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IMPULSIVE INTEGRAL EQUATIONS IN BANACH SPACES AND APPLICATIONS[1]

DAJUN GUO

Department of Mathematics
Shandong University
Jinan, PEOPLE'S REPUBLIC OF CHINA

ABSTRACT

In this paper, we first use the fixed point theory to prove two existence theorems of positive solutions for the impulsive Fredholm integral Equations in Banach spaces. And then, we offer some applications to the two-point boundary value problems for the second order impulsive differential equations in Banach spaces.

Key words: Impulsive Fredholm integral equation, impulsive differential equation, strict set contraction, fixed point theorem.

AMS (MOS) subject classification: 45N05, 47H07.

1. INTRODUCTION

The theory of impulsive differential equations is a new important branch of differential equations (see [5]). Of course, impulsive integral equations and impulsive differential equations are closely connected. Paper [3] discussed the impulsive Volterra integral equations in Banach spaces based on the iterative monotone technique and a comparison result. Since there is no comparison theorem for impulsive Fredholm integral equations, the method in [3] is not available for such equations. In this paper, we first use the fixed point theory to prove two existence theorems of positive solutions for impulsive Fredholm integral equations in Banach spaces. And then, we offer some applications to the two-point boundary value problems for second order impulsive differential equations in Banach spaces.

Let the real Banach space E be partially ordered by a cone P of E, i.e., $x \leq y$ iff $y - x \in P$. P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\| x \| \leq N \| y \|$, where θ denotes the zero element of E, and N is called the normal constant of P (see [4]). Consider the impulsive Fredholm integral equation in E:

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$$x(t) = \int_0^T H(t,s,x(s))ds + \sum_{0 < t_k < t} I_k(x(t_k)), \tag{1}$$

where $H \in C[J \times J \times P, P], J = [0,T]$, $I_k \in C[P,P]$ $(k = 1,2,\ldots,p)$ and $0 < t_1 < \ldots < t_k \ldots < t_p < T$. Assume that $H(t,s,\theta) = \theta$ for $t,s \in J$ and $I_k(\theta) = \theta$ $(k = 1,2,\ldots,p)$, then $x(t) \equiv \theta$ is the trivial solution of equation (1). Let $PC[J,E] = \{x$: x is a map from J into E such that $x(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$ and $\lim_{t \to t_k^+} x(t)$ exist for $k = 1,2,\ldots,p\}$. Evidently, $PC[J,E]$ is a Banach space with norm $\| x \|_p = \sup_{t \in J} \| x(t) \|$. Clearly, $Q = \{x \in PC[J,E]$: $x(t) \geq \theta$ for $t \in J\}$ is a normal cone in space $PC[J,E]$ if P is normal. A map $x \in PC[J,E]$ is called a positive solution of equation (1) if it satisfies (1) on J and $x \in Q, x \neq \theta$.

2. LEMMAS

Let $J_0 = [0,t_1], J_1 = (t_1,t_2], \ldots, J_{p-1} = (t_{p-1}, t_p]$, $J_p = (t_p, T]$. For $S \subset PC[J,E]$, we denote $S(t) = \{x(t): x \in S\} \subset E (t \in J)$.

Lemma 1: *If S is bounded and the elements of S are equicontinuous on each $J_k (k = 0,1,\ldots,p)$, then*

$$\alpha(S) = \sup_{t \in J} \alpha(S(t)), \tag{2}$$

where α denotes the Kuratowski measure of noncompactness.

Proof: Let $S^{(k)} = \{x \mid_{J_k}: x \in S\}$ $(k = 0,1,\ldots,p)$. Since $\lim_{t \to t_k^+} x(t)$ exists, $S^{(k)}$ may be regarded as a subset of the space $C[\bar{J}_k, E]$, where $\bar{J}_k$ is the closure of J_k, i.e., $\bar{J}_k = [t_{k-1}, t_k]$. Hence (see [6])

$$\alpha(S^{(k)}) = \sup_{t \in J_k} \alpha(S^{(k)}(t)) \ (k = 0,1,\ldots,p). \tag{3}$$

