

## CONTROLLABILITY OF NONLINEAR VOLTERRA INTEGRODIFFERENTIAL SYSTEMS WITH PRESCRIBED CONTROLS<sup>1</sup>

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### ABSTRACT

Sufficient conditions for the global controllability of nonlinear Volterra integrodifferential systems with prescribed controls are derived. The method is a transformation of the given control system into a boundary value problem and then the result is obtained by the application of the Schaefer fixed point theorem.

**Key words:** Controllability, Volterra Integrodifferential Systems, Boundary Value Problem, Fixed Point Theorem.

**AMS (MOS) subject classifications:** 93B05.

### 1. INTRODUCTION

Controllability of nonlinear systems has been studied by several authors by means of fixed point principles [4]. Anichini [1,2], Dauer [7], Kartsatos [8] and Lukes [9] respectively studied the controllability of nonlinear systems with prescribed controls by means of Schaefer's theorem, Fan's theorem, Tychonov's theorem and the Schauder theorem. With the use of Fan's fixed point theorem, Dauer [6] established sufficient conditions for the controllability of nonlinear systems between two fixed points. Chukwu and Silliman [5] examined the problem for nonlinear systems to a closed target set. Balachandran [3] studied the problem for the nonlinear Volterra integrodifferential systems.

In this paper, we seek to find sufficient conditions for the nonlinear control process,

$$\dot{x} = A(t)x + \int_0^t H(t,s)x(s)ds + B(t)u(t) + g(t,x(t)) \quad (1)$$

is completely controllable in a finite interval  $[t_0, T]$  by means of controls whose initial and final values can be assigned in advance. That is, we want to find conditions upon  $A(t)$ ,  $B(t)$ ,

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$H(t, s)$ ,  $g(t, x)$  which ensure that, for each  $t_0, T \in \mathbb{R}$ ,  $\alpha, \beta \in \mathbb{R}^m$ ,  $x_0, x_T \in \mathbb{R}^n$ , there exists a control  $u \in C([t_0, T]; \mathbb{R}^m)$  of (1), with  $u(t_0) = \alpha$ ,  $u(T) = \beta$  which produces a response  $x(t, u)$  satisfying the boundary conditions  $x(t_0, u) = x_0$  and  $x(T, u) = x_T$ .

This result will be obtained by an application of a fixed point argument to the linear boundary value problem

$$\begin{aligned} \dot{x} &= A(t)x + \int_0^t H(t, s)x(s)ds + B(t)u(t) + g(t, z(t)) & (2) \\ x(t_0) &= x_0, \quad x(T) = x_T \\ u(t_0) &= \alpha, \quad u(T) = \beta \end{aligned}$$

where  $z \in C([t_0, T]; \mathbb{R}^n)$ , the space of continuous functions with sup norm. For brevity let us take  $t_0 = 0$ ,  $\alpha = u_0$  and  $\beta = u_T$ .

This result generalizes the results of Anichini [1].

## 2. PRELIMINARIES

Consider the nonlinear Volterra integrodifferential system

$$\dot{x}(t) = A(t)x(t) + \int_0^t H(t, s)x(s)ds + B(t)u(t) + g(t, x(t))$$

where the state  $x$  is an  $n$ -vector and the control  $u$  is an  $m$ -vector.

The matrix functions  $A: J \rightarrow \mathbb{R}^{n^2}$ ,  $B: J \rightarrow \mathbb{R}^{nm}$ ,  $J = [0, T]$ ,  $H: \Delta \rightarrow \mathbb{R}^{n^2}$ ,  $\Delta = \{(t, s): 0 \leq t \leq s \leq T\}$  are assumed to be continuous. The function  $g: [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is such that  $g(t, x) \in C^1(J \times \mathbb{R}^n, \mathbb{R}^n)$ . The norm of a matrix and a function are taken as in [1]. Assume that, for  $t \in J$ , and  $(t, s) \in \Delta$ ,  $\|A(t)\| \leq a$ ,  $\|B(t)\| \leq b$ ,  $\|H(t, s)\| \leq c$  and  $\|g(t, x)\| \leq \gamma(t)$ , where  $\gamma(t)$  is a continuous function and  $a, b, c$  are positive constants. We observe that the hypothesis on  $A(t)$  and  $H(t, s)$  allow us to say that there exists a unique continuous matrix  $E(t, s)$  such that

$$\frac{\partial E(t, s)}{\partial s} + E(t, s)A(s) + \int_0^t E(t, w)H(w, s)dw = 0$$

with  $E(t, t) = I$ , the identity matrix.

