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EXISTENCE OF A SOLUTION OF A FOURIER NONLOCAL QUASILINEAR PARABOLIC PROBLEM[1]

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ABSTRACT

The aim of this paper is to give a theorem about the existence of a classical solution of a Fourier third nonlocal quasilinear parabolic problem. To prove this theorem, Schauder's theorem is used. The paper is a continuation of papers [1]-[8] and the generalizations of some results from [9]-[11]. The theorem established in this paper can be applied to describe some phenomena in the theories of diffusion and heat conduction with better effects than the analogous classical theorem about the existence of a solution of the Fourier third quasilinear parabolic problem.

Key words: Nonlocal quasilinear parabolic problem, integral equations, potential theory, existence of a solution, Schauder's theorem.

AMS (MOS) subject classifications: 35K20, 35K55, 45D05, 31B35, 47H10, 35K99.

1. INTRODUCTION

In paper [7], the author studied the uniqueness of solutions of parabolic semilinear nonlocal-boundary problems in the cylindrical domain. The coefficients of the nonlocal conditions had values belonging to the interval [-1,1] and, therefore, the problems considered were more general than the analogous parabolic initial-boundary and periodic-boundary problems. In this paper we study in the cylindrical domain, the existence of a classical solution of a Fourier third nonlocal quasilinear parabolic problem, which possesses tangent derivatives in the boundary condition. The coefficients of the nonlocal condition from this paper can belong not only to the interval [-1,1] but also to intervals containing the interval [-1,1]. Therefore, a larger class of physical phenomena can be described by the results of this paper

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than by the results of paper [7]. Moreover, this fundamental theorem of the paper, about the existence of the solution of the nonlocal problem, can be applied in the theories of diffusion and heat conduction with better effects than the analogous classical theorem. To prove this fundamental theorem, Schauder's theorem is used.

This paper is a continuation not only of paper [7] but also of papers [1]-[6] and [8]. The main result of the paper is the generalization of the Pogorzelski's result (see [11], Section 22.11), and generalizations of Chabrowski (see [9]) and Friedman (see [10], Section 7.4) results.

2. PRELIMINARIES

The notation, assumptions and definitions from this section are valid throughout this paper.

Let n be any integer greater than 2. Given two points, $x=(x_1,\ldots,x_n)\in R^n$ and $y=(y_1,\ldots,y_n)\in R^n$, the symbol $|x-y|$ means the Euclidian distance between x and y. The Euclidian distance between two points P_1 and P_2 belonging to R^n is also denoted by $\rho(P_1,P_2)$.

To prove a theorem about the existence of a classical solution of a Fourier's third nonlocal quasilinear parabolic problem, some assumptions will be used.

Assumptions:

I. $D:=D_0\times(0,T)$, where $0<T<\infty$ and D_0 is an open and bounded domain in R^n such that the boundary ∂D_0 satisfies the following Lyapunov conditions:

i) For each point belonging to ∂D_0 there exists the tangent plane at this point.

ii) For each points P_1 and P_2 belonging to ∂D_0 the angle $\kappa(n_{P_1}, n_{P_2})$ between the normal lines n_{P_1} and n_{P_2} to ∂D_0 at points P_1 and P_2 satisfies the inequality

$$|\kappa(n_{P_1},n_{P_2})| \le const.[\rho(P_1,P_2)]^{h_L},$$

where h_L is a constant satisfying the inequalities $0<h_L\le 1$.

iii) There exists $\delta>0$ such that for every point P belonging to ∂D_0, each line ℓ parallel to the normal line to ∂D_0 at point P has the property that $\partial D_0\cap K(P,\delta)\cap\ell$, ($K(P,\delta)$ is the ball of radius δ centered at point P) is equal at most to the one point.

II. For each point P belonging to ∂D_0 there exist q fields $\{t_P{}^{(1)}\},\ldots,\{t_P{}^{(q)}\}$ $(q\le n-1)$ of the tangent directions to ∂D_0 at P such that the following inequalities

$$|\kappa(t_{P_1}{}^{(i)},t_{P_2}{}^{(i)})| \le const.[\rho(P_1,P_2)]^{h_t} \quad (i=1,2,\ldots,q)$$

are satisfied, where $\kappa(t_{P_1}{}^{(i)}, t_{P_2}{}^{(i)})$ $(i = 1,2,\ldots,q)$ denote the angles between $t_{P_1}{}^{(i)}$ and $t_{P_2}{}^{(i)}$ $(i = 1,2,\ldots,q)$, respectively, P_1 and P_2 are arbitrary points belonging to ∂D_0, and h_t is a constant satisfying the inequalities $0 < h_t \le 1$.

III. The real functions $a_{ij}(x,t)$, $b_i(x,t)$ $(i,j = 1,2,\ldots,n)$, $c(x,t)$ are defined for $(x,t) \in \overline{D}$ and satisfy the Hölder conditions:

$$| a_{ij}(x,t) - a_{ij}(\overline{x}, \overline{t}) | \le const.(| x - \overline{x} |^{h_1} + | t - \overline{t} |^{h_2}) \quad (i,j = 1,2,\ldots,n),$$

$$| b_i(x,t) - b_i(\overline{x}, t) | \le const. | x - \overline{x} |^{h_1} \quad (i = 1,2,\ldots,n),$$

$$| c(x,t) - c(\overline{x}, t) | \le const. | x - \overline{x} |^{h_1}$$

for all $(x,t) \in \overline{D}, (\overline{x}, \overline{t}) \in \overline{D}$, where h_1 and h_2 are constants satisfying the inequalities $0 < h_1 \le 1$, $0 < h_2 \le 1$. Moreover, b_i $(i = 1,2,\ldots,n)$ and c are continuous functions with respect to t belonging to $[0,T]$.

IV. The quadratic form $\sum_{i,j=1}^{n} a_{ij}(x,t)\lambda_i\lambda_j$ is positive defined for all $(x,t) \in \overline{D}$, i.e., $\sum_{i,j=1}^{n} a_{ij}(x,t)\lambda_i\lambda_j > 0$ for every point $(x,t) \in \overline{D}$ and for every real vector $(\lambda_1,\ldots,\lambda_n) \ne (0,\ldots,0)$.

V. The real function $F(x,t,z_0,z_1,\ldots,z_n)$ is defined and continuous for $(x,t) \in D_0 \times (0,T]$ and $z_i \in R$ $(i = 0,1,\ldots,n)$. Moreover, F satisfies the inequality

$$| F(x,t,z_0,\ldots,z_n) | \le M_F \sum_{i=0}^{n} | z_i | + \overline{M}_F | x - P_x |^{-p} t^{-\mu_F} \tag{2.1}$$

for $(x,t) \in D_0 \times (0,T]$, $z_i \in R$ $(i = 0,1,\ldots,n)$, where in this paper P_x denotes the point belonging to ∂D_0 such that $\rho(x, P_x)$ attains the minimum, and F satisfies the Hölder condition

$$| F(x,t,z_0,\ldots,z_n) - F(\overline{x}, t, \overline{z}_0,\ldots,\overline{z}_n) |$$

$$\le C(D_0^*) | x - \overline{x} |^{h_F} t^{-\mu_F} + C_F \sum_{i=0}^{n} | z_i - \overline{z}_i |^{\widetilde{h}_F} \tag{2.2}$$

for all (x,t), $(\overline{x}, t) \in D_0^* \times (0,T]$, z_i, $\overline{z}_i \in R$ $(i = 0,1,\ldots,n)$, where D_0^* is an arbitrary closed subdomain of D_0; M_F, $\overline{M}_F$, C_F, h_F, $\widetilde{h}_F$, μ_F, p are constants which do not depend on D_0^* and satisfy the inequalities

$$M_F, \overline{M}_F, C_F > 0, \; 0 < h_F \le 1, \; 0 < \widetilde{h}_F \le 1, \; 0 \le \mu_F < 1, \; 0 \le p < 1,$$

and $C(D_0^*)$ is a positive constant that depends on D_0^*.