Obviously, $S(t) = S^{(k)}(t)$ for $t \in J_k$, $k = 0,1,\ldots,p$, so

$$\sup_{t \in J} \alpha(S(t)) = \max_k \sup_{t \in J_k} \alpha(S^{(k)}(t)). \tag{4}$$

Now, we show

$$\alpha(S) = \max_k \alpha(S^{(k)}). \tag{5}$$

It is evident,

$$\alpha(S^{(k)}) \leq \alpha(S) \ (k = 0,1,\ldots,p). \tag{6}$$

On the other hand, for any given $\epsilon > 0$, there exists partition $S^{(k)} = \bigcup_{i=1}^{n_k} S_i^{(k)}$ such that

$$diam(S_i^{(k)}) < \alpha(S^{(k)}) + \epsilon \leq b + \epsilon \quad (i = 1, 2, \ldots, n_k), \tag{7}$$

where $b = \max_k \alpha(S^{(k)})$. Let $S_i^{(k)} = S_{ki} \mid J_k$ and $S(i, j, \ldots, h) = S_{1i} \cap S_{2j} \cap \ldots \cap S_{ph}$, then $S = \cup \{S(i, j, \ldots, h): i = 1, 2, \ldots, n_1;\ j = 1, 2, \ldots, n_2; \ldots; h = 1, 2, \ldots, n_p\}$, and by (7),

$$dim(S(i, j, \ldots, h)) \leq b + \epsilon (i = 1, 2, \ldots, n_1; j = 1, 2, \ldots, n_2; \ldots; h = 1, 2, \ldots, n_p).$$

Hence

$$\alpha(S) \leq b + \epsilon,$$

which implies, since ϵ is arbitrary, that

$$\alpha(S) \leq b. \tag{8}$$

Consequently, (5) follows from (6) and (8), and finally, (2) follows from (3), (4) and (5). The proof is complete.

In the following, the closed balls in E and $PC[J, E]$ are denoted by $T_r = \{x \in E: \| x \| \leq r\}$ and $B_r = \{x \in PC[J, E]: \| x \|_p \leq r\}$ respectively.

Consider the operator

$$Ax(t) = \int_0^T H(t, s, x(s))ds + \sum_{0 < t_k < t} I_k(x(t_k)). \tag{9}$$

Lemma 2: *Let $H \in C[J \times J \times P, P]$ and $I_k \in C[P, P]$ $(k = 1, 2, \ldots, p)$. Suppose that, for any $r > 0$, H is uniformly continuous on $J \times J \times (P \cap T_r)$, I_k is bounded on $P \cap T_r$, and there exist nonnegative constants L_r and $M_r^{(k)}$ with*

$$2TL_r + \sum_{k=1}^{p} M_r^{(k)} < 1$$

such that

$$\alpha(H(t, s, D)) \leq L_r \alpha(D), \quad t, s \in J, \quad D \subset P \cap T_r \tag{10}$$

and

$$\alpha(I_k(D)) \leq M_r^{(k)} \alpha(D), \quad D \subset P \cap T_r, \quad k = 1, 2, \ldots, p. \tag{11}$$

Then, for any $r > 0$, operator A is a strict set contraction from $Q \cap B_r$ into Q, i.e., there exists a constant k_r with $0 \leq k_r < 1$ such that $\alpha(A(S)) \leq k_r \alpha(S)$ for any $S \subset Q \cap B_r$.

Proof: It is easy to see that the uniform continuity of H on $J \times J \times (P \cap T_r)$ implies the boundedness of H on $J \times J \times (P \cap T_r)$, and so A is a bounded and continuous operator from $Q \cap B_r$ into Q. By the uniform continuity of H and (10) and using [6], we have

$$\alpha(H(J \times J \times D)) = \max_{t, s \in J} \alpha(H(t,s,D)) \leq L_r \alpha(D), \; D \subset P \cap T_r. \tag{12}$$

