to the stability of the zero solution ($w(t) \equiv 0$) of system (4). For this reason, from now on we take $D = \{y \in R^n \mid |y| \leq b\}$, where $b > 0$ is a constant, little later we assume that $f(t, 0, 0) = 0$ and we study the stability of the zero solution of (1).

Let $t_0 \in I$ and $y_0 \in D^0$ (i.e., $|y_0| < b$). Assume that the function f satisfies the following

conditions:

(I) $f(t, y, z)$ is continuous with respect to $(t, y, z) \in I \times D \times S$,

(II) $f(t, y, z) \in S$ for all $(t, y, z) \in I \times D \times S$

and

(III) $|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq k_1 |y_1 - y_2| + k_2 |z_1 - z_2|$ for all $(t, y_1, z_1), (t, y_2, z_2) \in I \times D \times S$, where $k_1 \geq 0$ and $0 \leq k_2 < 1$ are constants.

Then we have the following existence result:

Result 2: If conditions (I)–(III) are satisfied then IVP (1), (2) has a unique solution existing on the interval $J = [t_0 - \alpha, t_0 + \alpha] \cap I$, where α is any real number such that

$$0 < \alpha < \min((1 - k_2)/k_1, (b - |y_0|)/c). \quad (6)$$

Proof. Consider the Banach space $C(J, R^n)$ equipped with the supremum norm $\|\cdot\|$ given by

$$\|z\| = \sup_{t \in J} |z(t)|, z \in C(J, R^n).$$

Let M be the closed subset of $C(J, R^n)$ defined by

$$M = \{z \in C(J, R^n) \mid \|z\| \leq c\}.$$

Denote by F the map on M defined by

$$(Fz)(t) = f(t, y_0 + \int_{t_0}^t z(s) ds, z(t)), z \in M \text{ and } t \in J.$$

Using conditions (I)–(III), it is easy to see that F is a contraction on M and by the contraction mapping theorem [6], F has a unique fixed point in M . Hence by Result 1, IVP (1), (2) has a unique solution existing on the interval J and the proof is complete.

The following corollary is an immediate consequence of Result 2:

Corollary 1. Let $0 < \epsilon < b$ be a constant and let conditions (I)–(III) be satisfied. Then for every y_0 with $|y_0| \leq \epsilon$, IVP (1), (2) has a unique solution existing on the interval $J = [t_0 - \alpha, t_0 + \alpha] \cap I$, where α is any real number, such that

$$0 < \alpha < \min((1 - k_2)/k_1, (b - \epsilon)/c). \quad (7)$$

In other words, every IVP (1), (2) with $|y_0| \leq \epsilon$ has a unique solution existing on the common

interval J .

The next result follows from condition (III) and the Gronwall's inequality [3], and can be easily established.

Result 3: Let $t_0 \in I$, and let $y_0, \tilde{y}_0 \in D^0$. Suppose that conditions (I) – (III) are true. If the solution $y(t, t_0, y_0)$ and $\tilde{y}(t, t_0, \tilde{y}_0)$ exist on a common interval J_1 then

$$|y(t, t_0, y_0) - \tilde{y}(t, t_0, \tilde{y}_0)| \leq |y_0 - \tilde{y}_0| e^{\frac{k_1}{1-k_2}|t-t_0|}, \tag{8}$$

for all $t \in J_1$.

Now, suppose that f satisfies the following condition:

(IV) f has all continuous partial derivatives upto order p with respect to $(t, y, z) \in I \times D \times S$, where $p > 1$ is an integer, and $|\frac{\partial f}{\partial y_j}| \leq \tilde{k}_1$, $|\frac{\partial f}{\partial z_j}| \leq \tilde{k}_2$ ($j = 1, 2, \dots, n$), where $\tilde{k}_1 \geq 0$ and $0 \leq \tilde{k}_2 < 1$ are constants.

