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# STRONG LAWS OF LARGE NUMBERS FOR ARRAYS OF ROWWISE CONDITIONALLY INDEPENDENT RANDOM ELEMENTS[1]

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**ABSTRACT**

Let $\{X_{nk}\}$ be an array of rowwise conditionally independent random elements in a separable Banach space of type $p$, $1 \leq p \leq 2$. Complete convergence of $n^{-1/r}\sum_{k=1}^{n} X_{nk}$ to 0, $0 < r < p \leq 2$ is obtained by using various conditions on the moments and conditional means. A Chung type strong law of large numbers is also obtained under suitable moment conditions on the conditional means.

**Key words:** Strong law of large numbers, type $p$, rowwise conditionally independent, complete convergence.

**AMS (MOS) subject classifications:** 60B12.

## I. INTRODUCTION AND PRELIMINARIES

Let $(\mathcal{S}, \|\cdot\|)$ be a real separable Banach space. Let $(\Omega, \mathcal{A}, P)$ denote a probability space. A random element (r.e.) $X$ in $\mathcal{S}$ is a function from $\Omega$ into $\mathcal{S}$ which is $\mathcal{A}$-measurable with respect to the Borel subsets $B(\mathcal{S})$. The $r$th absolute moment of a random element $X$ is $E\|X\|^r$ where $E$ is the expected value of the random variable $\|X\|^r$. The expected value of

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a random element $X$ is defined to be the Bochner integral (when $E \| X \| < \infty$) and is denoted by $EX$. The concepts of independence and identical distributions for real-valued random variables extend directly to $\mathcal{S}$. A separable Banach space is said to be of (Rademacher) type $p$, $1 \leq p \leq 2$, if there exist a constant $C$ such that

$$E \| \sum_{k=1}^{n} X_k \|^p \leq C \sum_{k=1}^{n} E \| X_k \|^p \tag{1.1}$$

for all independent random elements $X_1, \ldots, X_n$ with zero means and finite $p$th moments. The sequence of random elements $\{X_n\}$ is said to be conditionally independent if there exists a sub-$\sigma$-field $\zeta$ of $\mathcal{A}$ such that for each positive integer $m$

$$P\left[\bigcap_{i=1}^{m} [X_i \in B_i] \mid \zeta\right] = \prod_{i=1}^{m} P[X_i \in B_i \mid \zeta] \quad a.s.$$

where $P[X_i \in B_i \mid \zeta]$ denotes the conditional probability of the random element $X_i$ being in the Borel set $B_i$ given the sub-$\sigma$-field $\zeta$. Independent random elements are conditionally independent with respect to the trivial $\sigma$-field $\{\emptyset, \mathcal{A}\}$.

Throughout this paper $\{X_{nk}: 1 \leq k \leq n, \ n \geq 1\}$ will denote rowwise conditionally independent random elements in $\mathcal{S}$ such that $EX_{nk} = 0$ for all $n$ and $k$. The first major result of this paper shows that

$$\frac{1}{n^{1/n}} \sum_{k=1}^{n} X_{nk} \to 0 \text{ completely} \tag{1.2}$$

where complete convergence is defined (as in Hsu and Robbins [5]) by

$$\sum_{n=1}^{\infty} P\left[ \| \frac{1}{n^{1/r}} \sum_{k=1}^{n} X_{nk} \| > \epsilon \right] < \infty \tag{1.3}$$

for each $\epsilon > 0$. The second major result is a Chung type strong law of large numbers (SLLN) which provides

$$\frac{1}{a_n} \sum_{k=1}^{n} X_{nk} \to 0 \quad a.s. \tag{1.4}$$

where $a_n < a_{n+1}$ and $\lim_{n \to \infty} a_n = \infty$. For comparisons with (1.2) and (1.4), a brief partial review of previous results will follow.

