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EXISTENCE OF SOLUTION TO TRANSPIRATION CONTROL PROBLEM[1]

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ABSTRACT

Transpiration control can avoid change of the shape of a high-speed vehicle resulting from ablation of the nose, therefore also can avoid the change of the performance of Aerodynamics. Hence it is of practical importance. A set of mathematical equations and their boundary conditions are founded and justified by an example of non-ablation calculation in reference [1]. In [2], the ablation model is studied by the method of finite differences, the applicable margin of the equations is estimated through numerical calculation, and the dynamic responses of control parameters are analyzed numerically. In this paper we prove that the solution to transpiration control problem given in [1] exists uniquely under the assumption that the given conditions (i.e. given functions) are continuous.

Key words: Nonlinear PDE of parabolic type, transpiration control, heat transfer.

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I. THE CONSIDERED PROBLEM AND THE EQUIVALENT PROBLEM FOR AN INTEGRAL EQUATION

In this paper we consider the following problem:

$$\left.\begin{array}{ll} \frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial u}{\partial x} + \dot{s}(t)\frac{l-x}{l-s(t)}\frac{\partial u}{\partial x} & \text{for } t > 0,\ s(t) < x < l, \\ u(x,t)\,|_{t=0} = \varphi(x) & \text{for } 0 \le x \le l \text{ with } \varphi(0) = c, \\ u(x,t)\,|_{x=s(t)} = c & \text{for } 0 < t < \sigma \text{ with } s(0) = 0, \\ \frac{\partial u}{\partial x}\,|_{x=l} = -Q_1(t) & \text{for } 0 < t < \sigma \text{ with } Q_1(t) \ge 0, \\ \dot{s}(t) = k\frac{\partial u}{\partial x}\,|_{x=s(t)} + Q_2(t), & Q_2(t) > 0, \text{ for } 0 < t < \sigma, \end{array}\right\} \quad (1.1)$$

where $u(x,t)$ and $s(t)$ are unknown real functions, $\varphi(x)$, $Q_1(t)$ and $Q_2(t)$ are given functions and c, k are given constants.

Using the transformation $T = u - c$, the condition

$$u(x,t)\,|_{x=s(t)} = c$$

can be written in the form

$$T(x,t)\,|_{x=s(t)} = 0.$$

Thus, without losing generality we may assume that the constant c from (1.1) is equal to zero.

Below, we transform problem (1.1) into an equivalent problem which is formulated in the form of an integral equation.

Lemma 1.1:Suppose $s(t)$ is the Lipschitz continuous function for $t \in [0,\sigma]$ and $\rho(t)$ is the continuous function for $t \in [0,\sigma]$. Then we have

$$\lim_{x\to s(t)+0}\left\{\frac{\partial}{\partial x}\int_0^t \rho(\tau)K(x,t;s(\tau),\tau)d\tau\right\}$$

$$= -\frac{1}{2}\rho(t) + \int_0^t \rho(\tau)\frac{\partial}{\partial x}K(s(t),t;s(\tau),\tau)d\tau,$$

where

$$K(x,t;\xi,\tau) = \frac{1}{2\pi^{\frac{1}{2}}\alpha(t-\tau)^{\frac{1}{2}}}\, exp\left[-\frac{(x-\xi)^2}{4(t-\tau)\alpha^2}\right].$$

The proof of the above lemma is a consequence of standard computations.

Definition: A function $u = u(x,t)$ is said to be a solution of problem (1.1), where $s(t)$ is defined for $t \in (0,\sigma)$ $(0 < \sigma < \infty)$, if

(*i*) $\partial u/\partial t$, $\partial u/\partial x$ and $\partial^2 u/\partial x^2$ are continuous for $s(t) < x < l$, $0 < t < \sigma$;

(*ii*) u and $\partial u/\partial x$ are continuous for $s(t) \le x \le l$, $0 < t < \sigma$;

(*iii*) u is continuous for $t = 0$, $0 \le x \le l$;

(*iv*) $s(t)$ is continuously differentiable for $0 \le t \le \sigma$, and $\inf_{0 \le t \le \sigma} |l - s(t)| > 0$;

(*v*) problem (1.1) is satisfied.

From Lemma 1.1 and from Chapter 5 in [3], we have

Lemma 1.2: *Let $u(x,t)$ be a solution of (1.1) and let $\inf_{0 \le t \le \sigma} |l - s(t)| = d > 0$. Then there exists the fundamental solution $\Gamma(x,t;\xi,\tau)$ for equation $Lu = 0$ in $\Omega: = [-l,l] \times [0,\sigma]$. Moreover*

$$\lim_{x \to s(t)+0} \frac{\partial}{\partial x} \int_0^t \rho(\tau)\Gamma(x,t;s(\tau),\tau)d\tau = -\tfrac{1}{2}\rho(t) + \int_0^t \rho(\tau)\frac{\partial}{\partial x}\Gamma(s(t),t;s(\tau),\tau)d\tau,$$

$$|\Gamma(x,t;\xi,\tau)| \le M\frac{1}{(t-\tau)^{\frac{1}{2}}} \exp\left[-\frac{(x-\xi)^2}{8\alpha^2(t-\tau)}\right] \tag{1.2}$$

and

$$\left|\frac{\partial\Gamma(x,t;\xi,\tau)}{\partial x}\right| \le M\frac{1}{(t-\tau)} \exp\left[-\frac{(x-\xi)^2}{8\alpha^2(t-\tau)}\right], \tag{1.3}$$

where

$$Lu: = \alpha^2\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial u}{\partial x} + \dot{s}(t)\frac{l-x}{l-s(t)}\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t}, \tag{1.4}$$

$\rho(t)$ is the continuous function for $t \in [0,\sigma]$ and $M = M(\alpha, d, \sup_{0 \le t \le \sigma} |\dot{s}(t)|)$.

Next, let us get down to transform problem (1.1) into the equivalent integral equation problem.

Let us suppose the solution of problem (1.1) exists. By Lemma 1.2, there exists the fundamental solution $\Gamma(x,t;\xi,\tau)$ for $Lu = 0$ in Ω. We shall use the following sets:

$$B_\tau = \Omega \cap \{-l \le x \le l,\ t = \tau\}.$$

Let $V(x,t;\xi,\tau)$ be the solution of the following problem

$$\left.\begin{aligned} &LV = 0 \qquad \text{for } (x,t;\xi,\tau) \in \Omega\times\Omega,\ t > \tau, \\ &V|_{t=\tau} = 0, \\ &\frac{\partial V}{\partial x}\Big|_{x=l} = -\frac{\partial\Gamma}{\partial x}\Big|_{x=l},\ \frac{\partial V}{\partial x}\Big|_{x=-l} = \frac{\partial\Gamma}{\partial x}\Big|_{x=-l}, \end{aligned}\right\} \tag{1.5}$$

where $\Gamma(x,t;\xi,\tau)$ is the fundamental solution of $Lu=0$. From Chapter 5 in [3] we know that the solution of (1.5) exists. Let

$$G(x,t;\xi,\tau):=\Gamma(x,t;\xi,\tau)+V(x,t;\xi,\tau). \tag{1.6}$$

Then (see Chapter 3 in [3]), $G\in C^2([\Omega\times\Omega]\cap\{t>\tau\})$, and for any $f\in C[-l,l]$

$$u_\tau(x,t)=\int_{B_\tau} f(\xi)G(x,t;\xi,\tau)d\xi \tag{1.7}$$

satisfies $Lu_\tau=0$. Moreover

$$(\partial G/\partial x)\,|_{x\pm l}=0 \tag{1.8}$$

and

$$\lim_{t\to\tau+0}\int_{B_\tau} f(\xi)G(x,t;\xi,\tau)d\xi=f(x). \tag{1.9}$$

Consider the conjugate operator L^* of L given by the formula

$$L^*V=\alpha^2\frac{\partial^2 V}{\partial x^2}-(\beta+\dot{s}(t)\frac{l-x}{l-s(t)})\frac{\partial V}{\partial x}+\dot{s}(t)\frac{V}{l-s(t)}+\frac{\partial V}{\partial t}. \tag{1.10}$$

From Chapter 3, Section 7 in [3], the fundamental solution $\Gamma^*(x,t;\xi,\tau)$ of $L^*V=0$ exists in the domain Ω. Now, we shall study the following problem:

$$\left.\begin{aligned} L^*V^*=0 \text{ in } (x,t;\xi,\tau)\in\Omega\times\Omega,\ 0<t<\tau<T_0\\ V^*\,|_{t=\tau}=0,\\ (\frac{\partial V^*}{\partial x}-bV^*)\,|_{x=\pm l}=-(\frac{\partial\Gamma^*}{\partial x}-b(x,t)\Gamma^*)\,|_{x=\pm l},\end{aligned}\right\} \tag{1.11}$$

where $b(x,t):=\beta+\dot{s}(t)(l-x)/(l-s(t))$.