VI. The real function $G(x,t,z_0,z_1,\ldots,z_q)$ is defined and continuous for all

$(x,t) \in \partial D_0 \times (0,T]$ and $z_i \in R$ $(i = 0,1,\ldots,q)$. Moreover, G satisfies the inequality

$$|G(x,t,z_0,\ldots,z_q)| \leq M_G \sum_{i=0}^{q} |z_i| + \overline{M}_G t^{-\mu_G} \tag{2.3}$$

for $(x,t) \in \partial D_0 \times (0,T]$, $z_i \in R$ $(i = 0,1,\ldots q)$ and the Hölder-Lipschitz condition

$$|G(x,t,z_0,\ldots,z_q) - G(\overline{x},t,\overline{z}_0,\ldots,\overline{z}_q)|$$

$$\leq C_G[|x-\overline{x}|^{h_G} t^{-\mu_G} + |z_0 - \overline{z}_0|^{\widetilde{h}_G} + \sum_{i=1}^{q} |z_i - \overline{z}_i|] \tag{2.4}$$

for (x,t), $(\overline{x},t) \in \partial D_0 \times (0,T], z_i, \overline{z}_i \in R$ $(i = 0,1,\ldots,q)$, where M_G, $\overline{M}_G$, C_G, h_G, $\widetilde{h}_G$ and μ_G are constants satisfying the inequalities

$$M_G,\ \overline{M}_G,\ C_G > 0,\ 0 < h_G \leq 1,\ 0 < \widetilde{h}_G \leq 1,\ 0 \leq \mu_G < 1.$$

VII. The real function $g(x,t)$ defined, continuous and bounded for $(x,t) \in \partial D_0 \times (0,T]$ satisfies the Hölder condition

$$|g(x,t) - g(\overline{x},t)| \leq C_g |x - \overline{x}|^{h_g}$$

for (x,t), $(\overline{x},t) \in \partial D_0 \times (0,T]$, where C_g and h_g are constants satisfying the inequalities $C_g > 0$ and $0 < h_g \leq 1$.

VIII. The real function $f(x)$ defined and integrable for $x \in D_0$ satisfies the inequality

$$|f(x)| < \frac{M_f}{|x - P_x|^p} \text{ for } x \in D_0, \tag{2.5}$$

where M_f is a positive constant and p is a constant from Assumption V. Moreover, the set $\hat{D}_0$ of x belonging to D_0 such that $f(x)$ is the continuous function is nonempty.

IX. $T_1, T_2, \ldots, T_k$ are arbitrary positive numbers satisfying the inequalities $0 < T_1 < T_2 < \ldots < T_k \leq T$; Z is the set of real functions $z(x,t)$ defined and continuous for $(x,t) \in \overline{D}_0 \times (0,T]$,

$$A(x,t) := \frac{\sqrt{det[a_{ij}(x,t)]}}{(2\sqrt{\P})^n} \text{ for } (x,t) \in \overline{D}, \tag{2.6}$$

and $K: D_0 \times (0,T)^{k-1} \times (0,T] \times R^k \rightarrow R$ is a function such that

$$\widetilde{K}(x,T_1,\ldots,T_k,z(x,T_1),\ldots,z(x,T_k)) := A(x,0)K(x,T_1,\ldots,T_k,z(x,T_1),\ldots,z(x,T_k)) \tag{2.7}$$

$$\text{for } x \in D_0,\ z \in Z$$

is integrable with respect to $x \in D_0$ for each $z \in Z$ and

$$\int_{D_0} \widetilde{K}(y, T_1, \ldots, T_k, \lambda z(y, T_1) + (1-\lambda)\overline{z}(y, T_1), \ldots, \lambda z(y, T_k) + (1-\lambda)\overline{z}(y, T_k))dy$$

$$= \lambda \int_{D_0} \widetilde{K}(y, T_1, \ldots, T_k, z(y, T_1), \ldots, z(y, T_k))dy$$

$$+ (1-\lambda) \int_{D_0} \widetilde{K}(y, T_1, \ldots, T_k, \overline{z}(y, T_1), \ldots, \overline{z}(y, T_k))dy \tag{2.8}$$

for all z, $\overline{z} \in Z$, $\lambda \in (0,1)$. Moreover,

$$| K(x, T_1, \ldots, T_k, z(x, T_1), \ldots, z(x, T_k)) | < \frac{M_K}{| x - P_x |^p} \sum_{i=1}^{k} | z(x, T_i) | \tag{2.9}$$

for $x \in D_0$, $z \in Z$ and

$$| K(x, T_1, \ldots, T_k, z(x, T_1), \ldots, z(x, T_k)) - K(x, T_1, \ldots, T_k, \overline{z}(x, T_1), \ldots, \overline{z}(x, T_k)) |$$

$$\leq C_K \sum_{i=1}^{k} | z(x, T_i) - \overline{z}(x, T_i) | \tag{2.10}$$

for $x \in D_0$, z, $\overline{z} \in Z$, where M_K and C_K are positive constants. Finally, the set of x belonging to D_0, such that for each z belonging to Z function

$$K(x, T_1, \ldots, T_k, z(x, T_1), \ldots, z(x, T_k))$$

is continuous, is equal to set $\hat{D}_0$ from Assumption VIII.

To find a solution of a Fourier's third nonlocal quasilinear parabolic problem considered in the paper, we shall use the space X of all the systems

$$(w_0(x,t), w_1(x,t), \ldots, w_n(x,t), \phi_0(\eta, t))$$

of real functions, defined and continuous for $(x,t) \in \overline{D}_0 \times (0,T], (\eta, t) \in \partial D_0 \times (0,T]$, respectively, and such that

$$\sup_{(x,t) \in \overline{D}_0 \times (0,T]} t^{\alpha+\beta} | w_i(x,t) | < \infty \quad (i = 0, 1, \ldots, n),$$

$$\sup_{(\eta,t) \in \partial D_0 \times (0,T]} t^{\alpha+\beta} | \phi_0(\eta, t) | < \infty,$$

where α and β are characteristic for space X arbitrary fixed constants chosen according to the conditions

$$\alpha > 0,\ max(\mu_F, \mu_G, \tfrac{1}{2}(1+p)) < \beta < 1, \alpha + \beta < 1. \tag{2.11}$$

For two arbitrary systems $(w_0, w_1, \ldots, w_n, \phi_0), (\widetilde{w}_0, \widetilde{w}_1, \ldots, \widetilde{w}_n, \widetilde{\phi}_0)$ belonging to X and for each λ belonging to R, the addition in X and the scalar multiplication are defined by the formulae

$$(w_0, w_1, \ldots, w_n, \phi_0) + (\widetilde{w}_0, \widetilde{w}_1, \ldots, \widetilde{w}_n, \widetilde{\phi}_0) := (w_0 + \widetilde{w}_0, w_1 + \widetilde{w}_1, \ldots, w_n + \widetilde{w}_n, \phi_0 + \widetilde{\phi}_0),$$

$$\lambda(w_0, w_1, \ldots, w_n, \phi_0) := (\lambda w_0, \lambda w_1, \ldots, \lambda w_n, \lambda \phi_0).$$

The norm of $W = (w_0, w_1, \ldots, w_n, \phi_0)$ belonging to X is defined by the formula

$$\| W \| = \max_{i=0,1,\ldots,n} \sup_{(x,t) \in \overline{D}_0 \times (0,T]} t^{\alpha+\beta} | w_i(x,t) | + \sup_{(\eta,t) \in \partial D_0 \times (0,T]} t^{\alpha+\beta} | \phi_0(\eta,t) | . \tag{2.12}$$

It is easy to see that X is the Banach space.

To find a solution of a Fourier's third nonlocal quasilinear parabolic problem considered in the paper, we shall also need a set E of all $W = (w_0, w_1, \ldots, w_n, \phi_0)$ belonging to X, satisfying the inequalities

$$t^{\beta} | w_i(x,t) | \le \rho_1 \text{ for } (x,t) \in \overline{D}_0 \times (0,T] \ (i = 0,1,\ldots,n), \tag{2.13}$$

$$| w_0(x, T_i) | \le N_i \text{ for } x \in \overline{D}_0 \ (i = 1,2,\ldots,k), \tag{2.14}$$

$$t^{\beta} | \phi_0(\eta,t) | \le \rho_2 \text{ for } (\eta,t) \in \partial D_0 \times (0,T], \tag{2.15}$$

$$t^{\beta} | \phi_0(\eta,t) - \phi_0(\widetilde{\eta},t) | \le \kappa | \eta - \widetilde{\eta} |^{\gamma} \text{ for } (\eta,t), (\widetilde{\eta},t) \in \partial D_0 \times (0,T] \tag{2.16}$$

and such that

$$\int_{D_0} \widetilde{K}(y, T_1, \ldots, T_K, w_0(y,T_1), \ldots, w_0(y,T_k)) \Gamma(x,t,y,0) dy = B(x,t) \quad \text{for } (x,t) \in \overline{D}_0 \times (0,T], \tag{2.17}$$

where ρ_1, ρ_2, κ are arbitrary fixed positive constants, N_i $(i = 1,2,\ldots,k)$ are positive constants such that

$$N_i \le T_i^{-\beta} \rho_1 \ (i = 1,2,\ldots,k), \tag{2.18}$$

γ is a fixed constant chosen according to the condition

$$0 < \gamma < min\{h_G, \widetilde{h}_G, h_g, h_1, 2h_2, h_L, h_t, 1-p\}, \tag{2.19}$$

and $B(x,t)$ is a given real function defined and continuous for $(x,t) \in \overline{D}_0 \times (0,T]$ and such that the derivatives $\frac{\partial B(x,t)}{\partial x_i}$ $(i = 1,2,\ldots,n)$ are continuous for $(x,t) \in \overline{D}_0 \times (0,T]$ $(i = 1,2,\ldots,n)$.