Following Anichini [1,2], we say that (1) is  $[0, T]$ -controllable by means of a certain set  $U$  of controls iff, for every pair  $x_0, x_T \in \mathbb{R}^n$ , there exists  $u \in U$ , such that  $x(0, u) = x_0$ ,

$$x(T, u) = x_T.$$

For brevity let us denote

$$P(t; \theta) = \int_0^t E(\theta, \theta - s)B(\theta - s)ds$$

$$\bar{C}(t; T) = \int_{T-t}^T P(s; T)^* ds - (t/T) \int_0^T P(s; T)^* ds$$

$$S(t; T) = \int_0^t E(t, s)B(s)\bar{C}(s; T)ds$$

and define

$$M(0, t) = \int_0^t B(s)B(s)^* ds$$

$$\bar{S}(T) = \int_0^T P(s; \theta)P(s; \theta)^* ds - (1/T) \left[ \int_0^T P(s; \theta) ds \right] \left[ \int_0^T P(s; \theta)^* ds \right]$$

where the star denotes the matrix transpose. Observe that  $P(t; \theta)$ ,  $\bar{C}(t; T)$  and  $S(t; T)$  are continuous.

We use the following theorem, due to Schaefer.

**Proposition 1:** *Let  $X$  be a normed space,  $T$  a continuous mapping of  $X$  into  $X$ , which is compact on each bounded subset of  $X$ . Then, either (i) the equation  $x = \lambda T(x)$  has a solution for  $\lambda = 1$ , or (ii) the set of all such solutions  $x$  for  $0 < \lambda < 1$ , is unbounded.*

The following theorem is vital to the criterion of controllability.

**Theorem 1:** *Assume that the control process (1) satisfies the hypotheses. If the matrix  $M(0, t_1)$  is non-singular for some  $t_1 > 0$ , then the set of points attainable by the trajectories of the control process (1) is all of  $\mathbb{R}^n$ .*

**Proof:** For fixed  $u$ , the given system has a solution  $x(t; u)$  which satisfies

$$x(t; u) = x_0 + \int_0^t A(s)x(s; u)ds + \int_0^t \left[ \int_0^s H(s, \tau)x(\tau; u)d\tau \right] ds$$

$$+ \int_0^t B(s)u(s)ds + \int_0^t g(s, x(s; u))ds. \tag{3}$$

Let  $x_1$  be any given point in  $\mathbb{R}^n$ . We have to find a control  $v$  such that for some finite point  $t_1 > 0$ ,  $x(t_1; v) = x_1$ . Consider the controls of the form  $v(t) = B(t)^*q$ , where  $q \in \mathbb{R}^n$ . Define a mapping  $S: \mathbb{R}^n \rightarrow \mathbb{R}^n$  as follows:

$$S(q) = M^{-1}(0, t_1)[x_1 - K(q) - x_0] \text{ where}$$

$$K(q) = \int_0^{t_1} A(s)x(s; q)ds + \int_0^{t_1} \left[ \int_0^s H(s, \tau)x(\tau, q)d\tau \right] ds + \int_0^{t_1} g(s, x(s; q))ds.$$

Suppose that the mapping  $q \rightarrow S(q)$  has a fixed point. Then  $q = M^{-1}(0, t_1)[x_1 - K(q) - x_0]$  and from (3) it is very easy to verify that  $x(t_1, q) = x_1$ .

Now we shall prove the mapping  $q \rightarrow S(q)$  has a fixed point. Since all the functions involved in the definition of the operator  $S$  are continuous, this mapping is continuous. From (3) we have

$$\begin{aligned} \|x(t; u)\| &\leq \|x_0\| + \int_0^t a \|x(s; u)\| ds + \int_0^T \left[ \int_0^s c \|x(\tau; u)\| d\tau \right] ds \\ &\quad + \int_0^T \sup\{b \|u(s)\|, s \in [0, T]\} ds + \int_0^T \gamma(s) ds \\ &\leq \alpha_0 + \int_0^t \left[ a + \int_0^s c d\tau \right] \|x(\tau; u)\| ds \end{aligned}$$

where

$$\alpha_0 = \|x_0\| + b \int_0^T \sup\{\|u(s)\|, s \in [0, T]\} ds + \int_0^T \gamma(s) ds.$$

Therefore by Gronwall's inequality

$$\begin{aligned} \|x(t; u)\| &\leq \alpha_0 \exp\left[ \int_0^t \left( a + \int_0^s c d\tau \right) ds \right] \\ &\leq \alpha_0 \exp(aT + cT^2/2). \end{aligned}$$

Thus if  $\|q\| < +\infty$ , then  $\|x(t; u)\| < +\infty$ , which implies that  $\|K(q)\| < +\infty$  and hence  $\|S(q)\| < +\infty$ . Thus  $S(q)$  sends bounded sets into bounded sets. By a similar argument, we can show that the solutions of the equation  $q = \lambda S(q)$ , for  $0 < \lambda < 1$ , are bounded. Then by

Schaefer's theorem the mapping has a fixed point.