Again from Chapter 5 in [3], we obtain that the solution $V^*(x,t;\xi,\tau)$ of problem (1.11) exists. Let $G^*=\Gamma^*+V^*$. Then $G^*\in C^2$. Moreover for $t<\tau$ we have

$$L^*G^*=0 \tag{1.12}$$

and

$$\frac{\partial G^*}{\partial x}\,|_{x=\pm l}=(bG^*)\,|_{x=\pm l}. \tag{1.13}$$

If $f\in C[-l,l]$ then

$$\lim_{t\to\tau-0}\int_{B_\tau} f(\xi)G^*(x,t;\xi,\tau)d\xi=f(x). \tag{1.14}$$

It is easy to see that

$$G(x,t;\xi,\tau) = G^*(\xi,\tau;x,t). \tag{1.15}$$

Let $u(\xi,\tau)$ be the solution of problem (1.1) where (x,t) is replaced by (ξ,τ). We consider the Green identity

$$\begin{aligned} &GL_{(\xi,\tau)}u - uL^*_{(\xi,\tau)}G \\ &= \frac{\partial}{\partial\xi}[G\frac{\partial u}{\partial\xi} - u\frac{\partial G}{\partial\xi} + (\beta + \dot{s}(\tau)\frac{l-\xi}{l-s(t)})uG] - \frac{\partial}{\partial\tau}(uG) \equiv 0. \end{aligned} \tag{1.16}$$

Integrating this identity over the domain $D_\epsilon := \{0 < \tau \le t \le t-\epsilon, s(0) \le \sigma \le l\}$ and applying the Ostrogradski formula, we obtain

$$\begin{aligned} 0 &\equiv \int\int_{D_\epsilon} \left\{\frac{\partial}{\partial\xi}\left[(G\frac{\partial u}{\partial\xi} - u\frac{\partial G}{\partial\xi}) + (\beta - \dot{x}(\tau)\frac{l-\xi}{l-s(\tau)})u(\xi,\tau)G\right] - \frac{\partial}{\partial\tau}(uG)\right\}d\xi d\tau \\ &= \int_0^{t-\epsilon} G(x,t;s(\tau),\tau)\frac{\partial u(s(\tau),\tau)}{\partial\xi}d\tau + \int_0^{t-\epsilon} Q_1(\tau)G(x,t;l,\tau)d\tau \\ &\quad - \int_0^l \varphi(\xi)G(x,t;\xi,0)d\xi + \int_{s(t-\epsilon)}^l u(\xi,t-\epsilon)G(x,t;\xi,t-\epsilon)d\xi. \end{aligned} \tag{1.17}$$

Let $u(x,t) \equiv 0$ for $x < s(t)$ and $t \in (0,\sigma)$. Then, applying (1.9) and passing to the limit in (1.17) as $\epsilon \to 0$, we have

$$\lim_{\epsilon\to 0}\int_{s(t-\epsilon)}^l u(\xi,t-\epsilon)G(x,t;\xi,t-\epsilon)d\xi = u(x,t)$$

and

$$\begin{aligned} u(x,t) &= \int_0^l \varphi(\xi)G(x,t;\xi,0)d\xi - \int_0^t Q_1(\tau)G(x,t;l,\tau)d\tau \\ &\quad - \int_0^t G(x,t;s(\tau),\tau)\frac{\partial u(s(\tau),\tau)}{\partial x}d\tau. \end{aligned} \tag{1.18}$$

Then using Lemma 1.2, we obtain

$$\begin{aligned} \frac{\partial u(s(t),t)}{\partial x} &= \frac{1}{2}\frac{\partial u(s(t),t)}{\partial x} - \int_0^t \frac{\partial u(s(\tau),\tau)}{\partial x}\frac{\partial G}{\partial x}(s(t),t;s(\tau),\tau)d\tau \\ &\quad - \int_0^t Q_1(\tau)G_x(s(t),t;l,\tau)d\tau + \int_0^l \varphi(\xi)G_x(s(t),t;\xi,0)d\xi \end{aligned}$$

i.e.

$$u_x(s(t),t) = -2\int_0^t u_x(s(\tau),\tau)G_x(s(t),t;s(\tau),\tau)d\tau$$

$$-2\int_0^t Q_1(\tau)G_x(s(t),t;l,\tau)d\tau + 2\int_0^l \varphi(\xi)G_x(s(t),t;\xi,0)d\xi.$$

Let $W(t) := u_x(s(t),t)$. Then W satisfies the following integral equation:

$$W(t) = -2\int_0^t W(\tau)G_x(s(t),t;s(\tau),\tau)d\tau - 2\int_0^t Q_1(\tau)G_x(s(t),t;l,\tau)d\tau$$
$$+2\int_0^l \varphi(\xi)G_x(s(t)t;\xi,0)d\xi, \text{ for } t\in(0,\sigma), \tag{1.19}$$

where

$$s(t) = \int_0^t (W(\tau)+Q_2(\tau))d\tau, \text{ for } t\in(0,\sigma). \tag{1.20}$$

Obviously, if $u(x,t)$ is the solution of (1.1) and if u is continuous with respect to $t\in(0,\sigma)$, then $W(t)=u_x(s(t),t)$ is the continuous solution of the integral equation (1.19) on $[0,\sigma]$, where $s(t)$ is defined by (1.20). Conversely, suppose that W is the continuous solution of the integral equation on $[0,\sigma]$, where $s(t)$ is defined by (1.20) and $\inf_{t\in[0,\sigma]} |l-s(t)| = d > 0$. Then we can prove that $u(x,t)$ obtained above is the solution of (1.1). Substituting $W(\tau)$ into (1.18), we have

$$u(x,t) = -\int_0^t W(\tau)G(x,t;s(\tau),\tau)d\tau - \int_0^t Q_1(\tau)G(x,t;l,\tau)d\tau$$
$$+\int_0^l \varphi(\xi)G(x,t;\xi,0)d\xi, \text{ for } s(t)<x<l \text{ and } 0<t<\sigma, \tag{1.21}$$

where $G(x,t;\xi,\tau)$ is the Green function (1.6) obtained after determining $s(t)$ by (1.20). For function $u(x,t)$ determined by (1.21), it is easy to see that

$$Lu(x,t) = 0, \text{ for } 0<t<\sigma,\ s(t)<x<l,$$

and

$$\lim_{t\to 0} u(x,t) = \varphi(x), \text{ for } 0\le x\le l.$$

Moreover, (1.21) implies that

$$u_x(x,t) = -\int_0^t W(\tau)G_x(x,t;s(\tau),\tau)d\tau - \int_0^t Q_1(\tau)G_x(x,t;l,\tau)d\tau$$
$$+\int_0^l \varphi(\xi)G_x(x,t;\xi,0)d\xi, \text{ for } s(t) < x < l \text{ and } 0 < t < \sigma. \tag{1.22}$$

Passing through to the limit as $x \to s(t)+0$ in the above equation and applying Lemma 1.2, we obtain that

$$u_x(s(t),t) = W(t)/2 - \int_0^t W(\tau)G_x(s(t),t;s(\tau),\tau)d\tau$$
$$-\int_0^t Q_1(\tau)G_x(s(t),t;l,\tau)d\tau + \int_0^l \varphi(\xi)G_x(s(t),t;\xi,0)d\xi$$
$$= (W(t)+W(t))/2 = W(t), \text{ for } t \in (0,\sigma).$$