Now, let $S \subset Q \cap B_r$ be arbitrarily given. By virtue of (9), it is easy to show that the elements of $A(S)$ are equicontinuous on each $J_k (k = 0, 1, \ldots, p)$, and so, by Lemma 1,

$$\alpha(A(S)) = \sup_{t \in J} \alpha(A(S(t))). \tag{13}$$

Using (12), (11) and the obvious formula

$$\int_0^T y(s)ds \in T\, \overline{co}\{y(s): s \in J\}, \quad y \in PC[J, E],$$

we find

$$\begin{aligned}\alpha(A(S(t))) &\leq T\alpha(\overline{co}\{H(t,s,x(s)): x \in S, s \in J\}) + \sum_{0 < t_k < t} \alpha(\{I_k(x(t_k)): x \in S\}) \\ &\leq T\alpha(H(J \times J \times S(J)) + \sum_{k=1}^{p} \alpha(I_k(S(t_k))) \\ &\leq TL_r \alpha(S(J)) + \sum_{k=1}^{p} M_r^{(k)} \alpha(S(t_k)),\end{aligned} \tag{14}$$

where $S(J) = \{x(s): x \in S, s \in J\}$ and $S(t_k) = \{x(t_k): x \in S\}$. For any given $\epsilon > 0$, there exists a partition $S = \bigcup_{j=1}^{m} S_j$ such that

$$diam(S_j) < \alpha(S) + \epsilon, \; j = 1, 2, \ldots, m. \tag{15}$$

Since $S(t_k) = \bigcup_{j=1}^{m} S_j(t_k)$ and $diam(S_j(t_k)) \leq diam(S_j)$, we find by (15),

$$\alpha(S(t_k)) \leq \alpha(S) + \epsilon, k = 1, 2, \ldots, p. \tag{16}$$

On the other hand, choosing $x_j \in S_j \;\; (j = 1, 2, \ldots, m)$ and a partition $J_k = \bigcup_{i=1}^{n_k} J_k^{(i)}$ $(k = 0, 1, 2, \ldots, p)$ such that

$$\| x_j(t) - x_j(t') \| < \epsilon, \;\; j = 1, 2, \ldots, m; \; t, t' \in J_k^{(i)} \; (k = 0, 1, \ldots, p; i = 1, 2, \ldots, n_k) \tag{17}$$

we have

$$S(J) = \cup \{S_j(J_k^{(i)}): \;\; i = 1, 2, \ldots, n_k; \; k = 0, 1, \ldots, p; \; j = 1, 2, \ldots, m\}.$$

For $x(t), \overline{x}(t') \in S_j(J_k^{(i)})$ (i.e., x, $\overline{x} \in S_j$, t, $t' \in J_k^{(i)}$), we find by (17) and (15)

$$\| x(t) - \overline{x}(t') \| \leq \| x(t) - x_j(t) \| + \| x_j(t) - x_j(t') \| + \| x_j(t') - \overline{x}(t') \|$$
$$\leq \| x - x_j \|_p + \epsilon + \| x_j - x \|_p$$
$$< 2diam(S_j) + \epsilon < 2\alpha(S) + 3\epsilon,$$

which implies

$$\alpha(S(J)) \leq 2\alpha(S) + 3\epsilon. \tag{18}$$

Since ϵ is arbitrary, it follows from (16) and (18) that

$$\alpha(S(t_k)) \leq \alpha(S), \ k = 1, 2, \ldots, p \tag{19}$$

and

$$\alpha(S(J)) \leq 2\alpha(S). \tag{20}$$

Finally, (13), (14), (19) and (20) imply $\alpha(A(S)) \leq k_r \alpha(S)$, where $k_r = 2TL_r + \sum_{k=1}^{p} M_r^{(k)} < 1$, and the lemma is proved.

We also need the following result which is concerned with the fixed points of strict set contractions (see [1], [2]):

Lemma 3: *Let K be a cone of the real Banach space X and $K_{r,R} = \{x \in K: r \leq \| x \| \leq R\}$ with $R > r > 0$. Suppose that $B: K_{r,R} \to K$ is a strict set contraction such that one of the following two conditions is satisfied:*

a) $Bx \nleq x$ *for* $x \in K$, $\| x \| = r$ *and* $Bx \ngeq x$ *for* $x \in K$, $\| x \| = R$.

b) $Bx \ngeq x$ *for* $x \in K$, $\| x \| = r$ *and* $Bx \nleq x$ *for* $x \in K$, $\| x \| = R$.