Formulae (2.12) - (2.18), (2.10) and (2.8) imply the following:

Lemma 2.1: *E is the closed convex subset of the Banach space X.*

In this paper we shall also use the functions $\widetilde{F}$ and $\widetilde{f}$ given by the formulae:

$$\widetilde{F}(x,t,z_0,z_1,\ldots,z_n) := -A(x,t)F(x,t,z_0,z_1,\ldots,z_n) \quad \text{for } (x,t) \in D_0 \times (0,T],\ z_i \in R\ (i = 0,1,\ldots,n), \tag{2.20}$$

$$\widetilde{f}(x) := A(x,0)f(x) \text{ for } x \in D_0, \tag{2.21}$$

where function A is defined by (2.6).

Moreover, we shall need the following:

Assumption:

X. For all the systems of functions $(w_0, w_1, \ldots, w_n, \phi_0)$ belonging to E functions F, f and K satisfy the following condition:

$$\int_{D_0} \widetilde{K}(y,T_1,\ldots,T_k,\hat{w}_0(y,T_1),\ldots,\hat{w}_0(y,T_k))\Gamma(x,t,y,0)dy = B(x,t) \text{ for } (x,t) \in \overline{D}_0 \times (0,T],$$

where

$$\hat{w}_0(y,T_j) := \int_0^{T_j}\int_{D_0} \widetilde{F}(\xi,\tau,w_0(\xi,\tau),w_1(\xi,\tau),\ldots,w_n(\xi,\tau))\Gamma(y,T_j,\xi,\tau))d\xi d\tau$$

$$+ \int_0^{T_j}\int_{\partial D_0} \phi_0(\xi,\tau)\Gamma(y,T_j,\xi,\tau)dS_\xi d\tau$$

$$+ \int_{D_0} \widetilde{f}(\xi)\Gamma(y,T_j,\xi,0)d\xi$$

$$- \int_{D_0} \widetilde{K}(\xi,T_1,\ldots,T_k,w_0(\xi,T_1),\ldots,w_0(\xi,T_k))\Gamma(y,T_j,\xi,0)d\xi$$

$$\text{for } y \in \overline{D}_0 \quad (j = 1,2,\ldots,k),$$

functions $\widetilde{F}, \widetilde{f}, \widetilde{K}$ are given by formulae (2.20), (2.21) and (2.7), respectively, and Γ is the fundamental solution of the homogeneous parabolic differential equation

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u - \frac{\partial u}{\partial t} = 0.$$

In the paper Z_x^1 denotes the set of functions w belonging to Z such that the derivatives $\frac{\partial w}{\partial x_1},\ldots,\frac{\partial w}{\partial x_n}$ are continuous in D.

3. DEFINITION OF A FOURIER'S THIRD NONLOCAL QUASILINEAR PARABOLIC PROBLEM

The Fourier's third nonlocal quasilinear parabolic problem considered in the paper is formulated in the form:

For the given domain D satisfying Assumptions I, II and for the given functions a_{ij}, b_i $(i,j=1,2,\ldots,n)$, c, F, G, g, f, K satisfying Assumptions III-X, the Fourier's third nonlocal quasilinear parabolic problem in D consists in finding a function u belonging to Z_x^1, satisfying the differential equation

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u(x,t)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u(x,t)}{\partial x_i} + c(x,t)u(x,t) - \frac{\partial u(x,t)}{\partial t}$$
$$= F(x,t,u(x,t),\frac{\partial u(x,t)}{\partial x_1},\ldots,\frac{\partial u(x,t)}{\partial x_n}) \text{ for } (x,t)\in D, \tag{3.1}$$

satisfying the nonlocal condition

$$\lim_{t\to 0} u(x,t) + K(x,T_1,\ldots,T_k,u(x,T_1),\ldots,u(x,T_k)) = f(x) \text{ for } x\in \hat{D}_0 \tag{3.2}$$

and satisfying the boundary condition

$$\frac{du(x,t)}{d\nu_x} + g(x,t)u(x,t)$$
$$= G(x,t,u(x,t),\frac{du(x,t)}{dt_x^{(1)}},\ldots,\frac{du(x,t)}{dt_x^{(q)}}) \text{ for } (x,t)\in \partial D_0\times(0,T], \tag{3.3}$$

where for each $t\in(0,T]$ the symbol $\frac{du(x,t)}{d\nu_x}$ denotes the boundary value of the transversal derivative of function u at point x and for each $t\in(0,T]$ the symbols $\frac{du(x,t)}{dt_x^{(i)}}$ $(i=1,2,\ldots,q)$ denote the boundary values of the derivatives of function u in the tangent directions $t_x^{(i)}$ $(i=1,2,\ldots,q)$ at point x, respectively.

A function u possessing the above properties is called a solution in D of the Fourier's third nonlocal quasilinear parabolic problem (3.1) – (3.3).

4. THEOREM ABOUT EXISTENCE

In this section we prove a theorem about the existence of a solution of the Fourier's third nonlocal quasilinear parabolic problem (3.1) – (3.3) assuming that Assumptions I – X from Section 2 are satisfied.

For this purpose observe that, according to known theorems from the potential theory (see [11], Sections 22.8 and 22.10), to find a solution of problem (3.1) – (3.3) it is sufficient to find a function u belonging to Z^1_x and satisfying the integral equation:

$$\begin{aligned} u(x,t) = &\int_0^t \int_{D_0} \widetilde{F}\,(y,s,u(y,s),\frac{\partial u(y,s)}{\partial y_1},\ldots,\frac{\partial u(y,s)}{\partial y_n})\Gamma(x,t,y,s)dyds \\ &+ \int_0^t \int_{\partial D_0} \phi(y,s)\Gamma(x,t,y,s)dS_y ds \\ &+ \int_{D_0} \widetilde{f}\,(y)\Gamma(x,t,y,0)dy \\ &- \int_{D_0} \widetilde{K}\,(y,T_1,\ldots,T_k,u(y,T_1),\ldots,u(y,T_k))\Gamma(x,t,y,0)dy \\ &\text{for } (x,t) \in \overline{D}_0 \times (0,T], \end{aligned} \tag{4.1}$$

where $\phi(y,s)$ is an unknown function considered for $(y,s) \in \partial D_0 \times (0,T]$.

By the fact that function u satisfying equation (4.1) must satisfy boundary condition (3.3), we get the following differential-integral equation:

$$-\tfrac{1}{2}A(x,t)^{-1}\phi(x,t)$$

$$+ \int_0^t \int_{\partial D_0} \phi(y,s)\{\frac{d\Gamma(x,t,y,s)}{d\nu_x} + g(x,t)\Gamma(x,t,y,s)\}dS_y ds$$

$$+ \int_0^t \int_{D_0} \widetilde{F}\,(y,s,u(y,s),\frac{\partial u(y,s)}{\partial y_1},\ldots,\frac{\partial u(y,s)}{\partial y_n})\{\frac{d\Gamma(x,t,y,s)}{d\nu_x} + g(x,t)\Gamma(x,t,y,s)\}dyds$$

$$+ \int_{D_0} \widetilde{f}\,(y)\{\frac{d\Gamma(x,t,y,0)}{d\nu_x} + g(x,t)\Gamma(x,t,y,0)\}dy$$

$$-\int_{D_0} \widetilde{K}(y,T_1,\ldots,T_k,u(y,T_1),\ldots,u(y,T_k))\{\frac{d\Gamma(x,t,y,0)}{d\nu_x}+g(x,t)\Gamma(x,t,y,0)\}dy$$

$$=G(x,t,u(x,t),\frac{du(x,t)}{dt_x^{(1)}},\ldots,\frac{du(x,t)}{dt_x^{(q)}}) \text{ for } (x,t)\in\partial D_0\times(0,T], \tag{4.2}$$

where

$$\frac{du(x,t)}{dt_x^{(i)}}=\int_0^t\int_{D_0}\widetilde{F}(y,s,u(y,s),\frac{\partial u(y,s)}{\partial y_1},\ldots,\frac{\partial u(y,s)}{\partial y_n})\frac{d\Gamma(x,t,y,s)}{dt_x^{(i)}}dyds$$