Hence, (1.20) implies that

$$\dot{s}(t) = u_x(s(t),t) + Q_2(t), \text{ for } t \in (0,\sigma) \text{ and } s(0) = 0.$$

Below we shall prove that

$$u_x(l,t) = -Q_1(t) \text{ and } u(s(t),t) = 0.$$

Lemma 1.3: *Assume that there exists a continuous solution W of (1.19), (1.20) on $0 \le t \le \sigma$ and $\inf_{t \in [0,\sigma]} |l - s(t)| = d > 0$. Then the function u defined by (1.21) satisfies the condition*

$$u_x(l,t) = -Q_1(t) \textit{ for } t \in (0,\sigma).$$

Proof: In this proof we denote by M various constants dependent only on α, d and sup norm of $\dot{s}(t)$ on $0 \le t \ge \sigma$. Since

$$G(x,t;\xi,\tau) = \Gamma(x,t;\xi,\tau) + V(x,t;\xi,\tau)$$

then

$$G(x,t;l,\tau) = \Gamma(x,t;l,\tau) + \lim_{\xi \to l-0} V(x,t;\xi,\tau), \tag{1.23}$$

where

$$\Gamma(x,t;\xi,\tau) = K(x,t;\xi,\tau) + \int_\tau^t \int_{-l}^l K(x,t;y,\sigma)\Phi(y,\sigma;\xi,\tau)dyd\sigma, \tag{1.24}$$

$$K(x,t;\xi,\tau):=(2\pi^{\frac{1}{2}}\alpha(t-\tau)^{\frac{1}{2}})^{-1}exp[-(x-\xi)^2/(4\alpha^2(t-\tau))], \quad (1.25)$$

$$\Phi(x,t;\xi,\tau):=\sum_{j=1}^{\infty}(LK)_j(x,t;\xi,\tau),\ (LK)_1=LK,$$

$$(LK)_{j+1}(x,t;\xi,\tau)=\int_{\tau}^{t}\int_{-l}^{l}(LK)(x,t;y,\sigma)(LK)_j(y,\sigma;\xi,\tau)dyd\sigma,\ j=1,2,\ldots$$

and

$$|\Phi(x,t;\xi,\tau)|\le(M/(t-\tau))exp[-(x-\xi)^2/(8(t-\tau)\alpha^2)]. \quad (1.26)$$

By (1.8), $G_x(x,t;\xi,0)|_{x=l}=0$ for any $t>0$. Moreover, since $inf|l-s(t)|=d>0$, so $G_x(l,t;s(\tau),\tau)=0$ for $t\ge\tau$. Thus, (1.22) implies that for any $\epsilon>0$, we have the following equation:

$$\begin{aligned}u_x(l,t)&=\lim_{x\to l-0}\int_0^t(-Q_1(\tau))G_x(x,t;l,\tau)d\tau\\&=\lim_{x\to l-0}\int_{t-\epsilon}^t-(Q_1(\tau))G_x(x,t;l,\tau)d\tau,\ \text{for } 0<\epsilon<t.\end{aligned} \quad (1.27)$$

Additionally, since $(\partial K/\partial x)=-((x-\xi)(2\alpha^2(t-\tau))^{-1})K$, then

$$|\partial K/\partial x|\le(M/(t-\tau))exp[-(x-\xi)^2/(8(t-\tau)\alpha^2)]. \quad (1.28)$$

Applying (1.26), (1.28), we get

$$\begin{aligned}&\int_{\tau}^{t}\int_{-l}^{l}|K_x(x,t;y,\sigma)\Phi(y,\sigma;\xi,\tau)|dyd\sigma\\&\le\int_{\tau}^{t}\int_{-\infty}^{+\infty}K_x(x,t;y,\sigma)\Phi(y,\sigma;\xi,\tau)dyd\sigma\\&\le M(t-\tau)^{-\frac{1}{2}}exp[-(x-\xi)^2/(8\alpha^2(t-\tau))],\ \text{for } (x,t)\in(-l,l)\times(\tau,\sigma).\end{aligned}$$

Moreover, for $Q_1(t)\in C[0,\sigma]$,

$$\left|\int_{t-\epsilon}^{t}-Q_1(\tau)\int_{\tau}^{t}\int_{-l}^{l}K_x(x,t;y,\sigma)\Phi(y,\sigma;\xi,\tau)dyd\sigma d\tau\right|$$

$$\le M\int_{t-\epsilon}^{t}(t-\tau)^{-\frac{1}{2}}d\tau\le M\epsilon^{\frac{1}{2}}$$

and

$$\lim_{x \to l - 0} \int_{t-\epsilon}^{t} (\partial K(x,t;\xi,\tau)/\partial x)(-Q_1(\tau))d\tau = -Q_1(t)/2, \text{ for } t \in (0,\sigma).$$

Therefore, we obtain from (1.27) that

$$\lim_{x \to l - 0} \left| \int_{t-\epsilon}^{t} -Q_1(\tau)\Gamma_x(x,t;l,\tau)d\tau + Q_1(t)/2 \right| \le M\epsilon^{\frac{1}{2}}$$

i.e.

$$\lim_{x \to l - 0} \int_{t-\epsilon}^{t} (-Q_1(\tau))\Gamma_x(x,t;l,\tau)d\tau = -Q(t)/2 \quad (\epsilon \to 0).$$

By (1.27), to prove the conclusion of this lemma we have only to prove that

$$\lim_{x \to l - 0} \int_{t-\epsilon}^{t} (-Q_1(\tau))V_x(x,t;l,\tau)d\tau = -Q_1(t)/2 \quad (\epsilon \to 0),$$

where $V(x,t;\xi,\tau)$ is the solution of (1.5). We denote by μ the inward normal vector to the boundary of $[-l,l]$. Then the boundary condition in (1.5) can be written in the form:

$$(\partial V/\partial \mu)\big|_{\substack{x=l\\x=-l}} = \Gamma_x(x,t;\xi,\tau)\big|_{\substack{x=l\\x=-l}}.$$

By results from Chapter 5 of [3], we know that the solution of (1.5) is given by the formula

$$V(x,t;\xi,\tau) = \int_{\tau}^{t} \Gamma(x,t;l,\sigma)\Phi(l,\sigma;\xi,\tau)d\sigma + \int_{\tau}^{t} (x,t;-l,\sigma)\Phi(-l,\sigma;\xi,\tau)d\sigma, \qquad (1.29)$$

where

$$\Phi(\pm l,t;\xi,\tau) = 2\int_{\tau}^{t} \left[\frac{\partial\Gamma(\pm l,t;l,\sigma)}{\partial\mu}\Phi(l,\sigma;\xi,\tau) + \frac{\partial\Gamma(\pm l,t;-l,\sigma)}{\partial\mu}\Phi(-l,\sigma;\xi,\tau)\right]d\sigma$$
$$-2\Gamma_x(\pm l,t;\xi,\tau). \qquad (1.30)$$

Moreover, from (1.24) it is easy to see that

$$\left|\partial\Gamma(\pm l,t;\pm l,\sigma)/\partial\mu\right| \le M(t-\sigma)^{-\frac{1}{2}}. \qquad (1.31)$$

Thus, in spite of a singularity in the integrand, the integral in equation (1.30) is integrable. Since (1.30) is an integral equation whose unknown function is $\Phi(\pm l,t;\xi,\tau)$, hence if $\xi \neq \pm l$, then there exists a continuous solution $\Phi(\pm l,t;\xi,\tau)$ of (1.30). From (1.24), we also have that

$$\left|(\partial\Gamma(\pm l,t;\xi,\tau)/\partial x)\right| \le M\left|l \mp \xi\right|(4^2\pi\alpha^3(t-\tau)^3)^{-\frac{1}{2}}exp\left[-(l\mp\xi)^2(4\alpha^2(t-\tau))^{-1}\right]$$
$$+M(t-\tau)^{-\frac{1}{2}}exp\left[-(l\mp\xi)^2\cdot(8\alpha^2(t-\tau))^{-1}\right]. \qquad (1.32)$$