Then, it is easy to verify that condition (IV) implies conditions (I) and (III), and hence we have the following result:

Result 4: Suppose that conditions (II) and (IV) are satisfied. Then IVP (1),(2) has a unique solution y existing on the interval $J = [t_0 - \alpha, t_0 + \alpha] \cap I$, where α is given by (6) with $k_i = \tilde{k}_i$ ($i = 1, 2$). Moreover, y is $p + 1$ times continuously differentiable on J and

$$y'' = (I - (\frac{\partial f_i}{\partial z_j}))^{-1} (\frac{\partial f}{\partial t} + (\frac{\partial f_i}{\partial y_j})f), \tag{9}$$

where I is $n \times n$ identity matrix

$$\left(\frac{\partial f_i}{\partial z_j}\right) = \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \dots & \frac{\partial f_1}{\partial z_n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial f_n}{\partial z_1} & \dots & \frac{\partial f_n}{\partial z_n} \end{bmatrix}$$

and $\left(\frac{\partial f_i}{\partial y_j}\right)$ is defined similarly. Relation (9) follows from the quotient

$$\frac{y'(t+h) - y'(t)}{h} = \frac{f(t+h, y(t+h), y'(t+h)) - f(t, y(t), y'(t))}{h}$$

by applying the mean value theorem for vector valued functions on the right hand side function, by making use of condition (IV) and, finally, by allowing h tend to zero.

2. STABILITY THEOREMS

Throughout this section, we assume that the function f satisfies conditions (II) and (IV) and that $f(t, 0, 0) = 0$ for all $t \in I$.

Consider the system

$$y' = f(t, y, y'). \quad (10)$$

Definition 5: We say that $y(t) \equiv 0$ is a stable solution of (10) if for every $\epsilon > 0$ and any $t_0 \in I$ there exists a $\delta(\epsilon, t_0) > 0$ such that whenever $|y_0| < \delta$ then the solution $y(t, t_0, y_0)$ exists for all $t \geq t_0$ and $|y(t, t_0, y_0)| < \epsilon$ for all $t \geq t_0$.

Definition 6: We say that $y(t) \equiv 0$ is a uniformly stable solution of (10) if δ in Definition 5 is independent of t_0 .

Definition 7: We say that $y(t) \equiv 0$ is an asymptotically stable solution of (10) if

(i) $y(t) \equiv 0$ is stable

and

(ii) there exists a $\delta_0(t_0) > 0$ such that whenever $|y_0| < \delta_0$ then the solution $y(t, t_0, y_0)$ tends to zero as $t \rightarrow \infty$.

Definition 8: We say that $y(t) \equiv 0$ is a uniformly asymptotically stable solution of (10) if

(i) $y(t) \equiv 0$ is uniformly stable

and

(ii) there exists a $\delta_0 > 0$ and for every $\epsilon > 0$ there exists a $T(\epsilon) > 0$ such that whenever $|y_0| < \delta_0$ then $|y(t, t_0, y_0)| < \epsilon$ for all $t \geq t_0 + T(\epsilon)$.

Note 1: We note that uniform [asymptotic] stability always implies [asymptotic] stability of the zero solution of (10). Below, we shall see that the converse is also true if f is either periodic in t or autonomous.

Theorem 1: Let $f(t, y, z)$ be periodic in t of period w . If the zero solution of (10) is [asymptotically] stable then it is uniformly [asymptotically] stable.

The theorem can be proved easily, using Result 3, along the same lines as of Theorems 7.3 and 7.4. [8], and hence the proof will be omitted here.

Definition 9: We call a real valued function $V(t, y, z)$, defined on $I \times D \times S$, a Lyapunov function, if:

(i) $V(t, y, z)$ is continuous with respect to $(t, y, z) \in I \times D \times S$

and

(ii) $V(t, y, z)$ is continuously differentiable with respect to $(t, y, z) \in I \times D^0 \times S^0$.

Definition 10: We define the derivative of $V(t, y, z)$ with respect to system (10) by

$$V^*(t, y, z) = \frac{\partial V}{\partial t} + \sum_{j=1}^n \left(\frac{\partial V}{\partial y_j} \right) f_j + \sum_{j=1}^n \frac{\partial V}{\partial z_j} \left(\left(I - \left(\frac{\partial f_i}{\partial z_j} \right) \right)^{-1} \left(\frac{\partial f}{\partial t} + \left(\frac{\partial f_i}{\partial y_j} \right) f \right) \right)_j. \quad (11)$$

Note 2: Along a solution y of system (10), we have, by (9),

$$\frac{dV}{dt}(t, y(t), y'(t)) = V^*(t, y(t), y'(t)).$$

Definition 11: A real valued function $a(r)$ defined on the interval $[0, b]$ is called positive definite if $a(0) = 0$ and $a(r) > 0$ for $r \in (0, b]$.

Now, we shall present few theorems for [uniform] stability and [uniform] asymptotic stability of the zero solution of system (10).