$$+\int_0^t\int_{\partial D_0}\phi(y,s)\frac{d\Gamma(x,t,y,s)}{dt_x^{(i)}}dS_yds$$

$$+\int_{D_0}\widetilde{f}(y)\frac{d\Gamma(x,t,y,0)}{dt_x^{(i)}}dy$$

$$-\int_{D_0}\widetilde{K}(y,T_1,\ldots,T_k,u(y,T_1),\ldots,u(y,T_k))\frac{d\Gamma(x,t,y,0)}{dt_x^{(i)}}dy$$

$$\text{for } (x,t)\in\partial D_0\times(0,T]\ (i=1,2,\ldots,q). \tag{4.3}$$

Then, to solve problem (3.1) – (3.3) it is enough to solve the system of the differential-integral equations (4.1) and (4.2), where the variables $u(x,t)$ and $\phi(\eta,t)$ of this system are defined for $(x,t)\in\overline{D}_0\times(0,T]$ and $(\eta,t)\in\partial D_0\times(0,T]$, respectively. For this purpose, consider the system of $n+2$ integral equations

$$u_i(x,t)=\int_0^t\int_{D_0}\widetilde{F}(y,s,u_0(y,s),u_1(y,s),\ldots,u_n(y,s))\Gamma_i(x,t,y,s)dyds$$

$$+\int_0^t\int_{\partial D_0}\phi(y,s)\Gamma_i(x,t,y,x)dS_yds$$

$$+\int_{D_0}\widetilde{f}(y)\Gamma_i(x,t,y,0)dy$$

$$-\int_{D_0}\widetilde{K}(y,T_1,\ldots,T_k,u_0(y,T_1),\ldots,u_0(y,T_k))\Gamma_i(x,t,y,0)dy$$

$$\text{for } (x,t)\in\overline{D}_0\times(0,T]\ (i=0,1,\ldots,n), \tag{4.4}$$

$$-\tfrac{1}{2}A(x,t)^{-1}\phi(x,t)$$

$$+\int_0^t\int_{\partial D_0}\phi(y,s)\{\frac{d\Gamma(x,t,y,s)}{d\nu_x}+g(x,t)\Gamma(x,t,y,s)\}dS_y ds$$

$$+\int_0^t\int_{D_0}\widetilde{F}(y,s,u_0(y,s),u_1(y,s),\ldots,u_n(y,s))\{\frac{d\Gamma(x,t,y,s)}{d\nu_x}+g(x,t)\Gamma(x,t,y,s)\}dyds$$

$$+\int_{D_0}\widetilde{f}(y)\{\frac{d\Gamma(x,t,y,0)}{d\nu_x}+g(x,t)\Gamma(x,t,y,0)\}dy$$

$$-\int_{D_0}\widetilde{K}(y,T_1,\ldots,T_k,u_0(y,T_1),\ldots,u_0(y,T_k))\{\frac{d\Gamma(x,t,y,0)}{d\nu_x}+g(x,t)\Gamma(x,t,y,0)\}dy$$

$$=G(x,t,u_0(x,t),\overline{u}_1(x,t),\ldots,\overline{u}_q(x,t)) \text{ for } (x,t)\in\partial D_0\times(0,T], \tag{4.5}$$

where

$$u_0(x,t),u_1(x,t),\ldots,u_n(x,t),\phi(\eta,t)$$

are unknown functions defined for $(x,t)\in\overline{D}_0\times(0,T],(\eta,t)\in\partial D_0\times(0,T]$, respectively,

$$\Gamma_0:=\Gamma,\quad \Gamma_i:=\frac{\partial\Gamma}{\partial x_i}\quad (i=1,2,\ldots,n)$$

and

$$\overline{u}_i(x,t):=\int_0^t\int_{D_0}\widetilde{F}(y,s,u_0(y,s),u_1(y,s),\ldots,u_n(y,s))\frac{d\Gamma(x,t,y,s)}{dt_x^{(i)}}dyds$$
$$+\int_0^t\int_{\partial D_0}\phi(y,s)\frac{d\Gamma(x,t,y,s)}{dt_x^{(i)}}dS_y ds$$
$$+\int_{D_0}\widetilde{f}(y)\frac{d\Gamma(x,t,y,0)}{dt_x^{(i)}}dy$$
$$-\int_{D_0}\widetilde{K}(y,T_1,\ldots,T_k,u_0(y,T_1),\ldots,u_0(y,T_k))\frac{d\Gamma(x,t,y,0)}{dt_x^{(i)}}dy$$
$$\text{for } (x,t)\in\partial D_0\times(0,T]\ \ (i=1,2,\ldots,q). \tag{4.6}$$

System (4.4) and (4.5) will be solved in the class of

$$(u_0(x,t),u_1(x,t),\ldots,u_n(x,t),\phi(\eta,t))\in E, \tag{4.7}$$

i.e., in the class of $(u_0,u_1,\ldots,u_n,\phi)$ belonging to X and satisfying the conditions

$$t^{\beta}\,|\,u_i(x,t)\,|\,\le \rho_1 \ \text{ for } \ (x,t)\in \overline{D}_0\times(0,T] \ \ (i=0,1,\ldots,n), \tag{4.8}$$

$$|\,u_0(x,T_i)\,|\,\le N_i \ \text{ for } \ x\in \overline{D}_0 \ \ (i=1,2,\ldots,k), \tag{4.9}$$

$$t^{\beta}\,|\,\phi(\eta,t)\,|\,\le \rho_2 \ \text{ for } (\eta,t)\in \partial D_0\times(0,T], \tag{4.10}$$

$$t^{\beta}\,|\,\phi(\eta,t)-\phi(\widetilde{\eta},t)\,|\,\le \kappa\,|\,\eta-\widetilde{\eta}\,|^{\gamma} \ \text{ for } \ (\eta,t),(\widetilde{\eta},t)\in \partial D_0\times(0,T] \tag{4.11}$$

and

$$\int_{D_0}\widetilde{K}(y,T_1,\ldots,T_k,u_0(y,T_1),\ldots,u_0(y,T_k))\Gamma(x,t,y,0)dy = B(x,t) \quad \text{for } (x,t)\in \overline{D}_0\times(0,T]. \tag{4.12}$$

To solve system (4.4) and (4.5) in the class of $U=(u_0,u_1,\ldots,u_n,\phi)$ belonging to E define a transformation

$$\mathfrak{T}: E \to \mathfrak{T}E \tag{4.13}$$

by the formula

$$\mathfrak{T}U = V, \tag{4.14}$$

where

$$V=(v_0,v_1,\ldots,v_n,\psi), \tag{4.15}$$

$$\begin{aligned} v_i(x,t) = &\int_0^t\int_{D_0}\widetilde{F}(y,s,u_0(y,s),u_1(y,s),\ldots,u_n(y,s))\Gamma_i(x,t,y,s)dyds \\ &+\int_0^t\int_{\partial D_0}\phi(y,s)\Gamma_i(x,t,y,s)dS_y ds \\ &+\int_{D_0}\widetilde{f}(y)\Gamma_i(x,t,y,0)dy \\ &-\int_{D_0}\widetilde{K}(y,T_1,\ldots,T_k,u_0(y,T_1),\ldots,u_0(y,T_k))\Gamma_i(x,t,y,0)dy \\ &\text{for } (x,t)\in \overline{D}_0\times(0,T] \ (i=0,1,\ldots,n), \end{aligned} \tag{4.16}$$

$-\frac{1}{2}A(x,t)^{-1}\psi(x,t)$

$$+\int_0^t\int_{\partial D_0}\psi(y,s)\{\frac{d\Gamma(x,t,y,s)}{d\nu_x}+g(x,t)\Gamma(x,t,y,s)\}dS_y ds$$

$$=-\int_0^t\int_{D_0}\widetilde{F}(y,s,u_0(y,s),u_1(y,s),\ldots,u_n(y,s))\{\frac{d\Gamma(x,t,y,s)}{d\nu_x}+g(x,t)\Gamma(x,t,y,s)\}dyds$$

$$-\int_{D_0}\widetilde{f}(y)\{\frac{d\Gamma(x,t,y,0)}{d\nu_x}+g(x,t)\Gamma(x,t,y,0)\}dy$$

$$+\int_{D_0}\widetilde{K}(y,T_1,\ldots,T_k,u_0(y,T_1),\ldots,u_0(y,T_k))\{\frac{d\Gamma(x,t,y,0)}{d\nu_x}+g(x,t)\Gamma(x,t,y,0)\}dy$$

$$+G(x,t,v_0(x,t),\overline{u}_1(x,t),\ldots,\overline{u}_q(x,t)) \text{ for } (x,t)\in\partial D_0\times(0,T], \tag{4.17}$$

and $\overline{u}_i$ $(i=1,2,\ldots,q)$ are given by formulae (4.6).