Formula (1.32) shows that $|\Gamma_x(-l,t;\xi,\tau)| \le M$ as $\xi \to l-0$, and from the inequalities

$$\left| \int_\tau^\sigma \Gamma_\mu(l,\sigma;l,s) \frac{l-\xi}{(s-\tau)^{3/2}} \, exp\left[-\frac{(l-\xi)^2}{4\alpha^2(s-\tau)} \right] ds \right|$$

$$\le \left| \int_\tau^{(\sigma+\tau)/2} \right| + \left| \int_{(\sigma+\tau)/2}^\sigma \right| \le M(\sigma-\tau)^{-\frac{1}{2}}$$

and

$$\left| \int_\tau^\sigma \Gamma_\mu(l,\sigma;l,s)\cdot(s-\tau)^{-\frac{1}{2}} ds \right| \le M,$$

we get an upper bound of the solution of the integral equation (1.30) as

$$|\Phi(l,t;\xi,\tau)| \le M\left\{ 1 + \frac{1}{(t-\tau)^{\frac{1}{2}}} \, exp\left[-\frac{(l-\xi)^2}{8\alpha^2(t-\tau)} \right] + \frac{|l-\xi|}{(t-\tau)^{3/2}} \, exp\left[\frac{-(l-\xi)^2}{4\alpha^2(t-\tau)} \right] \right\}$$

$$|\Phi(-l,t;\xi,\tau)| \le M\left\{ 1 + \frac{1}{(t-\tau)^{\frac{1}{2}}} \cdot exp\left[-\frac{(l-\xi)^2}{8\alpha^2(t-\tau)} \right] \right\}. \tag{1.33}$$

Moreover, we have

$$\left| \int_\tau^\sigma \Gamma_\mu(\pm l,\sigma;\pm l,s)\Phi(\pm l,s;\xi,\tau) ds \right| \le M(1+(\sigma-\tau)^{-\frac{1}{2}}),$$

$$\left| \int_\tau^t \Gamma(x,t;l,\sigma) \int_\tau^\sigma \Gamma_\mu(\pm l,\sigma;\pm l,s)\Phi(\pm l,s;\xi,\tau) ds d\sigma \right| \le M.$$

Thus, we can define the following integrals:

$$\int_\tau^t \Gamma(x,t;l,\sigma) \int_\tau^\sigma \Gamma_\mu(\pm l,\sigma;l,s)\Phi(l,s;l,\tau) ds d\sigma$$

$$:= \lim_{\xi \to l-0} \int_\tau^t \Gamma(x,t;l,\sigma) \int_\tau^\sigma \Gamma_\mu(\pm l,\sigma;l,s)\Phi(l,s;\xi,\tau) ds d\sigma,$$

$$\int_\tau^t \Gamma(x,t;l,\sigma) \int_\tau^\sigma \Gamma_\mu(\pm l,\sigma;-l,s)\Phi(-l,s;l,\tau) ds d\sigma$$

$$:= \lim_{\xi \to l-0} \int_\tau^t \Gamma(x,t;l,\sigma) \int_\tau^\sigma \Gamma_\mu(\pm l,\sigma;-l,s)\Phi(-l,s;\xi,\tau) ds d\sigma.$$

So, (1.29) becomes

$$V(x,t;\xi,\tau) = -2\int_\tau^t \Gamma(x,t;l,\sigma)\Gamma_x(l,\sigma;\xi,\tau)d\sigma - 2\int_\tau^t \Gamma(x,t;l,\sigma)\Gamma_x(-l,\sigma;\xi,\tau)d\sigma$$

$$+2\int_\tau^t \Gamma(x,t;l,\sigma)\int_\tau^\sigma \{\Gamma_\mu(l,\sigma;l,s)\Phi(l,s;\xi,\tau)+\Gamma_\mu(l,\sigma;-l,s)\Phi(-l,s;\xi,\tau)\}dsd\sigma$$

$$+2\int_\tau^t \Gamma(x,t;l,\sigma)\int_\tau^\sigma \{\Gamma_x(-l,\sigma;l,s)\Phi(l,s;\xi,\tau)+\Gamma_\mu(-l,\sigma;-l,s)\Phi(-l,s;\xi,\tau)\}dsd\sigma. \tag{1.34}$$

Since $\Gamma_x(-l,t;\xi,\tau)$ is continuous on $0 \le \tau \le t \le \sigma$ as $\xi \to l-0$, we have that

$$\lim_{\xi\to l-0}\int_\tau^t \Gamma(x,t;l,\sigma)\Gamma_x(-l,\sigma;\xi,\tau)d\sigma = \int_\tau^t \Gamma(x,t;l,\sigma)\Gamma_x(-l,\sigma;l,\tau)d\sigma.$$

Thus

$$V(x,t;l,\tau) := \lim_{x\to l-0} V(x,t;\xi,\tau)$$

$$=\lim_{\xi\to l-0}\left\{2\int_\tau^t \Gamma(x,t;l,\sigma)\frac{l-\xi}{4(\pi\alpha^3(\sigma-\tau)^3)^{\frac{1}{2}}}\, exp\left[-\frac{(l-\xi)^2}{4\alpha^2(\sigma-\tau)}\right]d\sigma\right\}$$

$$-2\int_\tau^t \Gamma(x,t;l,\sigma)\Gamma_x(l,\sigma;l,\tau)d\sigma + V_1+V_2+V_3$$

$$=\Gamma(x,t;l,\tau) - 2\int_\tau^t \Gamma(x,t;l,\sigma)\Gamma_x(l,\sigma;l,\tau)d\sigma + V_1+V_2+V_3.$$

and

$$\int_{t-\epsilon}^t (-Q_1(\tau))V_x(x,t;l,\tau)d\tau$$

$$= -\int_{t-\epsilon}^t Q_1(\tau)\Gamma_x(x,t;l,\tau)d\tau + 2\int_{t-\epsilon}^t Q_1(\tau)\int_\tau^t \Gamma_x(x,t;l,\sigma)\Gamma_x(l,\sigma;l,\tau)d\sigma d\tau$$

$$+\int_{t-\epsilon}^t -Q_1(\tau)(V_{1_x}+V_{2_x}+V_{3_x})d\tau.$$

We have

$$\lim_{x\to l-0} -\int_{t-\epsilon}^t Q_1(\tau)\Gamma_x(x,t;l,\tau)d\tau = -\tfrac{1}{2}Q_1(t) \quad (\epsilon\to 0).$$

Moreover, we have

$$\left| \int_{t-\epsilon}^{t} Q_1(\tau) \int_{\tau}^{t} \Gamma_x(x,t;l,\sigma)\Gamma_x(l,\sigma;l,\tau)d\sigma d\tau \right| \leq M\epsilon^{\frac{1}{2}}$$

and

$$\left| \int_{t-\epsilon}^{t} -Q_1(\tau)(V_{1_x} + V_{2_x} + V_{3_x})d\tau \right| \leq M\epsilon^{\frac{1}{2}}.$$

Therefore,

$$\lim_{x \to l-0} \int_{t-\epsilon}^{t} -Q_1(\tau)V_x(x,t;l,\tau)d\tau = -Q_1(t)/2, \quad (\epsilon \to 0).$$

Consequently, the proof of Lemma 1.3 is complete.