### 3. MAIN RESULT

For  $z \in F = C(J, \mathbb{R}^n)$ , consider

$$\begin{aligned} x_z(t) = & E(t, 0)x_0 + P(t; t)u_0 + (1/T)(u_T - u_0) \int_0^t P(s; t)ds \\ & + S(t; T)Y_z(T) + \int_0^t E(t, s)g(s, z(s))ds \end{aligned} \quad (4)$$

$$u_z(t) = (1 - t/T)u_0 + (t/T)u_T + \bar{C}(t; T)Y_z(T) \quad (5)$$

where

$$\begin{aligned} Y_z(t) = & [\bar{S}(T)]^{-1}[x_T - E(T, 0)x_0 - P(T; t)u_0 - (1/T)(u_T - u_0) \\ & \times \int_0^T P(s; t)ds - \int_0^T E(T, s)g(s, z(s))ds]. \end{aligned}$$

By a similar argument parallel to that of [1], we have the following propositions.

**Proposition 2:**

$$\int_0^t E(t, s)B(s)u_z(s)ds = P(t; t)u_0 + (1/T)(u_T - u_0) \int_0^t P(s; t)ds + S(t; T)Y_z(T)$$

and  $S(T; T) = \bar{S}(T)$ .

**Proposition 3:** Consider the boundary value control process

$$\dot{x}(t) = A(t)x(t) + \int_0^t H(t, s)x(s)ds + B(t)u(t) + g(t, z) \quad (6)$$

with

$$x(0) = x_0, \quad x(T) = x_T$$

$$u(0) = u_0, \quad u(T) = u_T.$$

Then, if the matrix  $M(0, T)$  is nonsingular, every pair  $(x_z(t), u_z(t))$  defined in (4) and (5) provides a solution of boundary value control process (6).

Now we shall prove the main result on the controllability of nonlinear Volterra integrodifferential system.

**Theorem 2:** *Assume that the nonlinear control system (1) satisfies the hypotheses and that the matrix  $M(0, T)$  is nonsingular for  $T > 0$ . Then, for every  $\alpha, \beta, \gamma \in \mathbb{R}^m$ ,  $x_0, x_1, x_T \in \mathbb{R}^n$ , and every  $\omega \in [0, T]$ , there exists a control  $v$ , such that*

$$(a) \quad v(0) = \alpha, \quad v(\omega) = \beta, \quad v(T) = \gamma$$

$$(b) \quad \text{the response of (1), for which } x(0; v) = x_0, \text{ satisfies } x(\omega; v) = x_1, \quad x(T; v) = x_T.$$

**Proof:** Consider the mapping  $q: z \in F \rightarrow q(z) = x_z \in F$  where  $x_z = x_z(t)$  and  $x_z(t)$  and  $u_z(t)$  are defined in (4) and (5) respectively. Then the proof is based upon two applications of Proposition 3.

First setting  $\omega = T$ ,  $u_0 = \alpha$ ,  $u_T = \beta$ ,  $x_T = x_1$  we can obtain a response  $x(t; v)$  of (1) such that  $x(0; v) = x_0$  and  $x(\omega, v) = x_1$ .

Then, setting  $u_0 = \beta$ ,  $x_0 = x_1$ ,  $u_T = \gamma$  we can obtain a response  $x(t; v)$  of (1) such that  $x(\omega; v) = x_1$  and  $x(T; v) = x_T$ .

Thus we extend the response  $x(t; v)$  to whole interval  $[0, T]$  and hence the theorem is proved.

To show that the mapping  $q$  has a fixed point, we use Schaefer's theorem.