We shall find sufficient conditions that an arbitrary point $U=(u_0,u_1,\ldots,u_n,\phi)$ belonging to set E might be transformed by $\mathcal{T}$ into the point $\mathcal{T}U=V=(v_0,v_1,\ldots,v_n,\psi)$ belonging to this set.

For this purpose introduce the functions

$$\eta_i(x,t):=\int_0^t\int_{D_0}\widetilde{F}(y,s,u_0(y,s),u_1(y,s),\ldots,u_n(y,s))\Gamma_i(x,t,y,s)dyds$$
$$+\int_0^t\int_{\partial D_0}\phi(y,s)\Gamma_i(x,t,y,x)dS_y ds$$
$$+\int_{D_0}\widetilde{f}(y)\Gamma_i(x,t,y,0)dy \text{ for } (x,t)\in\overline{D}_0\times(0,T]\ \ (i=0,1,\ldots,n).$$

It is known (see [11], p. 131) that, by (2.20), (2.1), (2.21), (2.5), (4.8), (4.10), (4.11) and (2.11), the following inequalities

$$|\eta_i(x,t)|\le A_1(M_F\rho_1+\rho_2+\kappa+\overline{M}_F)t^{-\beta+1-\mu_*}+A_2M_f t^{-\beta}$$
$$\text{for } (x,t)\in\overline{D}_0\times(0,T]\ \ (i=0,1,\ldots,n) \tag{4.18}$$

hold, where μ_* is an arbitrary constant satisfying the inequality

$$max(1-\tfrac{1}{2}h_0,\ \tfrac{1}{2}(1+p))<\mu_*<1, \tag{4.19}$$

$$h_0:=min(h_1,2h_2,h_L), \tag{4.20}$$

and A_1, A_2 are positive constants which do not depend on functions F, ϕ, f, u_i $(i = 0, 1, \ldots, n)$ and constant T. It is obvious that constants A_1, A_2 do not depend also on functions G, g and K.

Simultaneously, by (2.7), (2.6), by Assumption III and by (2.9),

$$|\widetilde{K}(y, T_1, \ldots, T_k, u_0(y, T_1), \ldots, u_0(y, T_k))|$$

$$\leq \sup_{y \in \overline{D}_0} |A(y, 0)| \frac{M_K}{|y - P_y|^p} \sum_{i=1}^{k} |u_0(y, T_i)| \text{ for } y \in D_0. \tag{4.21}$$

From inequalities (4.21), (4.9), (2.11) and from known properties of the Poisson-Weierstrass integral (see [11], p. 106-107, Theorem 8), we get

$$\begin{aligned} &\left| \int_{D_0} \widetilde{K}(y, T_1, \ldots, T_k, u_0(y, T_1), \ldots, u_0(y, T_k)) \Gamma_i(x, t, y, 0) dy \right| \\ &\leq A_3 M_K t^{-\beta} \text{ for } (x, t) \in \overline{D}_0 \times (0, T] \quad (i = 0, 1, \ldots, n), \end{aligned} \tag{4.22}$$

where A_3 is a positive constant that does not depend on functions K, u_0 and constant T. It is easy to see that constant A_3 does not depend also on functions F, ϕ, G, g, f and u_i $(i = 1, 2, \ldots, n)$.

Combining (4.18) and (4.22), we obtain

$$\begin{aligned} |v_i(x, t)| &\leq A_1(M_F \rho_1 + \rho_2 + \kappa + \overline{M}_F) t^{-\beta + 1 - \mu_*} + (A_2 M_f + A_3 M_K) t^{-\beta} \\ &\text{for } (x, t) \in \overline{D}_0 \times (0, T] \ (i = 0, 1, \ldots, n). \end{aligned} \tag{4.23}$$

To investigate function $\psi(x, t)$, observe that equation (4.17) has the form of the following Volterra equation

$$\begin{aligned} &-\tfrac{1}{2} A(x, t)^{-1} \psi(x, t) \\ &+ \int_0^t \int_{\partial D_0} \psi(y, s) \{ \frac{d\Gamma(x, t, y, s)}{d\nu_x} + g(x, t) \Gamma(x, t, y, s) \} dS_y ds \\ &= \Xi(x, t, v_0, u_0, u_1, \ldots, u_n) \text{ for } (x, t) \in \partial D_0 \times (0, T], \end{aligned} \tag{4.24}$$

where

$$\Xi(x, t, v_0, u_0, u_1, \ldots, u_n)$$

$$:= -\int_0^t \int_{D_0} \widetilde{F}(y,s,u_0(y,s),u_1(y,s),\ldots,u_n(y,s))\{\frac{d\Gamma(x,t,y,s)}{d\nu_x} + g(x,t)\Gamma(x,t,y,s)\}dyds$$

$$-\int_{D_0} \widetilde{f}(y)\{\frac{d\Gamma(x,t,y,0)}{d\nu_x} + g(x,t)\Gamma(x,t,y,0)\}dy$$

$$+\int_{D_0} \widetilde{K}(y,T_1,\ldots,T_k,u_0(y,T_1),\ldots,u_0(y,T_k))\{\frac{d\Gamma(x,t,y,0)}{d\nu_x} + g(x,t)\Gamma(x,t,y,0)\}dy$$

$$+G(x,t,v_0(x,t),\overline{u}_1(x,t),\ldots,\overline{u}_q(x,t)) \text{ for } (x,t) \in \partial D_0 \times (0,T]. \tag{4.25}$$

It is known (see [11], p. 99, Theorem 2) that the kernel of equation (4.24) can be estimated by

$$\frac{const.}{(t-\tau)^{\mu}} \cdot \frac{1}{|x-y|^{n+1-2\mu-h_0}},$$

where μ is an arbitrary number satisfying the inequalities

$$1-\frac{h_0}{2} < \mu < 1$$

and number h_0 is defined by (4.20). Therefore, equation (4.24) has the only one solution ψ given by the formula

$$\psi(x,t) = -2A(x,t)\Xi(x,t,v_0,u_0,u_1,\ldots,u_n)$$

$$-2\int_0^t \int_{\partial D_0} \mathcal{N}(x,t,y,s)A(y,s)\Xi(y,s,v_0,u_0,u_1,\ldots,u_n)dS_y ds \tag{4.26}$$

$$\text{for } (x,t) \in \partial D_0 \times (0,T],$$

where $\mathcal{N}$ denotes the solving kernel of equation (4.24).

To find an estimation for function Ξ, observe that analogously as in the proof of formula (4.23), using theorems from the potential theory (see [11], Section 22.8, Theorems 5, 8 and Section 22.10, Theorem 1), we obtain, by (4.6), (2.20), (2.1), (2.21), (2.5), (2.7), (2.9) and (4.8) – (4.11) the following inequalities

$$|\overline{u}_i(x,t)| \leq \widetilde{B}_1(M_F\rho_1 + \rho_2 + \kappa + \overline{M}_F)t^{-\beta+1-\mu_*} + \widetilde{B}_2(M_f + M_K)t^{-\beta}$$
$$\text{for } (x,t) \in \partial D_0 \times (0,T] \quad (i=1,2,\ldots,q), \tag{4.27}$$

where constants β and μ_* satisfy inequalities (2.11), (4.19) and $\widetilde{B}_1$, $\widetilde{B}_2$ are positive constants which do not depend on functions F, ϕ, G, g, f, K, u_i $(i = 0, 1, \ldots, n)$ and constant T.

Consequently, by (4.25), (2.20), (2.1), (2.3), (2.21), (2.5), (2.7), (2.9), (4.8) – (4.11), (4.23) and (4.27), by Assumption VII and by known properties of the potentials

$$\begin{aligned} &| \Xi(x, t, v_0, u_0, u_1, \ldots, u_n) | \\ &\leq \overline{B}_1(M_F\rho_1 + \rho_2 + \kappa + \overline{M}_F)M_G t^{-\beta+1-\mu_*} + \overline{B}_2(M_F\rho_1 + \overline{M}_F)t^{-\beta+1-\mu_*} \\ &\quad + \overline{B}_3(\overline{M}_G + M_f + M_K + M_G M_f + M_G M_K)t^{-\beta} \text{ for } (x,t) \in \partial D_0 \times (0, T], \end{aligned} \tag{4.28}$$

where constants β and μ_* satisfy inequalities (2.11), (4.19) and $\overline{B}_i$ $(i = 1, 2, 3)$ are positive constants which do not depend on functions F, ϕ, G, g, f, K, u_i $(i = 0, 1, \ldots, n)$ and constant T.