Next, we shall show that $u(s(t),t) = 0$ for $t \in (0,\sigma)$. Integrating the following Green's identity:

$$\frac{\partial}{\partial \xi}\left[G\frac{\partial u}{\partial \xi} - u\frac{\partial G}{\partial \xi} + (\beta + \dot{s}(\tau)\frac{l-\xi}{l-s(\tau)})uG\right] - \frac{\partial}{\partial \tau}(uG) \equiv 0,$$

on the region $0 \leq \tau \leq t - \epsilon$, $s(\tau) \leq \xi \leq l$ and letting $\epsilon \to 0$, we get

$$\int_0^t u(s(\tau),\tau)\left[G_\xi(x,t;s(\tau),\tau) - \dot{s}(\tau)G(x,t;s(\tau),\tau)\right]d\tau = 0. \tag{1.35}$$

Obviously, we have

$$\lim_{x \to s(t)+0} \left\{ \int_0^t u(s(\tau),\tau)\dot{s}(\tau)G(x,t;s(\tau),\tau)d\tau \right\}$$

$$= \int_0^t u(s(\tau),\tau)\dot{s}(\tau)G(s(t),t;s(\tau),\tau)d\tau,$$

$$G_\xi(x,t;\xi,\tau) = G_\xi^*(\xi,\tau;x,t) = \Gamma_x^*(\xi,\tau;x,t) + V_\xi^*(\xi,\tau;x,t),$$

$$G_\xi(x,t;s(\tau),\tau) = G_\xi^*(s(\tau),\tau;x,t) = \Gamma_\xi^*(s(\tau),\tau;x,t) + V_\xi^*(s(\tau),\tau;x,t);$$

and

$$\lim_{x \to s(t)+0} \int_0^t u(s(\tau),\tau)V_x^*(s(\tau),\tau;x,t)d\tau = \int_0^t u(s(\tau),\tau)V_\xi^*(x(\tau),\tau;s(t),t)d\tau.$$

Analogously to Lemma 1.2 we have

$$\lim_{x \to s(t)+0} \int_0^t u(s(\tau),\tau)\Gamma_\xi^*(s(\tau),\tau;x,t)d\tau$$

$$= -\frac{1}{2}u(s(t),t) + \int_0^t u(s(\tau),\tau)\Gamma_\xi^*(s(\tau),\tau;s(t),t)d\tau.$$

Letting $x \to s(t)+0$ in (1.35) we get

$$\frac{1}{2}u(s(t),t) = \int_0^t u(s(\tau),\tau)\Big[G_\xi(s(t),t;s(\tau),\tau) - \dot{s}(\tau)G(s(t),t;s(\tau),\tau)\Big]d\tau. \tag{1.36}$$

Similarly to the proof of Lemma 1.3 we can show that

$$\left|G_\xi(s(t),t;s(\tau),\tau)\right| \le M/(t-\tau)^{\frac{1}{2}},\ \left|G(s(t),t;s(\tau),\tau)\right| \le M/(t-\tau)^{\frac{1}{2}}.$$

So, the integrand in (1.36) is integrable. Therefore $u(s(t),t) \equiv 0$.

Summing up, in this section we have showed that the solvability of problem (1.1) is equivalent to the solvability of the integral equation (1.19).

2. THE SOLVABILITY OF THE INTEGRAL EQUATION

In this section we shall prove that the solution of (1.19), (1.20) exists uniquely. Consider the mapping

$$\omega(t) = T(W(t)), \tag{2.1}$$

where

$$T(W(t)) := -2\int_0^t W(\tau)G_x(s(t),t;s(\tau),\tau)d\tau - 2\int_0^t Q_1(\tau)G_x(s(t),t;l,\tau)d\tau + 2\int_0^l \varphi(\xi)G_x(s(t),t;\xi,0)d\xi, \tag{2.2}$$

$$s(t) = \int_0^t (W(\tau)+Q_2(\tau))d\tau. \tag{2.3}$$

The function $G(x,t;\xi,\tau)$ in (2.2) is given by (1.6). Let

$$C_{\sigma,A} := \Big\{W(t): W(t) \in C[0,\sigma], |W(t)| \le A, A > 0\Big\}.$$

By the continuity of $Q_2(t)$, it is easy to see that for any fixed $A > 0$ and sufficiently small $\sigma > 0$, $|s(t)| < l/2$ holds for $t \in [0,\sigma]$. Thus the mapping $\omega(t) = T(W(t))$ given by (2.1), (2.2) and (2.3) is well defined in $C_{\sigma,A}$.

Theorem 2.1: *Let $\varphi \in C^1[0,l]$, $Q_1 \in C[0,T]$ and $Q_2 \in C[0,T]$. Then, for $A := 2\max_{0\le x\le l}|\varphi'(x)| + 1$ there exists $\sigma_0 > 0$, such that $\omega(t) = T(W(t))$ defined by (2.1), (2.2), (2.3) is a mapping from $C_{\sigma_0,A}$ into itself.*

Proof: Since $G_x = \Gamma_x + V_x$ then

$$2\int_0^l \varphi(\xi)G_x(s(t),t;\xi,0)d\xi$$

$$= 2\int_0^l \varphi(\xi)\int_0^t \int_{-l}^l K_x(s(t),t;y,\sigma)\Phi(y,\sigma;\xi,0)dyd\sigma d\xi$$

$$+2\int_0^l \varphi(\xi)V_x(s(t),t;\xi,0)d\xi + 2\int_0^l \varphi(\xi)K_x(s(t),t;\xi,0)d\xi.$$

Noting $K_x = -K_\xi$, we have

$$2\int_0^l \varphi(\xi)K_x(s(t),t;\xi,0)d\xi = -2\varphi(l)K(s(t),t;l,0) + 2\int_0^l \varphi'(\xi)K(s(t),t;\xi,0)d\xi.$$

Thus

$$T(W(t)) = -2\int_0^t W(\tau)G_x(s(t),t;s(\tau),\tau)d\tau - 2\int_0^t Q_1(\tau)G_x(s(t),t;l,\tau)d\tau$$

$$-2\varphi(l)K(s(t),t;l,0) + 2\int_0^l K(s(t),t;\xi,0)\varphi'(\xi)d\xi$$

$$+2\int_0^l\int_0^t\int_{-l}^l \varphi(\xi)K_x(s(t),t;y,\sigma)\Phi(y,\sigma;\xi,0)dyd\sigma d\xi \tag{2.4}$$

$$+2\int_0^l \varphi(\xi)V_x(s(t),t;\xi,0)d\xi = \sum_{i=1}^6 T_i,$$

where $s(t)$ is defined by (2.3), and $\omega(t)$ is defined in $C_{\sigma_0,A}$ for a fixed A and sufficiently small $\sigma_0 > 0$ (such that $|s(t)| < l/2$). Below we shall estimate T_i $(i = 1,\ldots,6)$. We denote by $M = M(A')$ a constant, where A' is Lipschitz constant, i.e. $|s(t) - s(\tau)| \leq A'|t-\tau|$. By the condition

$$\Gamma_x(s(t),t;s(\tau),\tau)$$

$$= \frac{s(t)-s(\tau)}{2(t-\tau)\alpha^2}K(s(t),t;s(\tau),\tau) + \int_\tau^t\int_{-l}^l K_x(s(t),t;y,\sigma)\Phi(y,\sigma;s(\tau),\tau)dyd\sigma$$

we obtain

$$|\Gamma_x(s(t),t;s(\tau),\tau)| \leq M(t-\tau)^{-\frac{1}{2}}.$$

From (1.29), (1.33), (1.3) and from the inequality $|l-s(\tau)| \geq l/2>0$ we see that $V_x(s(t),t;s(\tau),\tau)$ is bounded on $[0,\sigma_0]$, and

$$|V_x(s(t),t;s(\tau),\tau)| \leq M(t-\tau).$$

Thus

$$|T_1| \leq 2A' \int_0^t |G_x(s(t),t;s(\tau),\tau)| d\tau \leq M t^{\frac{1}{2}} \leq M\sigma_0^{\frac{1}{2}}. \tag{2.5}$$

In the same way we have

$$|T_2| \leq M\sigma_0^{\frac{1}{2}}. \tag{2.6}$$

Since $|l-s(t)| \geq l/2$, we get

$$|K(s(t),t;l,0)| \leq M\sigma_0^{\frac{1}{2}}$$

and

$$\left| \int_0^l K(s(t),t;\xi,0)\varphi'(\xi)d\xi \right| \leq \max_{\xi \in [0,l]} |\varphi'(\xi)|. \tag{2.7}$$

Thus

$$|T_3| + |T_4| \leq 2\max_{\xi \in [,l]} |\varphi'(\xi)| + M\sigma_0^{\frac{1}{2}}. \tag{2.8}$$