Then B has at least one fixed point in $K_{r,R}$.

3. MAIN THEOREMS

Let us list some conditions for convenience.

(H_1): $H \in C[J \times J \times P, P]$, $H(t, s, \theta) = \theta$ for t, $s \in J$, $I_k \in C[P, P]$, $I_k(\theta) = \theta$ $(k = 1, 2, \ldots, p)$. For any $r > 0$, H is uniformly continuous on $J \times J \times (P \cap T_r)$, I_k is bounded on $P \cap T_r$ and there exist nonnegative constants L_r and $M_r^{(k)}$ with

$$2TL_r + \sum_{k=1}^{p} M_r^{(k)} < 1$$

such that (10) and (11) hold. There exist also $t_p < a < b < T$ and $0 < c < 1$ such that

$$H(t,s,x) \geq cH(u,s,x),\ t \in J_0 = [a,b],\ u,s \in J,\ x \in P. \tag{21}$$

(H_2): $\| H(t,s,x) \| / \| x \| \to 0$ as $x \in P$ and $\| x \| \to 0$ uniformly in $t,s \in J$; $\| I_k(x) \| / \| x \| \to 0$ as $x \in P$ and $\| x \| \to 0$ $(k = 1,2,\ldots,p)$.

(H_3) $\| H(t,s,x) \| / \| x \| \to 0$ as $x \in P$ and $\| x \| \to \infty$ uniformly in $t,s \in J$; $\| I_k(x) \| / \| x \| \to 0$ as $x \in P$ and $\| x \| \to \infty$ $(k = 1,2,\ldots,p)$.

(H_4): there exists a $g \in P^*$ (P^* denotes the dual cone of P) such that $g(x) > 0$ for any $x > \theta$ and $g(H(t,s,x))/g(x) \to \infty$ as $x \in P$ and $\| x \| \to 0$ uniformly in $t,s \in J_0$.

(H_5): there exists a $g \in P^*$ such that $g(x) > 0$ for any $x > \theta$ and $g(H(t,s,x))/g(x) \to \infty$ as $x \in P$ and $\| x \| \to \infty$ uniformly in $t,s \in J_0$.

Theorem 1: *Let cone P be normal. Suppose that conditions (H_1), (H_2) and (H_5) are satisfied. Then, equation (1) has at least one positive solution.*

Proof: Let $K = \{x \in Q: x(t) \geq cx(s)$ for $t \in J_0, s \in J\}$. Then K is a cone of $PC[J,E]$ and $K \subset Q$. For any $x \in Q$, we have by (21): $t \in J_0$ and $u \in J$ simply

$$\begin{aligned}
Ax(t) &= \int_0^T H(t,s,x(s))ds + \sum_{0 < t_k < t} I_k(x(t_k)) \\
&\geq c\int_0^T H(u,s,x(s))ds + \sum_{0 < t_k < t} I_k(x(t_k)) \\
&= c\int_0^T H(u,s,x(s))ds + \sum_{k=1}^{p} I_k(x(t_k)) \quad (\text{since } a > t_p) \\
&\geq c\{\int_0^T H(u,s,x(s))ds + \sum_{0 < t_k < u} I_k(x(t_k))\} = cAx(u),
\end{aligned}$$

hence $Ax \in K$, and so

$$A(K) \subset K. \tag{2}$$

Choose

$$M > (b-a)^{-1}. \tag{23}$$

By (H_5), there exists $h > 0$ such that

$$g(H(t,s,x)) \geq Mg(x),\ t,\ s \in J_0,\ \ x \in P,\ \| x \| \geq h. \tag{24}$$