Theorem 2 (Stability of the zero solution): Suppose that there exists a Lyapunov function $V(t, y, z)$ defined on $I \times D \times S$ satisfying the following conditions:

(i) $V(t, 0, 0) = 0$ for all $t \in I$,

(ii) there exists a continuous positive definite function $a(r)$ defined on $[0, b]$ satisfying the inequality $a(|y|) \leq V(t, y, z)$ for all $(t, y, z) \in I \times D \times S$ such that $z = f(t, y, z)$

and

(iii) $V^*(t, y, z) \leq 0$ for all $(t, y, z) \in I \times D^0 \times S^0$ such that $z = f(t, y, z)$.

Then the zero solution of system (10) is stable.

We note that for every $(t, y) \in I \times D$, in view of conditions (II) and (IV), there exists a unique $z \in S$ such that $z = f(t, y, z)$.

Proof: Let ϵ be such that $0 < \epsilon < b$ and $(\tilde{k}_1/(1-\tilde{k}_2))\epsilon < c$. From condition (i) and from the continuity of V , it follows that for every $t_0 \in I$ there exists a number $\delta_0(t_0, \epsilon) > 0$ such that if $|y_0| < \delta_0 (< \epsilon)$ and $|z_0| < (\tilde{k}_1/(1-\tilde{k}_2))\delta_0$, then

$$V(t_0, y_0, z_0) < a(\epsilon). \quad (12)$$

By Corollary 1, the solution $y(t, t_0, y_0)$ exists initially on the interval $[t_0, t_0 + \alpha]$, where α is given by (7). We have, by condition (10), that

$$|y'(t_0)| \leq \left(\frac{\tilde{k}_1}{1-\tilde{k}_2} \right) |y_0| < \left(\frac{\tilde{k}_1}{1-\tilde{k}_2} \right) \delta_0.$$

Hence, from relation (12), we get that

$$V(t_0, y(t_0), y'(t_0)) < a(\epsilon).$$

Claim: $|y(t, t_0, y_0)| < \epsilon$ for all $t \in [t_0, t_0 + \alpha]$.

For, if possible, let there exist a $t_1 \in (t_0, t_0 + \alpha]$ such that $|y(t, t_0, y_0)| = \epsilon$. From condition (iii) and note 2, we get that

$$\frac{dV}{dt}(t, y(t, t_0, y_0), y'(t, t_0, y_0)) \leq 0 \text{ for all } t \in [t_0, t_1],$$

which upon integrating between t_0 and t , gives that

$$V(t, y(t, t_0, y_0), y'(t, t_0, y_0)) \leq V(t_0, y(t_0), y'(t_0)) < a(\epsilon), \quad (13)$$

for all $t \in [t_0, t_1]$. Then from condition (ii) and inequality (13), we get that

$$\begin{aligned} a(\epsilon) &\leq V(t_1, y(t_1, t_0, y_0), y'(t_1, t_0, y_0)) \\ &\leq V(t_0, y_0, y'(t_0)) < a(\epsilon), \end{aligned}$$

which is a contradiction. Hence our claim is true.

Next, let us denote the solution $y(t, t_0, y_0)$ by $y_1(t)$. Since $|y_1(t_0 + \alpha)| < \epsilon$, again by Corollary 1, the solution $y(t, t_0 + \alpha, y_1(t_0 + \alpha))$ of the following initial value problem

$$y' = f(t, y, y'), y(t_0 + \alpha) = y_1(t_0 + \alpha)$$

exists on the interval $[t_0, t_0 + 2\alpha]$. If we denote $y(t, t_0 + \alpha, y_1(t_0 + \alpha))$ by $y_2(t)$, we have, by relation (8), that

$$y_2(t) = y_1(t) \text{ for all } t \in [t_0, t_0 + \alpha],$$

and it can be shown as in the above that

$$|y_2(t)| < \epsilon \text{ for all } t \in [t_0, t_0 + 2\alpha].$$

Again, since $|y_2(t_0 + 2\alpha)| < \epsilon$, by Corollary 1, the solution $y_3(t)$ of the initial value problem

$$y' = f(t, y, y'), y(t_0 + 2\alpha) = y_2(t_0 + 2\alpha)$$

exists on the interval $[t_0 + \alpha, t_0 + 3\alpha]$, and we have that

$$y_3(t) = y_2(t) \text{ for } t \in [t_0 + \alpha, t_0 + 2\alpha]$$

and

$$|y_3(t)| < \epsilon \text{ for all } t \in [t_0 + \alpha, t_0 + 3\alpha].$$

Proceeding in this way, indefinitely, we get a sequence of solutions $\{y_n(t)\}$ and if we define