Since  $E(\cdot, \cdot)$ ,  $P(\cdot; \cdot)$  and  $\bar{S}(\cdot)$  are continuous and  $g(\cdot, z)$  is continuous with respect to  $z$  we can say that  $z \rightarrow Y_z(t)$  is continuous function with respect to  $z$ . Thus the map  $z \rightarrow x_z$  is continuous. Assume that  $\|z\| \leq r$ ,  $0 < r < +\infty$ . Then,

$$\|E(t, s)\| \leq \exp(a\omega + c\omega^2/2)$$

$$\|P(t; t)\| \leq b\omega \exp(a\omega + c\omega^2/2)$$

$$\|\bar{C}(t; T)\| \leq 2b\omega^2 \exp(a\omega + c\omega^2/2)$$

$$\|S(t; T)\| \leq 2b^2\omega^3 \exp(a\omega + c\omega^2/2) \equiv a_1$$

$$\|Y_z(\omega)\| \leq a_1[\|x_1\| + \exp(a\omega + c\omega^2/2)\|x_0\|$$

$$+ b\omega \exp(a\omega + c\omega^2/2)\alpha$$

$$+ |\alpha - \beta| b\omega \exp(a\omega + c\omega^2/2)$$

$$+ \omega \exp(a\omega + c\omega^2/2)\gamma]$$

$$\equiv a_2.$$

Thus, we have

$$\begin{aligned} \|q(z)\| = \|x_z(t)\| &\leq \|x_0\| \exp(a\omega + c\omega^2/2) + \alpha b\omega \exp(a\omega + c\omega^2/2) \\ &+ |\alpha - \beta| b\omega \exp(a\omega + c\omega^2/2) + a_1 a_2 \\ &+ \omega \exp(a\omega + c\omega^2/2) \gamma \equiv a_3. \end{aligned}$$

Let us now estimate,

$$\begin{aligned} \|x_z(t_1) - x_z(t_2)\| &\leq \|E(t_1, 0) - E(t_2, 0)\| \|x_0\| \\ &+ \|P(t_1; t_1) - P(t_2; t_2)\| |\alpha| \\ &+ \frac{|\alpha - \beta|}{T} \left\| \int_{t_2}^{t_1} (P(s; t_1) - P(s; t_2)) ds \right\| \\ &+ \|Y_z(\omega)\| \|S(t_1; \omega) - S(t_2; \omega)\| \\ &+ \left\| \int_{t_2}^{t_1} (E(t_1, s) - E(t_2, s)) g(s, z(s)) ds \right\|. \end{aligned} \tag{7}$$

From the previous inequalities, we have

$$\begin{aligned} \|E(t_1, 0) - E(t_2, 0)\| &\leq |t_1 - t_2| (a + c\omega) \exp(a\omega + \frac{c\omega^2}{2}) \\ \|P(t_1; t_1) - P(t_2; t_2)\| &\leq |t_1 - t_2| b \exp(a\omega + \frac{c\omega^2}{2}) \\ \left\| \int_{t_2}^{t_1} (P(s; t_1) - P(s; t_2)) ds \right\| &\leq |t_1 - t_2|^2 b \exp(a\omega + \frac{c\omega^2}{2}) \\ \|S(t_1; \omega) - S(t_2; \omega)\| &\leq |t_1 - t_2|^2 2(a + c\omega) b^2 \omega^2 \exp(2(a\omega + \frac{c\omega^2}{2})) \\ \left\| \int_{t_2}^{t_1} (E(t_1, s) - E(t_2, s)) g(s, z(s)) ds \right\| &\leq |t_1 - t_2|^2 \gamma (a + c\omega) \exp(a\omega + \frac{c\omega^2}{2}). \end{aligned}$$

Therefore,

$$\begin{aligned} \|x_z(t_1) - x_z(t_2)\| &\leq \|x_0\| |t_1 - t_2| (a + c\omega) \exp(a\omega + \frac{c\omega^2}{2}) \\ &+ |\alpha| |t_1 - t_2| b \exp(a\omega + \frac{c\omega^2}{2}) \\ &+ \frac{|\alpha - \beta|}{\omega} |t_1 - t_2|^2 b \exp(a\omega + \frac{c\omega^2}{2}) \\ &+ |t_1 - t_2|^2 (a + c\omega) \gamma \exp(a\omega + \frac{c\omega^2}{2}) \end{aligned}$$

$$+ a_2 |t_1 - t_2|^2 2b^2 \omega^2 (a + c\omega) \exp 2(a\omega + \frac{c\omega^2}{2}).$$

Thus  $z \rightarrow q(z)$  is an equibounded and equicontinuous mapping. Since the solutions of the equation  $z = \lambda q(z)$  are bounded for  $0 < \lambda < 1$ , then by Schaefer's theorem,  $q$  has a fixed point.

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