Erdös [4] showed that for an array of i.i.d. random variables $\{X_{nk}\}$, (1.2) holds if and only if $E \mid X_{11} \mid^{2r} < \infty$. Jain [8] obtained a uniform SLLN for sequences of i.i.d. r.e.'s in a separable Banach space of type 2 which would yield (1.2) with $r = 1$ for an array of r.e.'s $\{X_{nk}\}$. Woyczynski [12] showed that

$$\frac{1}{n^{1/r}} \sum_{k=1}^{n} X_k \to 0 \quad \text{completely} \tag{1.5}$$

for any sequence $\{X_n\}$ of independent r.e.'s in a Banach space of type $p$, $1 \le r < p \le 2$ with $EX_n = 0$ for all $n$ which is uniformly bounded by a random variable $X$ satisfying $E|X|^r < \infty$. Recall that an array $\{X_{nk}\}$ of r.e.'s is said to be uniformly bounded by a random variable $X$ if for all $n$ and $k$ and for every real number $t > 0$

$$P[\|X_{nk}\| > t] \le P[|X| > t]. \tag{1.6}$$

Hu, Moricz and Taylor [7] showed that Erdös' result could be obtained by replacing the i.i.d. condition by the uniformly bounded condition (1.6). Taylor and Hu [9] obtained complete convergence in type $p$ spaces, $1 < p \le 2$ for uniformly bounded, rowwise independent r.e.'s. Bozorgnia, Patterson and Taylor [1] obtained a more general result by replacing the assumption of uniformly bounded random elements with moment conditions. One complete convergence result of this paper, given in Section 2, is obtained by assuming a condition on the conditional means and extends the result in Bozorgnia et. al [1].

If $\{X_n\}$ is a sequence of independent (but not necessarily identically distributed) r.v.'s, Chung's SLLN yield (1.4) for r.v.'s if $\Psi(t)$ is a positive, even, continuous function such that either

$$\Psi(t)\downarrow \text{ as } |t|\uparrow\infty \tag{1.7}$$

and

$$\sum_{n=1}^{\infty} \frac{E\Psi(X_n)}{\Psi(a_n)} < \infty \tag{1.8}$$

where $\{a_n\}$ is a sequence of real numbers such that $a_n < a_{n+1}$ and $\lim_{n\to\infty} a_n = \infty$ hold, or

$$\frac{\Psi(t)}{|t|}\uparrow \text{ and } \frac{\Psi(t)}{|t|^2}\downarrow \text{ as } |t|\uparrow\infty \tag{1.9}$$

$EX_n = 0$ and (1.8) holds where $\uparrow$ and $\downarrow$ denote monotone increasing and monotone decreasing respectively.

Wu, Taylor and Hu [6] considered SLLN's for arrays of rowwise independent random variables, $\{X_{ni}: 1 \le i \le n,\ n \ge 1\}$. They obtained Chung type SLLN's under the more general conditions:

$$\frac{\Psi(|t|)}{|t|^r}\uparrow \text{ and } \frac{\Psi(|t|)}{|t|^{r+1}}\downarrow \text{ as } |t|\uparrow \tag{1.10}$$

where $\Psi(t)$ is a positive, even function and $r$ is a nonnegative integer,

$$EX_{ni} = 0, \tag{1.11}$$

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E(\Psi(X_{ni}))}{\Psi(a_n)} < \infty, \tag{1.12}$$

and

$$\sum_{n=1}^{\infty}\left[\sum_{i=1}^{n}E\left(\frac{X_{ni}}{a_n}\right)^2\right]^{2k}<\infty \tag{1.13}$$

where $k$ is a positive integer and $\{a_n\}$ is a sequence of positive real numbers defined in (1.4). Combinations of Conditions (1.10), (1.11), (1.12) and (1.13) for different values of $r$ will imply that

$$\frac{1}{a_n}\sum_{i=1}^{n}X_{ni}\to 0 \quad a.s. \tag{1.14}$$

Bozorgnia, Patterson and Taylor [2] obtained Banach space versions of Hu, Taylor and Wu's results using the modified conditions:

$$\frac{\Psi(|t|)}{|t|^r}\uparrow \text{ and } \frac{\Psi(|t|)}{|t|^{r+p-1}}\downarrow \text{ as } |t|\uparrow \tag{1.15}$$

for some nonnegative integer $r$, where the separable Banach space is of type $p$, $1\le p\le 2$,

$$EX_{ni}=0, \tag{1.16}$$

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E(\Psi(\|X_{ni}\|))}{\Psi(a_n)}<\infty, \tag{1.17}$$

and

$$\sum_{n=1}^{\infty}\left[\sum_{i=1}^{n}E\left(\|\frac{X_{ni}}{a_n}\|^p\right)\right]^{pk}<\infty \tag{1.18}$$

where $k$ is a positive integer.