Now, for any

$$R > Nhc^{-1} \text{ (N-normal constant of P)}, \tag{25}$$

we are going to show that

$$Ax \nleq x \text{ for } x \in K, \ \| x \|_p = R. \tag{26}$$

In fact, if there exists $x_0 \in K$ with $\| x_0 \|_p = R$ such that $Ax_0 \leq x_0$, then $x_0(t) \geq cx_0(s)$, and so

$$N \| x_0(t) \| \geq c \| x_0(s) \|, \ t \in J_0, \ s \in J,$$

which implies by (25)

$$\| x_0(t) \| \geq cN^{-1} \| x_0 \|_p = cN^{-1}R > h, \ t \in J_0. \tag{27}$$

Also, we have

$$x_0(t) \geq Ax_0(t) \geq \int_a^b H(t,s,x_0(s))ds, \ t \in J_0. \tag{28}$$

It follows from (28), (27) and (24) that

$$g(x_0(t)) \geq \int_a^b g(H(t,s,x_0(s))ds \geq M \int_a^b g(x_0(s))ds, \ t \in J_0,$$

and so

$$\int_a^b g(x_0(t))dt \geq M(b-a) \int_a^b g(x_0(s))ds. \tag{29}$$

It is easy to see that

$$\int_a^b g(x_0(t))dt > 0. \tag{30}$$

In fact, if this integral equals to zero, then $g(x_0(t)) = 0$, and so $x_0(t) = \theta$ for any $t \in J_0$, which implies by virtue of $x_0 \in K$ that $x_0(s) = \theta$ for $s \in J$, in contradiction with $\| x_0 \|_p = R$. Now, (29) and (30) imply $M(b-a) \leq 1$, which contradicts (23), and therefore (26) is true.

On the other hand, on account of (H_2) and $H(t,s,\theta) = \theta$, $I_k(\theta) = \theta$, we can find a $r > 0$ $(r < R)$ such that

$$\| H(t,s,x) \| \leq m \| x \|, \ x \in P, \ \| x \| \leq r; \ t,s \in J \tag{31}$$

and

$$\| I_k(x) \| \leq m \| x \|, \ x \in P, \ \| x \| \leq r; \ k = 1,2,\ldots,p, \tag{32}$$

where

$$m = [2N(T+p)]^{-1}. \tag{33}$$

Now, we verify that

$$Ax \not\geq x \text{ for } x \in K, \ \| x \|_p = r. \tag{34}$$

In fact, if there is $x_1 \in K$, $\| x_1 \|_p = r$ such that $Ax_1 \geq x_1$, then

$$\theta \leq x_1(t) \leq \int_0^T H(t,s,x_1(s))ds + \sum_{0 < t_k < t} I_k(x_1(t_k)), \ t \in J,$$

and so, by (31) and (32)

$$\| x_1(t) \| \leq Nm(\int_0^T \| x_1(s) \| ds + \sum_{k-1}^{p} \| x_1(t_k) \|)$$

$$\leq Nm(T+p) \| x_1 \|_p, \ t \in J,$$

hence,

$$r = \| x_1 \|_p \leq Nm(T+p) \| x_1 \|_p = Nm(T+p)r,$$

which contradicts (33), and therefore (34) is true.

By Lemma 2, A is a strict set contraction on $K_{r,R} = \{x \in K: r \leq \| x \|_p \leq R\}$. Observing (22), (26) and (34) and using Lemma 3, we see that A has a fixed point in $K_{r,R}$, which is a positive solution of (1). The proof is complete.

Theorem 2: *Let cone P be normal. Suppose that conditions (H_1), (H_3) and (H_4) are satisfied. Then, equation (1) has at least one positive solution.*

Proof: The proof is similar. First, (22) holds. In the same way as establishing (26) we can show: (H_4) implies that there exists $r > 0$ such that

$$Ax \not\leq x \text{ for } x \in K, \ \| x \|_p = r. \tag{35}$$

On the other hand, by (H_3) there is a $q > 0$ such that

$$\| H(t,s,x) \| \leq m \| x \|, \ x \in P, \ \| x \| \geq q, \ t, \ s \in J$$

and

$$\| I_k(x) \| \leq m \| x \|, \ x \in P, \ \| x \| \geq q; \ k = 1,2,\ldots,p,$$

where

$$m = [2N(T+p)]^{-1}.$$

Consequently,

$$\| H(t,s,x) \| \leq m \| x \| + M, \ x \in P, \ t,s \in J$$

and

$$\| I_k(x) \| \leq m \| x \| + M, \ x \in P,$$

where, by (H_1),

$$M = max\left\{ \sup_{J \times J \times (P \cap T_q)} \| H(t,s,x) \| , \sup_{P \cap T_q} \| I_k(x) \| , \ k = 1,2,\ldots,p \right\} < \infty.$$

Choose $R > max\{r, 2NM(T+p)\}$. Then, it is easy to show as establishing (34) that

$$Ax \not\geq x \text{ for } x \in K, \ \| x \|_p = R. \tag{36}$$

Hence, again Lemma 3 implies that A has a fixed point in $K_{r,R}$.

