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EXISTENCE OF SOLUTION FOR A MIXED NEUTRAL SYSTEM[1]

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ABSTRACT

We prove the existence of unique solution of the mixed neutral system

$$x'(t) = f(t,x) + \sum_{j=1}^{m} A_j(t,x)x'(t+p_j) + g(t,x,x'(t+h))$$
$$x(0) = x_0$$

and also prove the continuous dependence of the solution.

Key words: Functional differential equations, existence and uniqueness.

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1. INTRODUCTION

A differential system in which the expression for $x'(t)$ involves $x'(h(t))$ for some $h(t) \neq t$ is said to be of neutral type. A system of first order functional differential equations in which the present value of $x'(t)$ is expressed in terms of both past and future values of x is said to be of mixed type. So when both of these characteristics are present, the system is of mixed neutral type, or simply a mixed neutral system.

In this paper we consider a mixed neutral system of the form

$$x'(t) = f(t,x) + \sum_{j=1}^{m} A_j(t,x)x'(t+p_j) + g(t,x,x'(t+h)) \tag{1}$$

where f is an n-vector valued function and each A_j is an $n \times n$ matrix valued function defined on $R \times C(R,R^n) \times C(R,R^n)$ and each p_j and h are constant real numbers.

The literature contains many papers on the problem (1) in the case when $A_j = 0$ and

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$g = 0$ ([1], [3], [4], [6]-[9]). Driver [2] proved the existence and continuous dependence of solutions of a neutral functional differential equation. But little is known when the mixed equation is also of neutral type. In [5], he proved the existence of unique solution and its continuous dependence for the mixed neutral system (1) when $g = 0$.

In this paper we shall prove the existence of unique solution of the mixed neutral system (1), and also prove that the solution of (1) depends continuously on the initial condition x_0. System (1) is a greatly over-simplified model of a mixed neutral system arising in a two-body problem of classical electrodynamics.

2. BASIC ASSUMPTIONS

First we define the term solution for the system (1). A function $x: R \to R^n$ which is absolutely continuous locally and satisfies (1) almost everywhere is a solution of the mixed neutral system (1).

Let $|\cdot|$ be a chosen norm in R^n and let $\|\cdot\|$ be the corresponding induced matrix norm. Let $p = max\{max\,|p_j|, h\}$, and assume the existence of positive constants M_f, M_A, M_g, K_f, K_A, K_g, and N_g such that for $t \in R$ and x, $\bar{x}$, $\bar{x}' \in C(R, R^n)$

(*i*) f and each A_j are continuous on $R \times C(R, R^n)$ and g is continuous on $R \times C(R, R^n) \times C(R, R^n)$,

(*ii*) $|f| \le M_f$ and each $\|A_j\| \le M_A$ on $R \times C(R, R^n)$ and $|g| \le M_g$ on $R \times C(R, R^n) \times C(R, R^n)$,

(*iii*) $|f(t,x) - f(t,\bar{x})| \le K_f \max_{t-p \le s \le t+p} |x(s) - \bar{x}(s)|$,
$\|A(t,x) - A(t,\bar{x})\| \le K_A \max_{t-p \le s \le t+p} |x(s) - \bar{x}(s)|$ and
$|g(t,x,x'(t+h)) - g(t,\bar{x},\bar{x}'(t+h))| \le K_g \max_{t-p \le s < t+p} |x(s) - \bar{x}(s)|$
$+ N_g \max_{t-p \le s \le t+p} |x'(s+h) - \bar{x}'(s+h)|$

(*iv*) the constants p, M_f, M_A, M_g, K_f, K_A, K_g, and N_g are sufficiently small such that

$$e^{ap}\Big[\frac{1}{a}\Big(K_f + K_g + \frac{mK_A(M_f + M_g)}{1 - mM_A}\Big) + mM_A + N_g\Big] < 1 \tag{2}$$

for some constant $a > 0$, and

(*v*) the solution of (1) satisfies the conditions

$$\int_t^{t+1} |x'(s)|\, ds \text{ bounded for } t \in R.$$

Remark 1: If x is a solution of (1) satisfying the assumption (*v*) and if $B = \sup_{t \in R}$