$$y(t) = \left\{ \begin{array}{l} y_1(t), t \in [t_0, t_0 + \alpha], \\ y_2(t), t \in [t_0 + \alpha, t_0 + 2\alpha], \\ \cdot \\ \cdot \\ y_n(t), t \in [t_0 + (n-1)\alpha, t_0 + n\alpha], \\ \cdot \\ \cdot \end{array} \right.$$

then clearly $y(t)$ is a unique solution of system (10) with $y(t_0) = y_0$. Moreover, $y(t)$ exists on the entire interval $[t_0, +\infty)$ and $|y(t)| < \epsilon$ for all $t \geq t_0$. This shows that the zero solution of (10) is stable and the proof is complete.

Theorem 3 (Uniform stability of the zero solution): Let condition (ii) in Theorem 2 be replaced by the condition:

(ii') $a(|y|) \leq V(t, y, z) \leq b(|y|)$ for all $(t, y, z) \in I \times D \times S$ such that $z = f(t, y, z)$, where $a(r)$ and $b(r)$ are continuous, positive definite functions defined on $[0, b]$.

Then the zero solution of system (10) is uniformly stable.

Proof: Let ϵ be such that $0 < \epsilon < b$ and $(\tilde{k}_1/(1 - \tilde{k}_2))\epsilon < c$. Choose a $\delta(\epsilon) > 0$ such that $b(\delta) < a(\epsilon)$ and $\delta < \epsilon$. Now, let $t_0 \in I$ and $|y_0| < \delta$. By Corollary 1, the solution $y(t, t_0, y_0)$ exists initially on the interval $[t_0, t_0 + \alpha]$, where α is given by (7).

Claim: $|y(t, t_0, y_0)| < \epsilon$ for all $t \in [t_0, t_0 + \alpha]$.

For, if possible, let there exist a point $t_2 \in (t_0, t_0 + \alpha]$ such that $|y(t_2, t_0, y_0)| = \epsilon$. Since $|y_0| < \delta < \epsilon$, there exists a point $t_1 \in (t_0, t_2)$ such that $y(t_1, t_0, y_0) = \delta$. Now, from condition (iii) and note 2, we get

$$\frac{dV}{dt}(t, y(t, t_0, y_0), y'(t, t_0, y_0)) \leq 0 \text{ for all } t \in [t_0, t_2],$$

which upon integrating between t_1 and t_2 , gives that

$$V(t_2, y(t_2, t_0, y_0), y'(t_2, t_0, y_0)) \leq V(t_1, y(t_1, t_0, y_0), y'(t_1, t_0, y_0)). \quad (14)$$

Then from condition (ii') and inequality (14), we get that

$$\begin{aligned} b(\delta) &< a(\epsilon) \leq V(t_2, y(t_2, t_0, y_0), y'(t_2, t_0, y_0)) \\ &\leq V(t_1, y(t_1, t_0, y_0), y'(t_1, t_0, y_0)) \leq b(\delta), \end{aligned}$$

which is a contradiction. Hence our claim is true.

Rest of the proof follows as in Theorem 2 and hence will be omitted.

Theorem 4 (Asymptotic stability of the zero solution): Suppose that there exists a Lyapunov function $V(t, y, z)$ defined on $I \times D \times S$ satisfying the following properties:

- (i) $V(t, 0, 0) = 0$ for all $t \in I$,
- (ii) $a(|y|) \leq V(t, y, z)$ for all $(t, y, z) \in I \times D \times S$ such that $z = f(t, y, z)$, where $a(r)$ is a continuous, positive definite function defined on $[0, b]$

and

- (iii) $V^*(t, y, z) \leq -c(|y|)$ for all $(t, y, z) \in I \times D^0 \times S^0$ such that $z = f(t, y, z)$, where $c(r)$ is a continuous, positive definite function defined on $[0, b]$.

Then the zero solution of (10) is asymptotically stable.

Stability of the zero solution of (10) follows from Theorem 2, and proof of existence of a $\delta_0(t_0) > 0$ such that $|y_0| < \delta_0$ implies that the solution $y(t, t_0, y_0) \rightarrow 0$ as $t \rightarrow \infty$ can be given along the same lines as that of Theorem 8.5 [8] and hence will be omitted.