To estimate T_5, we need the following two lemmas from [3]:

Lemma 2.1 ([3]): *Suppose* $-\infty < \alpha < 3/2$ *and* $-\infty < \beta < 3/2$. *Then*

$$\int_\tau^t \int_{-\infty}^{+\infty} (t-\sigma)^{-\alpha} exp \left[-\frac{h(x-h)^2}{4(t-\sigma)} \right] (\sigma-\tau)^{-\beta} exp \left[-\frac{h(y-\xi)^2}{4(\sigma-\tau)} \right] dy d\sigma$$

$$= \left(\frac{4\pi}{h}\right)^{\frac{1}{2}} B(-\alpha+3/2, -\beta+3/2)(t-\tau)^{-\alpha-\beta+3/2} exp \left[-\frac{h(x-\xi)^2}{4(t-\tau)} \right],$$

where $B(\cdot,\cdot)$ *is the beta function.*

Lemma 2.2 ([3]): *Assume that the coefficients of the operator* L *are Lipschitz continuous in* Ω. *Then the fundamental solution for* $Lu=0$ *exists in* Ω, *and it is given by* (1.24), *where*

$$\Phi(x,t;\xi,\tau) = LK(x,t;\xi,\tau) + \int_\tau^t \int_{-l}^{l} LK(x,t;y,\sigma)\Phi(y,\sigma;\xi,\tau) dy d\sigma,$$

$\Phi(x,t;\xi,\tau)$ *is bounded and satisfies inequality* (1.26) *and the constant* M *in* (1.26) *depends only* *on Lipschitz constants and* Ω.

With the aid of Lemma 2.1-2.2 we get down to estimate T_5.

$$T_5 := 2\int_0^l \int_0^t \int_{-l}^l \varphi(\xi) K_x(s(t),t;y,\sigma) LK(y,\sigma;\xi,0)\,dy\,d\sigma\,d\xi$$

$$+2\int_0^l \int_0^t \int_{-l}^l \varphi(\xi) K_x(s(t),t;y,\sigma) \int_0^\sigma \int_{-l}^l LK(y,\sigma;\zeta,z)\Phi(\zeta,z;\xi,0)\,d\zeta\,dz\,dy\,d\sigma\,d\xi$$

$= T_{51} + T_{52}$.

For this purpose, observe that applying Lemma 2.1-2.2 we get

$$|T_{52}| \le M\sigma_0^{\frac{1}{2}}.$$

Noting $LK(x,t;\xi,\tau) = (\beta + \dot{s}(t)(l-x)/(l-s(t)))K_x(x,t;\xi,\tau)$ we get

$$T_{51} = -2\int_0^l \int_0^t \int_{-l}^l \varphi(\xi) K_y(s(t),t;y,\sigma) LK(y,\sigma;\xi,0)\,dy\,d\sigma\,d\xi$$

$$= T_{51}^1 + T_{51}^2 + T_{51}^3$$

with

$$T_{51}^1 := -2\int_0^l \int_0^t \left[\varphi(\xi) K(s(t),t;y,\sigma) LK(y,\sigma;\xi,0)\right]_{y=\pm l} d\sigma\,d\xi,$$

$$T_{51}^2 := 2\int_0^l \int_0^t \int_{-l}^l \varphi(\xi) K(s(t),t;y,\sigma) \frac{-\dot{s}(\sigma)}{l-s(\sigma)} K_x(y,\sigma;\xi,0)\,dy\,d\sigma\,d\xi$$

$$T_{51}^3 := 2\int_0^l \int_0^t \int_{-l}^l \varphi(\xi) K(s(t),t;y,\sigma)\left(\beta + \frac{\dot{\sigma}(\sigma)(l-y)}{l-s(\sigma)}\right) K_{xx}(y,\sigma;\xi,0)\,dy\,d\sigma\,d\xi.$$

Since $|K(s(t),t;y,\sigma)| \le Mt$, for $y = \pm l$ and

$$\int_0^t |LK(y,\sigma;\xi,0)|\,d\sigma \le M,$$

thus we have $|T_{51}^1| \le M\sigma_0$.

Apply Lemma 2.1 and using the boundedness of $\varphi(\xi)$ and $\frac{\dot{s}(\sigma)}{l-s(\sigma)}$ we get $|T_{51}^2| \le M\sigma_0^{\frac{1}{2}}$. Moreover from $K_{xx}(y,\sigma;\xi,0) = -K_{x\xi}(y,\sigma;\xi,0)$ we have

$$T_{51}^3 = -2\int_{-l}^{l}\int_0^t \left[K(s(t),t;y,\sigma)(\beta+\frac{\dot{s}(\sigma)(l-y)}{l-s(\sigma)})\varphi(\xi)K_x(y,\sigma;\xi,0)\right]_{\xi=0}^{l} dyd\sigma$$

$$+2\int_{-l}^{l}\int_0^t\int_0^l K(s(t),t;y,\sigma)(\beta+\frac{\dot{s}(\sigma)(l-y)}{l-s(\sigma)})\varphi'(\xi)K_x(y,\sigma;\xi,0)dyd\sigma d\xi$$

$$= I_1 + I_2.$$

Since $\varphi(0)=0$ then

$$|I_1| \le \left| -2\int_0^t\int_{-l}^{l} \varphi(l)K(s(t),t;y,\sigma)(\beta+\frac{\dot{s}(\sigma)(l-y)}{l-s(\sigma)}K_x(y,\sigma;l,0)dyd\sigma\right|$$

$$\le M\ exp\left[-(s(t)-l)^2/(8\alpha^2 t)\right]$$

$$\le M\sigma_0^{\frac{1}{2}} \text{ (because } |s(t)-l| \ge l/2).$$

Applying Lemma 2.1 in a similar way we can obtain $|I_2| \le M\sigma_0^{\frac{1}{2}}$. Therefore,

$$|T_5| \le M\sigma_0^{\frac{1}{2}}. \tag{2.9}$$

Next, we shall estimate T_6. From (1.34) in Section 1 we get V_x and substituting $V_x(s(t),t;\xi,0)$ in to the expression of T_6 we obtain

$$T_6 = V_1 + V_2 + V_3 + V_4 \tag{2.10}$$

with

$$V_1 := -4\int_0^l \varphi(\xi)\int_0^t \Gamma_x(s(t),t;l,0)\Gamma_x(l,\sigma;\xi,0)d\sigma d\xi,$$

$$V_2 := -4\int_0^l \varphi(\xi)\int_0^t \Gamma_x(s(t),t;l,\sigma)\Gamma_x(-l,\sigma;\xi,0)d\sigma d\xi,$$

$$V_3 := 4\int_0^l \varphi(\xi)\int_0^t \Gamma_x(s(t),t;l,\sigma)\int_0^\sigma [\Gamma_\mu(l,\sigma;l,z)\Phi(l,z;\xi,0)$$

$$+\Gamma_\mu(l,\sigma;-l,z)\Phi(-l,z;\xi,0)]dzd\sigma d\xi,$$

$$V_4 := 4\int_0^l \varphi(\xi)\int_0^t \Gamma_x(s(t),t;l,\sigma)\int_0^\sigma [\Gamma_\mu(-l,\sigma;l,z)\Phi(l,z;\xi,0)$$

$$+\Gamma_\mu(-l,\sigma;-l,z)\Phi(-l,z;\xi,0)]dzd\sigma d\xi.$$

Since

$$|\Gamma_x(s(t),t;l,\sigma)| \le M(t-\sigma) \le M\sigma_0 \quad (\text{by } |s(t)-l| \ge l/2 > 0),$$

and

$$\int_0^t |\Gamma_x(l,\sigma;\xi,0)| d\sigma \le \int_0^t \frac{|l-\xi|}{\sigma} K(l,\sigma;\xi,0)d\sigma \le M,$$

then $|V_1| \le M\sigma_0$ and $|V_2| \le M\sigma_0$. By (1.33) we have

$$\left|\int_0^\sigma \Gamma_\mu(\pm l,\sigma;\pm l,z)\Phi(\pm l,z;\xi,0)dz\right| \le M(1+\sigma^{-\frac{1}{2}})$$

and $|\Gamma_x(s(t),t;l,\sigma)+ \le M\sigma_0$. Thus we get

$$|V_3| + \le M\sigma_0 \text{ and } |V_4| \le M\sigma_0.$$

Therefore, we obtain

$$|T_6| \le M\sigma_0. \tag{2.11}$$

Combining (2.5), (2.6), (2.8), (2.9), (2.11) we have

$$|T(W(t))| \le 2\max_{x\in[0,l]}|\varphi'(x)| + M\sigma_0^{\frac{1}{2}},$$

where constant M depends only on A', l, α, β, $\max_{x\in[0,l]}|\varphi|$, $\max_{t\in[0,\sigma]}|Q_1|$ and $\max_{t\in[0,\sigma]}|Q_2|$.