$\int_{t-r}^{t+r} | x'(s) | ds$ for some choice of $r > 0$ then for each t,

$$\int_{t-r}^{t+r} | x'(s) | ds \le \int_{t-r}^{t+r} | f(s,x) | ds + \sum_{j=1}^{m} \int_{t-r}^{t+r} \| A_j(s,x) \| \, | x'(s+p_j) | ds$$

$$+ \int_{t-r}^{t+r} | g(s,x,x'(s+h) | ds$$

$$\le 2rM_f + mM_A B + 2rM_g.$$

So if $mM_A < 1$,

$$B \le \frac{2r(M_f + M_g)}{1 - mM_A}.$$

3. EXISTENCE AND UNIQUENESS

Theorem 1: *Under the assumptions $(i)-(v)$ there exists a unique solution for the mixed neutral system* (1).

Proof: Let $mM_A < 1$ and choose $r > 0$. Define the set

$$S = \{\omega \in L^1_{loc}(R, R^n): \int_{t-r}^{t+r} | \omega(s) | ds \le 2r(M_f + M_g)/(1 - mM_A) \text{ for all } t\}$$

Then S is the space of allowable derivatives of solutions of (1).

For any constant $a > 0$, define the metric

$$d(\omega, \bar{\omega}) = \sup_{t \in R} [e^{-a|t|} \int_{t-r}^{t+r} | \omega(s) - \bar{\omega}(s) | ds] \text{ for } \omega, \bar{\omega} \in S.$$

Clearly (S,d) is a complete metric space. For $\omega \in S$, define $x(t) = x_0 + \int_0^t \omega(s)ds$ for all t and then

$$(T\omega)(t) = f(t,x) + \sum_{j=1}^{m} A_j(t,x)\omega(t+p_j) + g(t,x,\omega(t+h))$$

for all t. Now

$$\int_{t-r}^{t+r} | (T\omega)(s) | ds \le 2rM_f + mM_A \frac{2r(M_f + M_g)}{1 - mM_A} + 2rM_g$$

$$\le \frac{2r(M_f + M_g)}{1 - mM_A}.$$

Therefore T maps S into S. Now let $\omega, \bar{\omega} \in S$. Then for $t \ge 0$

$$\int_0^t |\omega(s)-\bar{\omega}(s)|\,ds = \int_0^{2r} + \int_{2r}^{4r} + \cdots + \int_{2r[t/2r]}^{t} |\omega(s)-\bar{\omega}(s)|\,ds$$
$$\le d(\omega,\bar{\omega})(e^{ar}+e^{3ar}+\ldots+e^{2r+2ar[t/2r]},$$
$$\le d(\omega,\bar{\omega})\frac{e^{3ar+at}}{e^{2ar}-1}.$$

Similarly, for $t \le 0$

$$\int_t^0 |\omega(s)-\bar{\omega}(s)|\,ds \le d(\omega,\bar{\omega})\frac{e^{3ar+a|t|}}{e^{2ar}-1}.$$

Using this,

$$|(T\omega)(t)-(T\bar{\omega})(t)| \le |f(t,x)-f(t,\bar{x})| + \sum_{j=1}^{m} \|A_j(t,x)-A_j(t,\bar{x})\|\,|\omega(t+p_j)|$$
$$+\sum_{j=1}^{m}\|A_j(t,\bar{x})\|\,|\omega(t+p_j)-\bar{\omega}(t+p_j)|$$
$$+|g(t,x,\omega(t+h))-g(t,\bar{x},\bar{\omega}(t+h))|$$
$$\le \max_{t-p\le s\le t+p}\{\int_{t-p}^{0}|\omega(s)-\bar{\omega}(s)|\,ds, \int_0^{t+p}|\omega(s)-\bar{\omega}(s)|\,ds\}$$
$$\times[K_f+K_g+K_A\sum_{j=1}^{m}|\omega(t+p_j)|]+\sum_{j=1}^{m}M_A|\omega(t+p_j)-\bar{\omega}(t+p_j)|$$
$$+N_g \max_{t-p\le s\le t+p}|\omega(s+h)-\bar{\omega}(s+h)|$$
$$\le d(\omega,\bar{\omega})\frac{e^{3ar+ap+a|t|}}{e^{2ar}-1}[K_f+K_g+K_A\sum_{j=1}^{m}|\omega(s+p_j)|]$$
$$+\sum_{j=1}^{m}M_A|\omega(s+p_j)-\bar{\omega}(s+p_j)|+N_g\max_{t-p\le s\le t+p}|w(s+h)-\bar{\omega}(s+h)|.$$