Then, by formulae (4.26), (4.28) and (2.6), by Assumption III and by known properties of the solving kernel $\mathcal{N}$, function ψ satisfies the inequality

$$\begin{aligned} &| \psi(x, t) | \\ &\leq B_1(M_F\rho_1 + \rho_2 + \kappa + \overline{M}_F)M_G t^{-\beta+1-\mu_*} + B_2(M_F\rho_1 + \overline{M}_F)t^{-\beta+1-\mu_*} \\ &\quad + B_3(\overline{M}_G + M_f + M_K + M_G M_f + M_G M_K)t^{-\beta} \text{ for } (x,t) \in \partial D_0 \times (0, T], \end{aligned} \tag{4.29}$$

where B_i $(i = 1, 2, 3)$ are positive constants which do not depend on functions F, ϕ, G, g, f, K, u_i $(i = 0, 1, \ldots, n)$ and constant T.

Now, we shall find the Hölder inequality for function ψ. For this purpose observe that, by (4.6), (2.20), (2.1), (2.21), (2.5), (2.7), (2.9), (4.8) – (4.11), by Assumption II and by some properties of the potentials (see [11], Section 22.8, Theorems 6, 8 and Section 22.10, Theorem 2),

$$\begin{aligned} &| \overline{u}_i(x, t) - \overline{u}_i(\widetilde{x}, t) | \leq \widetilde{C}(M_F\rho_1 + \rho_2 + \kappa + \overline{M}_F + M_f + M_K)t^{-\beta} | x - \widetilde{x} |^{\gamma} \\ &\qquad \text{for } (x, t), (\widetilde{x}, t) \in \partial D_0 \times (0, T] \ \ (i = 0, 1, \ldots, n), \end{aligned} \tag{4.30}$$

where $\widetilde{C}$ is a positive constant which does not depend on functions F, ϕ, G, g, f, K, u_i $(i = 0, 1, \ldots, n)$ and constant T; and β, γ are constants satisfying conditions (2.11), (2.19), respectively.

Then, from (4.24), (4.29), (4.25), (2.4), (4.16), (4.30), (2.20), (2.1), (2.21), (2.5), (2.7), (2.9), (4.8) – (4.11), from Assumption VII and from known properties of the potential theory (see [11], Section 8) we get that the solution ψ of the integral equation (4.24) satisfies the Hölder inequality

$$| \psi(x,t) - \psi(\widetilde{x},t) |$$
$$\leq \{C_1[t^{2(1-\mu_*)}(C_g+1)M_G + C_G]\cdot(M_F\rho_1 + \rho_2 + \kappa + \overline{M}_F) + C_2 t^{2(1-\mu_*)}(C_g+1)\cdot(M_F\rho_1 + \overline{M}_F)$$
$$+ C_3[t^{1-\mu_*}(C_g+1)+1]\cdot(\overline{M}_G + M_f + M_K + M_G M_f + M_G M_K)$$
$$+ C_4(M_F\rho_1 + \overline{M}_F + C_G + M_f + M_K + C_G M_f + C_G M_K)\} t^{-\beta} | x - \widetilde{x} |^{\gamma}$$
$$\text{for } (x,t), (\widetilde{x},t) \in \partial D_0 \times (0,T], \tag{4.31}$$

where C_i $(i = 1,2,3,4)$ are positive constants which do not depend on functions F, ϕ, G, g, f, K, u_i $(i = 1,2,\ldots,n)$ and constant T.

Comparing inequalities (4.23), (4.29) and (4.31) with inequalities (2.13), (2.15) and (2.16), it is easy to see that if the system of the following inequalities

$$T^{1-\mu_*}A_1(M_F\rho_1 + \rho_2 + \kappa + \overline{M}_F) + A_2 M_f + A_3 M_K \leq \rho_1, \tag{4.32}$$

$$T^{1-\mu_*}[B_1(M_F\rho_1 + \rho_2 + \kappa + \overline{M}_F)M_G + B_2(M_F\rho_1 + \overline{M}_F)]$$
$$+ B_3(\overline{M}_G + M_f + M_K + M_G M_f + M_G M_K) \leq \rho_2, \tag{4.33}$$

$$C_1[T^{2(1-\mu_*)}(C_g+1)M_G + C_G]\cdot(M_F\rho_1 + \rho_2 + \kappa + \overline{M}_F) + C_2 T^{2(1-\mu_*)}(C_g+1)\cdot(M_F\rho_1 + \overline{M}_F)$$
$$+ C_3[T^{1-\mu_*}(C_g+1)+1]\cdot(\overline{M}_G + M_f + M_K + M_G M_f + M_G M_K)$$
$$+ C_4(M_F\rho_1 + \overline{M}_F + C_G + M_f + M_K + C_G M_f + C_G M_K) \leq \kappa \tag{4.34}$$

is satisfied, then the inequalities

$$t^{\beta} | v_i(x,t) | \leq \rho_1 \text{ for } (x,t) \in \overline{D}_0 \times (0,T] \ (i = 0,1,\ldots,n), \tag{4.35}$$

$$t^{\beta} | \psi(\eta,t) | \leq \rho_2 \text{ for } (\eta,t) \in \partial D_0 \times (0,T], \tag{4.36}$$

$$t^{\beta} | \psi(\eta,t) - \psi(\widetilde{\eta},t) | \leq \kappa | \eta - \widetilde{\eta} |^{\gamma} \text{ for } (\eta,t), (\widetilde{\eta},t) \in \partial D_0 \times (0,T] \tag{4.37}$$

hold.

Moreover, if the system of the inequalities

$$A_1(M_F\rho_1 + \rho_2 + \kappa + \overline{M}_F)T_i^{-\beta+1-\mu_*} + (A_2 M_f + A_3 M_K)T_i^{-\beta} \leq N_i \ (i = 1,2,\ldots,k) \tag{4.38}$$

is satisfied, then from (4.23),

$$| v_0(x,T_i) | \leq N_i \text{ for } x \in \overline{D}_0 \ (i = 1,2,\ldots,k). \tag{4.39}$$

Finally, by Assumption X and by formulae (4.7), (4.16),

$$\int_{D_0} \widetilde{K}(y,T_1,\ldots,T_k,v_0(y,T_1),\ldots,v_0(y,T_k))\Gamma(x,t,y,0)dy = B(x,t)$$

$$\text{for } (x,t)\in \overline{D}_0\times(0,T].$$

Consequently, from (4.35) – (4.37) and (4.39), and from the above condition,

$$\mathcal{T}E\subset E. \tag{4.40}$$

Now, assuming that not only Assumptions I-X are satisfied but also inequalities (4.32) – (4.34), (4.38) are satisfied we shall prove two lemmas:

Lemma 4.1: *Transformation* $\mathcal{T}$ *defined by formulae* (4.13) – (4.17) *is continuous in space* X.

Proof: Let $\{U^{(m)}\}$ be a sequence of points $U^{(m)}=(u_0{}^{(m)},u_1{}^{(m)},\ldots,u_n{}^{(m)},\phi^{(m)})$ belonging to E such that

$$\|U^{(m)}-U\| = \max_{i=0,1,\ldots,n}\ \sup_{(x,t)\in\overline{D}_0\times(0,T]} t^{\alpha+\beta}\,|u_i{}^{(m)}(x,t)-u_i(x,t)|$$

$$+\sup_{(x,t)\in\partial D_0\times(0,T]} t^{\alpha+\beta}\,|\phi^{(m)}(x,t)-\phi(x,t)|\to 0 \text{ as } m\to\infty, \tag{4.41}$$

where $U=(u_0,u_1,\ldots,u_n,\phi)$ is a point belonging also to E.

To prove Lemma 4.1, it is sufficient to show that

$$\lim_{m\to\infty}\|V^{(m)}-V\|=0, \tag{4.42}$$

where $V^{(m)}=(v_0{}^{(m)},v_1{}^{(m)},\ldots,v_n{}^{(m)},\psi^{(m)})$ and $V=(v_0,v_1,\ldots,v_n,\psi)$ are values of transformation $\mathcal{T}$ at points $U^{(m)}$ and U, respectively.

