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ON A COUNTEREXAMPLE OF GAROFALO - LIN FOR A UNIQUE CONTINUATION OF SCHRÖDINGER EQUATION*

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ABSTRACT

Garofalo and Lin have given a counterexample for which unique continuation fails for the Schrödinger equation

$$-\Delta u + \frac{c}{|x|^{2+\varepsilon}} u = 0, \ \varepsilon > 0.$$

Their counterexample consists of a Bessel function of the third kind $K_v(|x|)$ with the restriction that v cannot be an integer. In this note we have removed the restriction.

Key Words: Schrödinger Equation, Unique Continuation, Bessel Function of the Third Kind, Inverse Square Potential

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Consider the following Schrödinger equation:

$$-\Delta u + V u = 0. \tag{1}$$

Garofalo and Lin [1] have shown that $V = c/|x|^2$, the inverse square potential, is optimal for unique continuation of (1). Indeed they have given a counterexample for which unique continuation fails once the square inverse potential is replaced by $c/|x|^{2+\varepsilon}$ for any $\varepsilon > 0$ [1: pp. 265-266]. Their counterexample is given by

$$u(x) = |x|^{-(n-2)/2} K_{(n-2)/\varepsilon}\left(\frac{2\sqrt{c}}{\varepsilon} |x|^{-\varepsilon/2}\right)$$

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with the restriction that $(n-2)/\varepsilon$ cannot be an integer for any $\varepsilon > 0$

In this note we show that this restriction is unnecessary. We use the same notations as in [1].

Theorem: *For the following Schrödinger equation,*

$$-\Delta u + \frac{c}{|x|^{2+\varepsilon}} u = 0 \text{ in } B_1, \tag{2}$$

where B_1 *is a unit ball in* R^n, *we have a radial solution given by*

$$u(x) = |x|^{-(n-2)/2} K_{(n-2)/\varepsilon}\left(- \frac{2\sqrt{c}}{\varepsilon} |x|^{-\varepsilon/2} \right), \tag{3}$$

where $K_{(n-2)/\varepsilon}$ *is a Bessel function of the third kind for any real number* $(n-2)/\varepsilon$ *and* $\varepsilon > 0$.

Proof. Since $\Delta_x u = u_{rr} + \frac{n-1}{r} u_r + \frac{1}{r^2} \Delta_\theta u$, a radial solution of (2) must satisfy

$$r^2 u''(r) + (n-1)\ r u'(r) - c\, r^{-\varepsilon}\ u(r) = 0,\ 0 < r < 1. \tag{4}$$

It is known that $I_v(z)$ and $I_{-v}(z)$, Bessel functions of imaginary argument, satisfy the differential equation [3: p. 77]

$$z^2 \frac{d^2 u}{dz^2} + z \frac{du}{dz} - (z^2 + v^2)\ u = 0. \tag{5}$$

Notice from (4) and (5) that $I_v(z)$ and $I_{-v}(z)$ (when v is an integer, see equation (9) for the definition) cannot be a solution of (4) because of the coefficient $n-1$ of $u'(r)$ in (4). Thus we look for a solution of a form $z^\alpha I_{\pm v}(\beta z^y)$, where α, β, and γ are constants to be determined. An easy computation reveals that $z^\alpha I_{\pm v}(\beta z^y))$ satisfies:

$$\begin{aligned} z^2 \frac{d^2}{dz^2} (z^\alpha I_{\pm v}(\beta z^y)) + (1 - 2\alpha)\, z \frac{d}{dz} (z^\alpha I_{\pm v}(\beta z^y)) \\ - (\gamma^2 \beta^2 z^{2y} + \gamma^2 v^2 - \alpha^2)\, (z^\alpha I_{\pm v}(\beta z^y)) = 0. \end{aligned} \tag{6}$$

Comparison of (6) with (4) shows that

(7) $$\alpha = -(n-2)/2,\ \ \beta = -2\sqrt{c}/\varepsilon,\ \ \gamma = -\varepsilon/2,\ \text{ and }\ \nu = (n-2)/\varepsilon.$$

Define the third kind of Bessel function $K_\nu(z)$ according to Watson [3: p. 78] by

(8) $$K_\nu(z) = \frac{\pi}{2}\,\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi}\ ,\quad \nu \neq \text{integer}$$

(9) $$K_n(z) = \lim_{\nu \to n} K_\nu(z), \qquad \nu = \text{integer}.$$

Then $K_\nu(z)$ is defined for all real values of ν. Conjunction of equations (4)-(9) yields the solution (3) to Schrödinger equation (2). This completes the proof.

Remark.

Since $K_\nu(r) \sim r^{-1/2} e^{-r}$ for all ν as $r \to \infty$ [3: p.202], $K_\nu(r)$ vanishes of infinite order as $r \to \infty$. Consequently unique continuation of Schrödinger equation (2) fails for any $\varepsilon > 0$, which implies that $V = c/|x|^2$, the inverse square potential, is optimal for unique continuation of solutions of Schrödinger equation (1).

References

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[3] Watson, G.N., *A Treatise on the Theory of Bessel Functions*, Cambridge Univ. Press (1966).