Remark 1: In particular case of one dimensional space, $E = R$, $P = R_+$ and $P^* = P = R_+$. In this case, (10) and (11) are satisfied automatically and we may choose $g = 1$ in (H_4) and (H_5). Hence, from Theorems 1 and 2, we get the following

Conclusion: Let $H \in C[J \times J \times R_+, R_+]$, $H(t,s,0) = 0$, $I_k \in C[R_+, R_+]$, $I_k(0) = 0$ $(k = 1,2,\ldots,p)$ and there exist $t_p < a < b < T$ and $0 < c < 1$ such that

$$H(t,s,x) \geq cH(u,s,x), \ t \in J_0 = [a,b], \ u,s \in J, \ x \geq 0.$$

Suppose that one of the following two conditions is satisfied:

$a)$ $H(t,s,x)/x \to 0$ as $x \to +0$ uniformly in $t,s \in J$; $I_k(x)/x \to 0$ as $x \to +0$ $(k = 1,2,\ldots,p)$,

and $H(t,s,x)/x \to +\infty$ as $x \to +\infty$ uniformly in $t,s \in J_0$.

$b)$ $H(t,s,x)/x \to 0$ as $x \to +\infty$ uniformly in $t,s \in J$; $I_k(x)/x \to 0$ as $x \to +\infty$ $(k = 1,2,\ldots,p)$, and $H(t,s,x)/x \to +\infty$ as $x \to +0$ uniformly in $t,s \in J_0$.

Then, equation (1) has at least one positive solution.

4. APPLICATIONS

Consider the two-point boundary value problem for second order impulsive differential equation in E:

$$\begin{cases} -x'' = f(t,x), t \neq t_k (k = 1,2,\ldots,p), \\ \Delta x \mid_{t = t_k} = I_k(x), (k = 1,2,\ldots,p), \\ x(0) = x'(1) = \theta, \end{cases} \tag{37}$$

where $0 < t_1 < \ldots < t_k < \ldots < t_p < 1$, $f \in C[J \times P, P]$, $J = [0,1]$, $f(t,\theta) = \theta$, $I_k \in C[P,P]$, $I_k(\theta) = \theta$ $(k = 1,2,\ldots,p)$. Evidently, $x(t) \equiv \theta$ is the trivial solution of (37). Let $J' = J \backslash \{t_1, t_2, \ldots, t_p\}$. A map $x \in PC[J,E] \cap C^2[J',E]$ is called a positive solution of (37) if it satisfies (37) and $x \in Q, x \neq \theta$ (Q is defined as before, i.e., $Q = \{x \in PC[J,E]: x(t) \geq \theta$ for

$t \in J\}$).

We list the following conditions:

(H_1'): $f \in C[J \times P, P], J = [0,1], f(t,\theta) = \theta$ for $t \in J$, $I_k \in C[P,P]$, $I_k(\theta) = \theta$ $(k = 1,2,\ldots,p)$. For any $r > 0$, f is uniformly continuous on $J \times (P \cap T_r)$, I_k is bounded on $P \cap T_r$, and there exist nonnegative constants L_r and $M_r^{(k)}$ with

$$2TL_r + \sum_{k=1}^{p} M_r^{(k)} < 1$$

such that

$$\alpha(f(t,D)) \leq L_r \alpha(D), \ t \in J, \ D \subset P \cap T_r$$

and

$$\alpha(I_k(D)) \leq M_r^{(k)} \alpha(D), \ D \subset P \cap T_r, \ k = 1,2,\ldots,p.$$

(H_2'): $\| f(t,x) \| / \| x \| \to 0$ as $x \in P$ and $\| x \| \to 0$ uniformly in $t \in J$; $\| I_k(x) \| / \| x \| \to 0$ as $x \in P$ and $\| x \| \to 0$ $(k = 1,2,\ldots,p)$.