In Section 2 of this paper, SLLN's for arrays of rowwise conditionally independent r.e.'s are obtained for Banach spaces under conditions similar to those of Chung [3], Hu, Taylor and Wu [6] and Bozorgnia, Patterson and Taylor [2] with appropriate conditions on the conditional means. These new results address the question of possible exchangeability extensions in the affirmative, and in addition, provide a class of new results for conditionally independent random elements. A generic constant, $C$, will be used throughout the paper.

## 2. MAIN RESULTS

A lemma by Wozczynski [12], is crucially used in the proofs of the major results, Theorems 2.2 and 2.3, and is stated here for future reference.

**Lemma 2.1:** *Let $1\le p\le 2$ and $q\ge 1$. The following properties are equivalent:*

(*i*) *The separable Banach space, $\mathcal{E}$, is of type $p$.*

(*ii*) *There exist a constant $C$ such that for all independent r.e.'s $X_1,\ldots,X_n$ in $\mathcal{E}$ with*

*$EX_i = 0$, and $E\|X_i\|^q < \infty$, $i = 1,2,\ldots,n$*

$$E\left\|\sum_{i=1}^{n} X_i\right\|^q \le CE\left(\sum_{i=1}^{n}\|X_i\|^p\right)^{q/p}. \qquad ///$$

The constant $C$ depends only on the Banach space $\mathcal{E}$ and not on $n$. Moreover, throughout this section $C$ will denote a generic constant which is not necessarily the same each time used but is always independent of $n$.

**Theorem 2.2:** *Let $\{X_{nk}\}$ be an array of rowwise conditionally independent random elements in a separable Banach space of type p, $1 \le p \le 2$. If*

(*i*) $\sup_{1\le k\le n} E\|X_{nk}\|^{\nu} = O(n^{\alpha})$, $\alpha \ge 0$ *where* $\nu\left(\frac{1}{r}-\frac{1}{p}\right) > \alpha + 1$, $0 < r < p \le 2$

*and*

(*ii*) *for all* $\eta > 0$,

$$\sum_{n=1}^{\infty} P\left(\left\|\frac{1}{n^{1/r}}\sum_{k=1}^{n} E_{\zeta}X_{nk}\right\| > \eta\right) < \infty \tag{2.1}$$

*where $E_\zeta$ is the conditional expectation with respect to an appropriate $\sigma$-field that gives conditional independence, then*

$$\frac{1}{n^{1/r}}\sum_{k=1}^{n} X_{nk} \to 0 \text{ completely.}$$

**Proof:** Let $\epsilon > 0$ be given. By Markov's inequality,

$$\sum_{n=1}^{\infty} P\left(\left\|\frac{1}{n^{1/r}}\sum_{k=1}^{n} X_{nk}\right\| > \epsilon\right) \le \sum_{n=1}^{\infty} P\left(\left\|\frac{1}{n^{1/r}}\sum_{k=1}^{n}(X_{nk} - E_\zeta X_{nk})\right\| > \frac{\epsilon}{2}\right) + \sum_{n=1}^{\infty} P\left(\left\|\frac{1}{n^{1/r}}\sum_{k=1}^{n} E_\zeta X_{nk}\right\| > \frac{\epsilon}{2}\right)$$

$$\le C\sum_{n=1}^{\infty} EE_\zeta\left\|\frac{1}{n^{1/r}}\sum_{k=1}^{n}(X_{nk} - E_\zeta X_{nk})\right\|^{\nu} + \sum_{n=1}^{\infty} P\left(\left\|\frac{1}{n^{1/r}}\sum_{k=1}^{n} E_\zeta X_{nk}\right\| > \frac{\epsilon}{2}\right). \tag{2.2}$$

By Lemma 2.1 and Hölder's inequality, the first term in (2.2) is bounded by

$$C\sum_{n=1}^{\infty} E\left(E_\zeta\left\|\frac{1}{n^{1/r}}\sum_{k=1}^{n}(X_{nk} - E_\zeta X_{nk})\right\|^{\nu}\right) \le C\sum_{n=1}^{\infty}\frac{n^{\nu/p-1}}{n^{\nu/r}}E\left(\sum_{k=1}^{n}\|X_{nk} - E_\zeta X_{nk}\|^{\nu}\right)$$

$$\le C\sum_{n=1}^{\infty}\frac{n^{\nu/p-1}}{n^{\nu/r}}\cdot 2^{\nu}\sum_{k=1}^{n} E\|X_{nk}\|^{\nu}$$