For this purpose consider the difference

$$v_i{}^{(m)}(x,t)-v_i(x,t)$$

$$=\int_0^t\int_{D_0}\{\widetilde{F}(y,s,u_0{}^{(m)}(y,s),u_1{}^{(m)}(y,s),\ldots,u_n{}^{(m)}(y,s))$$

$$-\widetilde{F}(y,s,u_0(y,s),u_1(y,s),\ldots,u_n(y,s))\}\Gamma_i(x,t,y,s)dyds$$

$$+\int_0^t\int_{\partial D_0}[\phi^{(m)}(y,s)-\phi(y,s)]\Gamma_i(x,t,y,s)dS_y ds$$
$$-\int_{D_0}[\widetilde{K}(y,T_1,\ldots,T_k,u_0^{(m)}(y,T_1),\ldots,u_0^{(m)}(y,T_k))$$
$$-\widetilde{K}(y,T_1,\ldots,T_k,u_0(y,T_1),\ldots,u_0(y,T_k))]\Gamma_i(x,t,y,0)dy$$
$$\text{for } (x,t)\in \overline{D}_0\times(0,T] \quad (i=0,1,\ldots,n). \tag{4.43}$$

Since $\{U^{(m)}\}\subset E$ and $U\in E$ then, by (4.43), (2.17) and (4.12),

$$v_i^{(m)}(x,t)-v_i(x,t)$$
$$=\int_0^t\int_{D_0}\{\widetilde{F}(y,s,u_0^{(m)}(y,s),u_1^{(m)}(y,s),\ldots,u_n^{(m)}(y,s))$$
$$-\widetilde{F}(y,s,u_0(y,s),u_1(y,s),\ldots,u_n(y,s))\}\Gamma_i(x,t,y,s)dyds$$
$$+\int_0^t\int_{\partial D_0}[\phi^{(m)}(y,s)-\phi(y,s)]\Gamma_i(x,t,y,s)dS_y ds$$
$$\text{for } (x,t)\in \overline{D}_0\times(0,T] \quad (i=0,1,\ldots,n). \tag{4.44}$$

Consequently, using the argumentation from Section 22.11 from [11] we conclude, by (4.44) and (2.2), by known properties of the potentials and by (4.41), that

$$\lim_{m\to\infty}\ \sup_{(x,t)\in\overline{D}_0\times(0,T]} t^{\alpha+\beta}\,|\,v_i^{(m)}(x,t)-v_i(x,t)\,| = 0. \tag{4.45}$$

Now, consider the difference

$$\psi^{(m)}(x,t)-\psi(x,t)$$
$$=-2A(x,t)[\Xi(x,t,v_0^{(m)},u_0^{(m)},u_1^{(m)},\ldots,u_n^{(m)})-\Xi(x,t,v_0,u_0,u_1,\ldots,u_n)]$$
$$-2\int_0^t\int_{\partial D_0}\mathcal{N}(x,t,y,s)A(y,s)[\Xi(y,s,v_0^{(m)},u_0^{(m)},u_1^{(m)},\ldots,u_n^{(m)})$$
$$-\Xi(y,s,v_0,u_0,u_1,\ldots,u_n)]dS_y ds \quad \text{for } (x,t)\in\partial D_0\times(0,T], \tag{4.46}$$

where

$$\Xi(x,t,v_0^{(m)},u_0^{(m)},u_1^{(m)},\ldots,u_n^{(m)})-\Xi(x,t,v_0,u_0,u_1,\ldots,u_n)$$
$$=-\int_0^t\int_{D_0}[\widetilde{F}(y,s,u_0^{(m)}(y,s),u_1^{(m)}(y,s),\ldots,u_n^{(m)}(y,s))$$

$$-\widetilde{F}(y,s,u_0(y,s),u_1(y,s),\dots,u_n(y,s))]\{\frac{d\Gamma(x,t,y,s)}{d\nu_x}+g(x,t)\Gamma(x,t,y,s)\}dyds$$

$$+\int_{D_0}[\widetilde{K}(y,T_1,\dots,T_k,u_0{}^{(m)}(y,T_1),\dots,u_0{}^{(m)}(y,T_k))$$

$$-\widetilde{K}(y,T_1,\dots,T_k,u_0(y,T_1),\dots,u_0(y,T_k))]\{\frac{d\Gamma(x,t,y,0)}{d\nu_x}+g(x,t)\Gamma(x,t,y,0)\}dy$$

$$+G(x,t,v_0{}^{(m)}(x,t),\overline{u}_1{}^{(m)}(x,t),\dots,\overline{u}_q{}^{(m)}(x,t))-G(x,t,v_0(x,t),\overline{u}_1(x,t),\dots,\overline{u}_q(x,t))$$

$$\text{for } (x,t)\in\partial D_0\times(0,T], \tag{4.47}$$

$$\overline{u}_i{}^{(m)}(x,t)=\int_0^t\int_{D_0}\widetilde{F}(y,s,u_0{}^{(m)}(y,s),u_1{}^{(m)}(y,s),\dots,u_n{}^{(m)}(y,s))\frac{d\Gamma(x,t,y,s)}{dt_x{}^{(i)}}dyds$$

$$+\int_0^t\int_{\partial D_0}\phi^{(m)}(y,s)\frac{d\Gamma(x,t,y,s)}{dt_x{}^{(i)}}dS_yds$$

$$+\int_{D_0}\widetilde{f}(y)\frac{d\Gamma(x,t,y,0)}{dt_x{}^{(i)}}dy$$

$$-\int_{D_0}\widetilde{K}(y,T_1,\dots,T_k,u_0{}^{(m)}(y,T_1),\dots,u_0{}^{(m)}(y,T_k))\frac{d\Gamma(x,t,y,0)}{dt_x{}^{(i)}}dy$$

$$\text{for } (x,t)\in\partial D_0\times(0,T]\ \ (i=1,2,\dots,q) \tag{4.48}$$

and functions $\overline{u}_i$ $(i=1,2,\dots,q)$ are given by formulae (4.6).

Since $\{U^{(m)}\}\subset E$ and $U\in E$ then, by (4.47), (2.17) and (4.12),

$$\Xi(x,t,v_0{}^{(m)},u_0{}^{(m)},u_1{}^{(m)},\dots,u_n{}^{(m)})-\Xi(x,t,v_0,u_0,u_1,\dots,u_n)$$

$$=-\int_0^t\int_{D_0}[\widetilde{F}(y,s,u_0{}^{(m)}(y,s),u_1{}^{(m)}(y,s),\dots,u_n{}^{(m)}(y,s))$$

$$-\widetilde{F}(y,s,u_0(y,s),u_1(y,s),\dots,u_n(y,s))]\{\frac{d\Gamma(x,t,y,s)}{d\nu_x}+g(x,t)\Gamma(x,t,y,s)\}dyds$$

$$+G(x,t,v_0{}^{(m)}(x,t),\overline{u}_1{}^{(m)}(x,t),\dots,\overline{u}_q{}^{(m)}(x,t))-G(x,t,v_0(x,t),\overline{u}_1(x,t),\dots,\overline{u}_q(x,t))$$

$$\text{for } (x,t)\in\partial D_0\times(0,T]. \tag{4.49}$$

Consequently, using the argument from Section 22.11 from [11], we obtain, by (4.46), (4.49), (2.4), (4.44), (4.48), (4.6), (2.20), (2.2), (2.17) and (4.12), by Assumption VII, by known properties of the potentials and by (4.41), the condition

$$\lim_{m\to\infty} \sup_{(x,t) \in \partial D_0 \times (0,T]} t^{\alpha+\beta} | \psi^{(m)}(x,t) - \psi(x,t) | = 0. \tag{4.50}$$

Then, formulae (4.45), (4.50) and (2.12) imply (4.42). Therefore, the proof of Lemma 4.1 is complete.