Choose $\sigma_0 > 0$ sufficiently small such that $M\sigma_0^{\frac{1}{2}} < 1$ for $A = 2\max_{x\in[0,l]}|\varphi'(x)|+1$. In this case we have $|T(W(t))| \le A$ for $W \in C_{\sigma_0, A}$. Therefore Theorem 2.1 is proved.

To solve the problem, we have only to prove that $T(W(t))$ is a contraction. We denote $\|Q_1(t)\| = \max_{t\in[0,\sigma]}|Q_1(t)|$, $\|Q_2(t)\| = \max_{t\in[0,\sigma]}|Q_2(t)|$, $\|\varphi(x)\| = \max_{x\in[0,l]}|\varphi(x)|$.

Theorem 2.2: *Suppose* $\varphi \in C^1[0,l]$, $Q_1 \in C[0,T]$, $Q_2 \in C[0,T_0]$ *and* $A = 2\max_{x\in[0,l]}|\varphi'(x)|+1$. *Then there exists a* $\sigma_1 > 0$ $(\sigma_1 < T_0)$ *such that the mapping* $\omega = T(W(t))$ *defined by* (2.1), (2.2), (2.3) *is a contraction on* $C_{\sigma_1, A}$.

Proof: Let $\|W(t)\| = \max_{t\in[0,\sigma_1]}|W(t)|$. Then $\|W(t)\| < A$.

Moreover let L be the operator defined by (1.4) and M be constants dependent only on A, $\| \varphi \|$, $\| Q_1 \|$, l, α. Additionally, let

$$b(x, W(t)) = \beta + [\dot{s}(t)(l-x)/(l-s(t))], \tag{2.12}$$

where $s(t)$ is given by (2.3). For any $W_j \in C_{\sigma_0, A} (j = 1,2)$ we consider the following equations:

$$L_j u = \alpha^2 \frac{\partial^2 u}{\partial x^2} + b(x, W_j(t)) \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0 \quad (j = 1,2) \tag{2.13}$$

and their fundamental solutions

$$\Gamma_j(x,t;\xi,\tau) = K(x,t;\xi,\tau) + \int_\tau^t \int_0^l K(x,t;y,\sigma)\Phi_j(y,\sigma;\xi,\tau)dyd\sigma, \ (j = 1,2) \tag{2.14}$$

where $K(x,t;\xi,\tau)$ is given by (1.25) and $\Phi_j(x,t;\xi,\tau)$ satisfying

$$\Phi_j(x,t;\xi,\tau) = L_j K(x,t,\xi,\tau) + \int_\tau^t \int_{-l}^l L_j K(x,t;y,\sigma)\Phi_j(y,\sigma;\xi,\tau)dyd\sigma \ (j = 1,2). \tag{2.15}$$

Lemma 2.3: *The functions $\Phi_j(x,t;\xi,\tau)$ $(j = 1,2)$ defined by (2.15) satisfy the estimation*

$$|\Phi_1(x,t;\xi,\tau) - \Phi_2(x,t;\xi,\tau)| \le M \| W_1 - W_2 \| (\tfrac{1}{t-\tau}) \ exp\ [-(x-\xi)^2/(8\alpha^2(t-\tau))] \tag{2.16}$$

for $0 \le \tau \le t \le \sigma_0 < 1$ and $-l \le x,\ \xi \le l$.

Lemma 2.4: *If $|x| \le l$ and $|\xi| \le l$ then*

$$\int_\tau^t \int_{-\infty}^{+\infty} K_{xx}(x,t;y,\sigma)b(y,W(\sigma))K_\xi(y,\sigma;\xi,\tau)dyd\sigma \le \frac{M}{t-\tau}\ exp\left[\frac{-(x-\xi)^2}{8\alpha^2(t-\tau)}\right].$$

Lemma 2.5: *Suppose that $-l < x,\ \xi < l$, $|x| \le l-d$ (or $|\xi| \le l-d$) and $d > 0$, then*

$$\left| \int_\tau^t \int_{-l}^l K_{xx}(x,t;y,\sigma)\bar{b}\ (y,\sigma)K_x(y,\sigma;\xi,\tau)dyd\sigma \right|$$

$$\le M\left\{1 + \frac{1}{t-\tau}\ exp[-(x-\xi)^2/(8\alpha^2(t-\tau))]\right\};$$

$$\left| \int_\tau^t \int_{-l}^l K_x(x,t;y,\sigma)\bar{b}\ (y,\sigma)K_{y\xi}(y,\sigma;\xi,\tau)dyd\sigma \right|$$

$$\leq M\Big\{1+\frac{1}{t-\tau}\,exp[-(x-\xi)^2/(8\alpha^2(t-\tau))]\Big\},$$

where $\overline{b}\,(y,\sigma)=b(y,W(\sigma))=\beta+\dot{s}(\sigma)(l-y)/(l-s(\sigma))$.

Lemma 2.6: *Assume that* $|x|\leq l/2$, $|\xi|\leq l/2$ *and* $W\in C_{\sigma_0,A}$. *Then*

$$\left|\int_{\tau}^{t}\int_{-l}^{l}K_{xx}(x,t;y,\sigma)\Phi(y,\sigma;\xi,\tau)dyd\sigma\right|\leq M\left(1+\frac{1}{t-\tau}\,exp\left[\frac{-(x-\xi)^2}{8\alpha^2(t-\tau)}\right]\right),$$

where $\Phi(x,t;\xi,\tau)$ *is given by* (2.15).

Lemma 2.7: *Suppose that* $-l\leq y,\ \xi\leq l$, $|\xi|<l/2$ *and* $W(t)\in C_{\sigma_0,A}$. *Then*

$$\left|\int_{\tau}^{\sigma}\int_{-l}^{l}LK(y,\sigma;\eta,\zeta)\Phi_{\xi}(\eta,\zeta;\xi,\tau)d\eta d\zeta\right|\leq M\Big\{1+\frac{1}{\sigma-\tau}\,exp[-(y-\xi)^2/(8\alpha^2(\sigma-\tau))]\Big\},$$

where $\Phi(x,t;y,\sigma)$ *is given by* (2.15).

Now let us get down to prove Theorem 2.2.

By (2.2), (1.6), (1.24) and by letting

$$K_1(x,t;\xi,\tau)=\int_{\tau}^{t}\int_{-l}^{l}K(x,t;y,\sigma)\Phi(y,\sigma;\xi,\tau)dyd\sigma,$$

we may write $T(W(t))$ in the way as

$$T(W(T))=T_1(W(t))+T_2(W(t))+T_3(W(t)),$$

where

$$T_1(W(t)):=-2\int_0^t W(\tau)K_x(s(t),t;s(\tau),\tau)d\tau-2\int_0^t Q_1(\tau)K_x(s(t),t;l,\tau)d\tau$$

$$+2\int_0^l\varphi(\xi)K_x(s(t),t;\xi,0)d\xi,$$

$$T_2(W(t)):=-2\int_0^t W(\tau)K_{1_x}(s(t),t;s(\tau),\tau)d\tau-2\int_0^t Q_1(\tau)K_{1_x}(s(t),t;l,\tau)d\tau$$

$$+2\int_0^l\varphi(\xi)K_{1_x}(s(t),t;\xi,0)d\xi$$

$$= N_1 + N_2 + N_3$$

and

$$T_3(W(t)) := -2\int_0^t W(\tau)V_x(s(t),t;s(\tau),\tau)d\tau - 2\int_0^t Q_1(\tau)V_x(s(t),t;l,\tau)d\tau$$

$$+2\int_0^l \varphi(\xi)V_x(s(t),t;\xi,0)d\xi$$

$$= N_4 + N_5 + N_6.$$

By a result in [4] we know that T_1 is a contraction mapping in $C_{\sigma_1,A}$ for sufficiently small $\sigma_1 > 0$.