(H_3'): $\| f(t,x) \| / \| x \| \to 0$ as $x \in P$ and $\| x \| \to \infty$ uniformly in $t \in J$; $\| I_k(x) \| / \| x \| \to 0$ as $x \in P$ and $\| x \| \to \infty$ $(k = 1,2,\ldots p)$.

(H_4'): there exists $t_p < a < b < 1$ and $g \in P^*$ such that $g(x) > 0$ for any $x > \theta$ and $g(f(t,x))/g(x) \to \infty$ as $x \in P$ and $\| x \| \to 0$ uniformly in $t \in J_0 = [a,b]$.

(H_5'): there exists $t_p < a < b < 1$ and $g \in P^*$ such that $g(x) > 0$ for any $x > \theta$ and $g(f(t,x))/g(x) \to \infty$ as $x \in P$ and $\| x \| \to \infty$ uniformly in $t \in J_0 = [a,b]$.

Theorem 3: *Let cone P be normal. Suppose that conditions $(H_1'), (H_2')$ and (H_5') are satisfied. Then, BVP (37) has at least one positive solution.*

Theorem 4: *Let cone P be normal. Suppose that conditions $(H_1'), (H_3')$ and (H_4') are satisfied. Then, BVP (37) has at least one positive solution.*

Proof of Theorems 3 and 4: It is easy to see that $x \in PC[J,E] \cap C^2[J',E]$ is a positive solution of (37) if and only if $x \in PC[J,E]$ is a positive solution of the following impulsive integral equation:

$$x(t) = \int_0^1 G(t,s) f(s,x(s)) ds + \sum_{0 < t_k < t} I_k(x(t_k)), \tag{38}$$

where $G(t,s)$ is the Green function for the differential operator $-x''$ under boundary condition $x(0) = x'(1) = 0$, i.e.,

$$G(t,s) = min\{t,s\} = \begin{cases} t, t \leq s; \\ s, t > s. \end{cases}$$

Evidently, (38) is an equation of the form (1) with $T = 1$ and $H(t,s,x) = G(t,s)f(s,x)$. It is easy to show that $0 \leq G(t,s) \leq 1$ $(t,s \in J = [0,1])$, and for any $t_p < a < b < 1$,

$$G(t,s) \geq aG(u,s), \ t \in [a,b], u, \ s \in J.$$

Hence, (21) is satisfied for any $t_p < a < b < 1$ with $c = a$. Consequently, it is clear that (H_1'), (H_2') and (H_5') imply (H_1) , (H_2) and (H_5), and (H_1'), (H_3') and (H_4') imply (H_1), (H_3) and (H_4). Thus, Theorems 3 and 4 follow from Theorems 1 and 2 respectively.

Remark 2: In one dimensional case, $E = R$, $P = P^* = R_+$ and $g = 1$, we get the following result from Theorems 3 and 4:

Conclusion: Let $f \in C[J \times R_+, R_+]$, $J = [0,1]$, $f(t,0) = 0$, $I_k \in C[R_+, R_+]$, $I_k(0) = 0$ $(k = 1,2,\ldots,p)$. Suppose that one of the following two conditions is satisfied:

$a')$ $f(t,x)/x \to 0$ as $x \to +0$ uniformly in $t \in J$, $I_k(x)/x \to 0$ as $x \to +0$ $(k = 1,2,\ldots,p)$ and there exist $t_p < a < b < 1$ such that $f(t,x)/x \to +\infty$ as $x \to +\infty$ uniformly in $t \in [a,b]$.

$b')$ $f(t,x)/x \to 0$ as $x \to +\infty$ uniformly in $t \in J$, $I_k(x)/x \to 0$ as $x \to +\infty$ $(k = 1,2,\ldots,p)$ and there exist $t_p < a < b < 1$ such that $f(t,x)/x \to +\infty$ as $x \to +0$ uniformly in $t \in [a,b]$.

Then, BVP (37) has at least one positive solution.

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