Lemma 4.2: *$\mathcal{T}E$ is precompact.*

Proof: Let $\{V^{(m)}\}$ be a sequence of points

$$V^{(m)} = (v_0{}^{(m)}, v_1{}^{(m)}, \ldots, v_n{}^{(m)}, \psi^{(m)})$$

in $\mathcal{T}E$. Then

$$t^{\beta} | v_i{}^{(m)}(x,t) | \leq \rho_1 \text{ for } (x,t) \in \overline{D}_0 \times (0,T] \;\; (i = 0,1,\ldots,n), \tag{4.51}$$

$$| v_0{}^{(m)}(x,T_i) | \leq N_i \text{ for } x \in \overline{D}_0 \;\; (i = 1,2,\ldots,k), \tag{4.52}$$

$$t^{\beta} | \psi^{(m)}(\eta,t) | \leq \rho_2 \text{ for } (\eta,t) \in \partial D_0 \times (0,T], \tag{4.53}$$

$$t^{\beta} | \psi^{(m)}(\eta,t) - \psi^{(m)}(\widetilde{\eta},t) | \leq \kappa | \eta - \widetilde{\eta} |^{\gamma} \text{ for } (\eta,t),(\widetilde{\eta},t) \in \partial D_0 \times (0,T] \tag{4.54}$$

and

$$\int_{D_0} \widetilde{K}(y,T_1,\ldots,T_k,v_0{}^{(m)}(y,T_1),\ldots,v_0{}^{(m)}(y,T_k))\Gamma(x,t,y,0)dy = B(x,t) \quad \text{for } (x,t) \in \overline{D}_0 \times (0,T]. \tag{4.55}$$

Inequalities (4.51) and (4.53) imply that the sequences

$$\{t^{\alpha+\beta} v_i{}^{(m)}\} \text{ and } \{t^{\alpha+\beta} \psi^{(m)}\} \tag{4.56}$$

are equi-bounded and equi-continuous in $\overline{D}_0 \times (0,T]$ and $\partial D_0 \times (0,T]$, respectively. Consequently, by the Ascoli-Arzela theorem, it is possible to choose uniformly convergent subsequences

$$\{t^{\alpha+\beta} v_i{}^{(m_j)}\} \text{ and } \{t^{\alpha+\beta} \psi^{(m_j)}\} \tag{4.57}$$

of sequences (4.56). This uniformly convergence implies that subsequences (4.57) are convergent in the sense of norm (2.12). Since functions $v_i{}^{(m_j)}$ $(i = 0,1,\ldots,n)$ and $\psi^{(m_j)}$ satisfy conditions (4.51) – (4.55), where $v_i{}^{(m)}$ $(i = 0,1,\ldots,n)$ and $\psi^{(m)}$ are replaced by $v_i{}^{(m_j)}$ $(i = 0,1,\ldots,n)$ and $\psi^{(m_j)}$, respectively, then the proof of Lemma 4.2 is complete.

Now we shall give the fundamental theorem about the existence of a solution in D of the Fourier's third nonlocal quasilinear parabolic problem (3.1) – (3.3).

Theorem 4.1: *If boundary ∂D_0 of domain D_0 satisfies Assumptions I and II, if coefficients a_{ij}, b_i $(i,j = 1,2,\ldots,n)$, c of equation* (3.1) *satisfy Assumptions III and IV, if functions F, G, g, f and K satisfy Assumptions V-X, if constants T, M_F, $\overline{M}_F$, M_G, $\overline{M}_G$, M_f, M_K, C_G and C_g satisfy inequalities* (4.32) – (4.34), (4.38) *and if constants T_i, N_i $(i = 1,2,\ldots,k)$ satisfy inequalities* (2.18), *then the Fourier's third nonlocal quasilinear parabolic problem* (3.1) – (3.3) *has a solution in D.*

Proof: From Lemma 2.1, from formula (4.40) and from Lemmas 4.1 and 4.2, it is easy to see that all the assumptions of the Schauder's theorem (see [10], p. 189) are satisfied. Therefore, there exists a point

$$U^* = (u_0^*, u_1^*, \ldots, u_n^*, \phi^*) \in E$$

which is invariable with respect to the transformation given by (4.13) – (4.17). This point is a solution of the integral equations (4.4) and (4.5). From known properties of the potentials (see [11], Section 22.8) and from equations (4.4), (4.5) we get

$$u_i^*(x,t) = \frac{\partial u_0^*(x,t)}{\partial x_i} \quad (i = 1,2,\ldots,n) \text{ for } (x,t) \in D.$$

Then functions u_0^* and ϕ^* satisfy the system of the functional-differential equations (4.1) and (4.2). It is obvious that function u_0^* satisfies the boundary condition (3.3) and the nonlocal condition (3.2). Moreover, by (2.20), (2.6), by Assumption III, by (2.2), (4.4), (2.20), (2.21), (2.6), (2.1), (2.9), (2.13) – (2.16), (2.11) and by some properties of the potentials (see [11], Section 22.8, Theorems 1, 6, 8), the function

$$\Phi(x,t) := \widetilde{F}(x,t,u_0^*(x,t),u_1^*(x,t),\ldots,u_n^*(x,t)), \quad (x,t) \in D_0^* \times (0,T]$$

satisfies the Hölder condition

$$|\Phi(x,t) - \Phi(\widetilde{x},t)| \leq \frac{C_*(D_0^*)}{t^\beta} |x - \widetilde{x}|^{\overline{\gamma}} \text{ for } (x,t),(\widetilde{x},t) \in D_0^* \times (0,T],$$

where $C_*(D_0^*)$ is a positive constant which depends on D_0^* and $\overline{\gamma} := min\{\gamma, h_F, \widetilde{h}_F\}$. Consequently, by theorems from the potential theory (see [11], Section 22.8, Theorems 7 and 8) function u_0^* satisfies equation (3.1) in domain D. Therefore, this function is a solution in D of the Fourier's third nonlocal quasilinear parabolic problem (3.1) – (3.3).

5. REMARK

If $K(x,T_1,\ldots,T_k,z(x,T_1),\ldots,z(x,T_k))$ is defined by the formula

$$K(x,T_1,\dots,T_k,z(x,T_1),\dots,z(x,T_k)) = \sum_{i=1}^{k} \xi_i(x)z(x,T_i) \text{ for } x \in D_0,\ z \in Z, \tag{5.1}$$

where $\xi_i(x)$ $(i = 1,2,\dots,k)$ are given functions defined and integrable for $x \in D_0$, then condition (2.8) holds.

Moreover, if

$$|\xi_i(x)| \le \frac{M_{\xi_i}}{|x-P_x|^p} \text{ for } x \in D_0 \ \ (i = 1,2,\dots,k),$$

where M_{ξ_i} $(i = 1,2,\dots,k)$ are positive constants and p is a constant from Assumption V, then condition (2.9) is satisfied, where the constant M_K is given by the formula

$$M_K{:} = max\{M_{\xi_1},\dots,M_{\xi_k}\}.$$

Additionally, if

$$|\xi_i(x)| \le C_{\xi_i} \text{ for } x \in D_0 \ \ (i = 1,2,\dots,k),$$

where C_{ξ_i} $(i = 1,2,\dots,k)$ are positive constants such that

$$C_{\xi_i} \le \frac{M_{\xi_i}}{|x-P_x|^p} \text{ for } x \in D_0 \ \ (i = 1,2,\dots,k),$$

then condition (2.10) holds, where the constant C_K is defined by the formula

$$C_K{:} = max\{C_{\xi_1},\dots,C_{\xi_k}\}.$$

It is easy to see that if $\xi_i(x) = 0$ for $x \in D_0$ $(i = 1,2,\dots,k)$, then the Fourier's third nonlocal quasilinear parabolic problem (3.1) – (3.3), where function K is given by formula (5.1), is reduced to the classical initial-boundary problem. Moreover, if $k = 1$, $T_1 = T$, $\xi_1(x) = -1$ for $x \in D_0$, $f(x) = 0$ for $x \in D_0$ and $C_{\xi_1} \ge 1$, then problem (3.1) – (3.3), where function K is given by formula (5.1), contains the periodic problem, i.e., the Fourier's third nonlocal quasilinear parabolic problem (3.1) – (3.3), where condition (3.2) is replaced by the condition

$$\lim_{t \to 0} u(x,t) = u(x,T) \text{ for } x \in \hat{D}_0.$$

It is obvious, from the above considerations, that it is sensible to consider the Fourier's third nonlocal quasilinear parabolic problem (3.1) – (3.3) since this problem is always more general than the analogous classical Fourier's third quasilinear parabolic problem and, additionally, if $k = 1$,

$$f(x) = 0 \text{ for } x \in D_0$$

and

$$1 \leq C_{\xi_1} \leq \frac{M_{\xi_1}}{|x - P_x|^p} \text{ for } x \in D_0,$$

where p is a constant from Assumption V, then this problem is also more general than the analogous Fourier's third periodic quasilinear parabolic problem.

If $u(x,t)$ is, for example, interpreted as the temperature of a physical substance then Theorem 4.1 can be applied for all the physical phenomena from the theory of the heat conduction, where the temperatures

$$u(x,0), u(T_1,x), \ldots, u(T_k,x), u(T,x) \tag{5.2}$$

satisfy condition (3.2) in the general sense or maybe in a particular sense considered in this section.

It is obvious, that to use Theorem 4.1 it is not necessary to know quantities (5.2). It is only necessary to know relations between these quantities. Therefore, the physical interpretations of nonlocal problems are significant and the author is of the opinion that, in general, nonlocal problems possess deep physical and philosophical meanings. It is the reason for which, in the opinion of the author, nonlocal problems should be developed and applied in the near future.

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