With the aid of the above lemmas, we obtain

$$|N_1(W_1) - N_1(W_2)| \le M\sigma \| W_1 - W_2 \| \text{ for } \sigma \le \sigma_0;\ W_1, W_2 \in C_{\sigma_0,A};$$

$$|N_2(W_1) - N_2(W_2)| \le M \| W_1 - W_2 \| \sigma^{\frac{1}{2}} \text{ for } \sigma \le \sigma_0;\ W_1, W_2 \in C_{\sigma_0,A};$$

$$|N_3(W_1) - N_3(W_2)| \le M\sigma^{\frac{1}{2}} \| W_1 - W_2 \| \text{ for } \sigma \le \sigma_0;\ W_1, W_2 \in C_{\sigma_0,A}.$$

To complete the proof of Theorem 2.2 we need the following lemmas:

Lemma 2.8: *Suppose that W_1, $W_2 \in C_{\sigma_0,A}$. Then*

$$|\Gamma_{1_x}(s_1(t),t;\pm l,\tau) - \Gamma_{2_x}(s_2(t),t;\pm l,\tau)| \le M \| W_1 - W_2 \| \sigma$$

and

$$|\Gamma_{1_x}(\pm l,t;s_1(\tau),\tau) - \Gamma_{2_x}(\pm l,t;s_2(\tau),\tau)| \le M \| W_1 - W_2 \| \sigma.$$

Lemma 2.9: *Assume that W_1, $W_2 \in C_{\sigma_0,A}$ and $\Phi_j(\pm l,t;\xi,\tau)$ is given by (1.30) for $W_j (j = 1,2)$, i.e.,*

$$\Phi_j(\pm l,t;s_j(\tau),\tau) = 2\int_\tau^t \Gamma_{j_\mu}(\pm l,t;l,\sigma)\Phi_j(l,\sigma;s_j(\tau),\tau)d\sigma$$

$$+2\int_\tau^t \Gamma_{j_\mu}(\pm l,t;-l,\sigma)\Phi_j(-l,\sigma;s_j(\tau),\tau)d\sigma - 2\Gamma_{j_x}(l,t;s_j(\tau),\tau).$$

Then

$$|\Phi_1(\pm l,t;s_1(\tau),\tau) - \Phi_2(\pm l,t;s_2(\tau),\tau)| \le M \| W_1 - W_2 \| .$$

Lemma 2.10: *Suppose that* W_1, $W_2 \in C_{\sigma_0, A}$. *Then* $\Phi_j(\pm l, \sigma; \xi, \tau)$ *given by* (1.30) *satisfy*

$$|\Phi_1(\pm l,\sigma;\xi,\tau) - \Phi_2(\pm l,\sigma;\xi,\tau)| \leq M(\sigma-\tau)^{-\frac{1}{2}} \| W_1 - W_2 \|.$$

Below we go on with the proof of Theorem 2.2 by considering N_4, N_5, N_6 and we have

$$|N_4(W_1) - N_4(W_2)| \leq M \| W_1 - W_2 \| \sigma^2$$

$$|N_5(W_1) - N_5(W_2)| \leq M\sigma^{3/2} \| W_1 - W_2 \|,$$

$$|N_6(W_1) - N_6(W_2)| \leq M \| W_1 - W_2 \| \sigma.$$

Combining the estimates for N_1, N_2, N_3, N_4, N_5 and N_6 we have proved that

$$|T(W_1(t)) - T(W_2(t))| \leq M\sigma^{\frac{1}{2}} \| W_1 - W_2 \| \text{ for } \sigma \leq \sigma_0 \text{ and } \sigma_0 < 1,$$

where M depends only on A, α, $\|\varphi\|$, $\|Q_1\|$, $\|Q_2\|$. Choose $0 < \sigma_1 < \sigma_0$ such that $M\sigma_1^{\frac{1}{2}} < 1$. Then we get that T is a contraction of $C_{\sigma_1, A}$ into $C_{\sigma_1, A}$. Therefore, Theorem 2.2 is proved.

Theorem 2.2 implies that there exists unique fixed point. Thus, we have the following existence theorem for problem (1.1).

Theorem 2.3 (Existence): *Suppose that* $\varphi \in C^{(1)}[0,l]$, $\varphi(0) = 0$, $Q_1 \in C[0,T_0]$, $Q_2 \in C[0,T_0]$, *constant* $T_0 > 1$. *Then, there exists* $0 < \sigma_1 < 1$ *such that the solution* $u(x,t)$, $s(t)$ *of the problem* (1.1) *exists on* $[0,\sigma_1]$.

Theorem 2.4 (Uniqueness): *Assume that* $\varphi \in C^{(1)}[0,l]$, $\varphi(0) = 0$, $Q_1 \in C[0,T_0]$, $Q_2 \in C[0,T_0]$ *and constant* $T_0 > 1$. *Then, the solution of problem* (1.1) *is unique on* $[0,\sigma_1]$ *and the constant* σ_1 *is the same as in Theorem 2.3.*

Proof: Let $u_0(x,t)$ be another solution of (1.1) on $[0,\sigma_1]$, with $s(t)$ replaced by $s_0(t)$ and let $W_0(t)$ be the solution of corresponding integral equation. Moreover, let

$$\bar{A} = max\left\{A, \sup_{0 \leq t \leq \sigma_1} |W_0(t)|\right\}.$$

Choose σ_2 sufficiently small such that for any $W \in C_{\sigma_2, \bar{A}}$, where

$$C_{\sigma_2, \bar{A}} = \{W(t): W(t) \in C[0,\sigma_2], |W(t)| \leq \bar{A}\},$$

mapping $T(W)$ is a contraction of $C_{\sigma_2, \bar{A}}$ into $C_{\sigma_2, \bar{A}}$. On $[0,\sigma_2]$ we thus have that $W(t) = u_x(s(t),t) \equiv W_0(t) = u_{0_x}(s_0(t),t)$, i.e., in the region $D_{0,2} = \{s(t) \leq x < l, \; 0 \leq t \leq \sigma_2\}$ the solution of (1.1) is unique. For $D_{2,1} = \{s(t) \le x < l,\ \sigma_2 \le t \le \sigma_1\}$ we consider the following problem $(1.1)^*$:

$$\left.\begin{array}{ll} \bar{u}_t = \alpha^2 \bar{u}_{xx} + \beta \bar{u}_x + \dot{s}(t)(l-x)(l-s(t))^{-1}\bar{u}_x & \text{in } D_{2,1}. \\ \bar{u}(s(\sigma_2), \sigma_2) = 0,\ \bar{u}(x, \sigma_2) = u(x, \sigma_2) & \text{for } s(t) \le x < l, \\ \bar{u}(l,t) = -Q_1(t) & \text{for } \sigma_2 \le t \le \sigma_1, \\ \bar{u}(s(t),t) \equiv 0 & \text{for } \sigma_2 \le t \le \sigma_1, \\ \dot{s}(t) = \bar{u}_x(s(t),t) + Q_2(t), & \text{for } \sigma_2 \le t \le \sigma_1. \end{array}\right\} \quad (1.1)^*$$

Repeating the same procedure as above, we can prove that there exists a constant $\sigma_3 > 0$, $\sigma_2 < \sigma_3 \le \sigma_1$, such that the solution of problem $(1.1)^*$ exists uniquely on $[\sigma_2, \sigma_3]$. Therefore, we have proved that for any $0 < \sigma^* < \sigma_1$ the solution of problem (1.1) exists uniquely on $[0, \sigma^*]$. Therefore, Theorem 2.4 is proved.

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