Thus for each t

$$\int_{t-r}^{t+r}|(T\omega)(s)-(T\bar{\omega})(s)|\,ds \le d(\omega,\bar{\omega})\frac{e^{4ar+ap+a|t|}}{e^{2ar}-1}[2r(K_f+K_g)+\frac{mK_A(M_f+M_g)}{1-mM_A}]$$
$$+mM_A d(\omega,\bar{\omega})e^{a|t|+ap}+N_g d(\omega,\bar{\omega})e^{a|t|+ap}.$$

Therefore $d(T\omega,T\bar{\omega}) \le \rho d(\omega,\bar{\omega})$ where

$$\rho = e^{ap}[\frac{2re^{4ar}}{e^{2ar}-1}(K_f+K_g+\frac{mK_A(M_f+M_g)}{1-mM_A})+mM_A+N_g]. \tag{3}$$

By the contraction mapping theorem, T will have a unique fixed point in S if $\rho<1$.

Letting $r \to 0$ the sufficient condition becomes

$$e^{ap}[\frac{1}{a}(K_f + K_g + \frac{mK_A(M_f + M_g)}{1 - mM_A} + mM_A + N_g] < 1.$$

If the above condition holds, then T will be a contraction mapping for some choice of sufficiently small $r > 0$. This yields the desired unique solution of (1) and the theorem is proved.

Remark 2: To make the sufficient condition (2) more specific, we can set $a = \frac{1}{p}$. By setting $a = \frac{1}{p}$ in (2) we get

$$p(K_f + K_g + \frac{mK_A(M_f + M_g)}{1 - mM_A}) + mM_A + N_g < \frac{1}{e}.$$

4. CONTINUOUS DEPENDENCE

We complete this paper by giving a continuous dependence result. That is, we shall obtain an estimate for the change in the solution of the mixed neutral system (1) due to a change in the initial condition x_0. Assuming we actually measured $\bar{x}_0$ instead of x_0, the question is this: If $\bar{x}_0$ is close to x_0, will the corresponding solution $\bar{x}$ be close to the solution x? The following theorem asserts that the answer is 'yes' if the conditions $(i)-(v)$ are satisfied.

Theorem 2: *Under the same hypotheses as in Theorem 1, the solution depends continuously on x_0. More precisely, assume the smallness condition (2) for some $a > 0$, and let $\bar{x}$ be the unique solution of (1) with $\bar{x}(0) = \bar{x}_0$ and with $\int_t^{t+1} |x'(s)|\,ds$ bounded. Then for any $T > 0$,*

$$\sup_{|t| < T} [\,|x(t) - \bar{x}(t)| + \int_t^{t+1} |x'(s) - \bar{x}'(s)|\,ds] \to 0 \text{ as } |x_0 - \bar{x}_0| \to 0.$$

Proof: Choose a sufficiently small $r > 0$ so that $\rho < 1$ as in equation (3) of Theorem 1.

Then a computation quite analogous to that in the Theorem 1 yields

$$d(x', \bar{x}') \le |x_0 - \bar{x}_0| e^{-a|t|} 2r[K_f + K_g + \frac{mK_A(M_f + M_g)}{1 - mM_A}] + \rho d(x', \bar{x}').$$

But since $\rho < 1$, we have

$$d(x', \bar{x}') \le |x_0 - \bar{x}_0| 2r[K_f + K_g + \frac{mK_A(M_f + M_g)}{1 - mM_A}].$$

So $d(x', \bar{x}') \rightarrow 0$ as $|x_0 - \bar{x}_0| \rightarrow 0$.

The assertion of the theorem is a straightforward consequence of this.

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