Theorem 5 (Uniform asymptotic stability of the zero solution): Suppose that there exists a Lyapunov function $V(t, y, z)$ defined on $I \times D \times S$ satisfying the following conditions:

- (i) $a(|y|) \leq V(t, y, z) \leq b(|y|)$ for all $(t, y, z) \in I \times D \times S$ such that $z = f(t, y, z)$, where $a(r)$ and $b(r)$ are continuous, positive definite functions defined on $[0, b]$

and

- (ii) $V^*(t, y, z) \leq -c(|y|)$ for all $(t, y, z) \in I \times D^0 \times S^0$ such that $z = f(t, y, z)$, where $c(r)$ is a

continuous, positive definite function defined on $[0, b]$.

Then the zero solution of (10) is uniformly asymptotically stable.

Proof: By Theorem 3, the zero solution of (10) is uniformly stable. Choose $\eta > 0$ such that $\eta < b$ and $(\tilde{k}_1/(1-\tilde{k}_2))\eta < c$. Then there exists a $\delta_0 > 0$ such that if $t_0 \in I$ and $|y_0| < \delta_0$, then $|y(t, t_0, y_0)| < \eta$ for all $t \geq t_0$. Also for a given $\epsilon > 0$ with $\epsilon < \eta$, there exists a $\delta(\epsilon) > 0$ such that if $t_0 \in I$ and $|y_0| < \delta$, then $|y(t, t_0, y_0)| < \epsilon$ for all $t \geq t_0$.

Let

$$\begin{aligned} M_1 &= \inf\{a(r) \mid \delta/2 \leq r \leq b\}, \\ M_2 &> \sup\{b(r) \mid \delta/2 \leq r \leq b\}, \end{aligned}$$

and

$$M_3 = \inf\{c(r) \mid \delta/2 \leq r \leq b\}.$$

Clearly $M_i > 0, (i = 1, 2, 3)$ and, from condition (ii), we have that $M_1 < M_2$. Now, let $T = (M_2 - M_1)/M_3$ and note that $T > 0$ and depends only on ϵ .

Claim: For every solution $y(t, t_0, y_0)$ of (10) such that $t_0 \in I$ and $|y_0| < \delta_0$, there exists at least one point $t_1 \in [t_0, t_0 + T]$ such that $|y(t_1, t_0, y_0)| < \delta$.

For, if possible, let there exist a solution $y(t, t_0, y_0)$ of (10), for some $t_0 \in I$ and some y_0 with $|y_0| < \delta_0$, such that $|y(t, t_0, y_0)| \geq \delta$ for all $t \in [t_0, t_0 + T]$. Then, there exists a $\tilde{t} > t_0 + T$ such that $|y(t, t_0, y_0)| > \delta/2$ for all $t \in [t_0, \tilde{t}]$. Now, from conditions (i) and (ii) and note 2, we get that

$$\begin{aligned} M_1 &\leq a(|y(t, t_0, y_0)|) \leq V(t, y(t, t_0, y_0), y'(t, t_0, y_0)) \\ &\leq V(t_0, y_0, y'(t_0)) - M_3(t - t_0) \\ &\leq b(|y_0|) - M_3(t - t_0) \\ &\leq M_2 - M_3(t - t_0) \end{aligned}$$

which implies that

$$t - t_0 \leq (M_2 - M_1)/M_3 = T \text{ for all } t \in [t_0, \tilde{t}],$$

which is certainly a contradiction. Hence our claim is true.

Thus, we have that if $t_0 \in I$ and $|y_0| < \delta_0$, then $|y(t_1, t_0, y_0)| < \delta$ for some $t_1 \in [t_0, t_0 + T]$ and consequently $|y(t, t_0, y_0)| < \epsilon$ for all $t \geq t_0 + T$. The proof is complete.

The following corollary follows immediately from Theorem 5.

Corollary 2: *Let condition (ii) in Theorem 5 be replaced by*

(ii') $V^*(t, y, z) \leq -cV(t, y, z)$ for all $(t, y, z) \in I \times D^0 \times S^0$ such that $z = f(t, y, z)$, where $c > 0$ is a constant.

Then the zero solution of (10) is uniformly asymptotically stable.

3. EXAMPLES

1. In the study of free oscillations of positively damped systems, we encounter the following differential equation [1]:

$$u'' + w^2u + \epsilon(u')^3(u'')^2 = 0, \quad (15)$$

where w is a nonzero constant and $0 < \epsilon < 1$.

Equation (15) is equivalent to the system

$$y' = f(t, y, y') \quad (16)$$

where

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ -(w^2y_1 + \epsilon z_1^3 z_2^2) \end{pmatrix}$$

and

$$y_1 = u.$$

Choose the constants $b > 0$ and $c > 0$ such that

$$3\epsilon < 1/c^4 \text{ and } b < c(1 - \epsilon c^4)/(1 + w^2). \quad (17)$$

Let $D = \{y \in R^2 \mid |y| \leq b\}$ and $S = \{z \in R^2 \mid |z| \leq c\}$. We can easily verify that $f(t, y, z)$ is defined on $I \times D \times S$ and satisfies the following conditions:

- A) $f(t, 0, 0) = 0$,
- B) $f(t, y, z) \in S$ for $(t, y, z) \in I \times D \times S$,
- C) $f(t, y, z)$ is continuously differentiable with respect to $(t, y, z) \in I \times D \times S$, and

$$\left| \frac{\partial f}{\partial y_1} \right| \leq w^2, \quad \left| \frac{\partial f}{\partial y_2} \right| \leq 1, \quad \left| \frac{\partial f}{\partial z_1} \right| \leq 3\epsilon c^4 \text{ and } \left| \frac{\partial f}{\partial z_2} \right| \leq 2\epsilon c^4.$$

Thus f satisfies conditions (II) and (IV) on $I \times D \times S$ with $\tilde{k}_1 = \max(1, w^2)$ and $\tilde{k}_2 = 3\epsilon c^4$.

Now, consider the function $V(t, y, z)$ defined on $I \times D \times S$ by $V(t, y, z) = w^2 y^2_1 + y^2_2$. Clearly, V is continuously differentiable on $I \times D \times S$ and satisfies the following conditions:

- (i) $V(t, 0, 0) = 0$,
- (ii) $a(|y|) \leq V(t, y, z) \leq b(|y|)$ for all $(t, y, z) \in I \times D \times S$, where $a(r) = \alpha r^2$, $b(r) = \alpha r^2$, $\alpha = \min(1, w^2)$, $\beta = \max(1, w^2)$

and

$$\begin{aligned}
 \text{(iii)} \quad V^*(t, y, z) &= \frac{\partial V}{\partial y_1} f_1 + \frac{\partial V}{\partial y_2} f_2 \\
 &= 2y_1 y_2 w^2 - 2y_2 (w^2 y_1 + \epsilon z^3_1 z^2_2) \\
 &= -2\epsilon y_2 z^3_1 z^2_2 = -2\epsilon z^4_1 z^2_2 \leq 0
 \end{aligned}$$

for $(t, y, z) \in I \times D \times S$ such that $z = f(t, y, z)$.

Hence, by Theorem 3, the zero solution of (16) and hence of (15) is uniformly stable.

2. Let

$$g(y) = \begin{cases} \sin y, & -\pi/2 \leq y \leq \pi/2 \\ 1, & \pi/2 < y \\ -1, & y < -\pi/2. \end{cases}$$

Clearly, g is a continuously differentiable function on R , $g(0) = 0$, $yg(y) > 0$ for $y \neq 0$ and $|g(y)| \leq 1$.

Consider the equation

$$y' = -\alpha g(y)(\cos y')^2, \tag{18}$$

where $0 < \alpha \leq 1/2$. Let

$$D = \{y \in R \mid |y| \leq \pi\} \text{ and } S = \{z \in R \mid |z| \leq 1\}.$$

It is easy to verify that the function $f(t, y, z) = -\alpha g(y)(\cos z)^2$ satisfies conditions (II) and (IV) on $I \times D \times S$ and $f(t, 0, 0) = 0$ for all $t \in I$. Now, consider $V(t, y, z)$ defined on $I \times D \times S$ by $V(t, y, z) = y^2$. Clearly, V is continuously differentiable on $I \times D \times S$ and satisfies the following conditions:

- (i) $a(|y|) \leq V(t, y, z) \leq b(|y|)$ for all $(t, y, z) \in I \times D \times S$, where $a(r) = b(r) = r^2$,
- (ii) $V(t, 0, 0) = 0$ for all $t \in I$
- and
- (iii) $v^*(t, y, z) = \frac{\partial V}{\partial y} f = -2\alpha y g(y)(\cos z)^2 \leq -c(|y|)$, for all $(t, y, z) \in I \times D \times S$, where $c(r) = 2\alpha r g(r)(\cos 1)^2$.

Hence, by Theorem 5, the zero solution of (18) is uniformly asymptotically stable.

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