$$\le C\sum_{n=1}^{\infty}\frac{n^{\nu/p}\cdot n^{\alpha}}{n^{\nu/r}}$$

$$= C\sum^{\infty} n^{-\nu(\frac{1}{r}-\frac{1}{p})+\alpha} < \infty.$$

The second term of (2.1) is finite by (*ii*). Thus, the result follows. ///

**Remark 1:** Condition (*ii*) can be replaced by the condition $E\,\|\,E_\zeta X_{n1}\,\|^{\,p} = O(n^{-\beta})$, $\beta > \frac{2r-p}{r}$, if the r.e.'s are conditionally i.i.d. or rowwise infinitely exchangeable. If $0 < r < 1$, and $p/r > 2$, then $\beta$ can be nonpositive and the bound for each row can increase.

**Remark 2:** If the random elements are independent with zero means, then condition (*ii*) is identically zero when $\zeta$ is chosen to be the trivial $\sigma$-field, $\{\emptyset, \mathcal{A}\}$. Thus, Theorem 2.2 generalizes the results of Bozorgnia et. al [1].

**Remark 3:** Condition (*i*) implies condition (6.2.2) in Theorem 6.2.3 of Taylor, Daffer and Patterson [10]. Condition (6.2.2) was given as:

$$\sum_{n=1}^{\infty} \frac{E(\,\|\,X_{n1}\,\|^{\,pq})}{n^{q(p-1)}} < \infty.$$

Letting $\nu = pq$ and $r = 1$, it follows that the third inequality in the proof of Theorem 2.2 is majorized by

$$C\sum_{n=1}^{\infty} \frac{n \cdot n^{\nu/p-1}}{n^{\nu/r}} \cdot \sup_{1 \le k \le n} E\,\|\,X_{nk}\,\|^{\,\nu} = C\sum_{n=1}^{\infty} \frac{n^{\nu/p}}{n^{\nu}} \cdot E\,\|\,X_{n1}\,\|^{\,\nu}$$

$$\le C\sum_{n=1}^{\infty} \frac{n^{q} \cdot n^{\alpha}}{n^{pq}}$$

$$= C\sum_{n=1}^{\infty} n^{-q(p-1)+\alpha}$$

which is a substantial improvement of Theorem 6.2.3 of Taylor et. al [10]. Moreover, Condition (6.2.1) of Theorem 6.2.3 in Taylor et. al [10] implies Condition (*ii*) of Theorem 2.2 since for $\{X_{nk}\}$ rowwise infinitely exchangeable and $r = 1$,

$$\sum_{n=1}^{\infty} P\left[\,\|\,\frac{1}{n}\sum_{k=1}^{n} E_\zeta X_{nk}\,\| > \eta^{1/\nu}\right] \le \sum_{n=1}^{\infty} P(\,\|\,E_\zeta X_{n1}\,\|^{\,\nu} > \eta)$$

$$= \sum_{n=1}^{\infty} \int P_\zeta(\,\|\,E_\zeta X_{n1}\,\|^{\,\nu} > \eta)d\mu_n(P_\zeta)$$

$$= \sum_{n=1}^{\infty} \mu_n\{P_\zeta\colon \|\,E_\zeta X_{n1}\,\|^{\,\nu} > \eta\} < \infty,$$

where $\mu_n$ denotes the mixing measure for the exchangeable sequence $\{X_{n1}, X_{n2}, \ldots\}$ and $P_\zeta$ denotes the conditional probability.

The next result is a Chung type SLLN for arrays of rowwise conditionally independent r.e.'s in a separable Banach space of type $p$, $1 \le p \le 2$. Let $\{a_n\}$ be a sequence of positive real numbers such that $a_n < a_{n+1}$ and $\lim_{n\to\infty} a_n = \infty$. Let $\Psi(t)$ be the positive, even function defined in (1.15).

**Theorem 2.3:** *Let $\{X_{ni}\}$ be an array of rowwise, conditionally independent random elements in a separable Banach space of type p, $1 \le p \le 2$ such that $EX_{ni} = 0$ for all n and i. Let $\Psi(t)$ satisfy (1.15) for some $r \ge 2$. If $\{a_n\}$ is a sequence of positive real numbers such that $a_n < a_{n+1}$ and $\lim_{n\to\infty} a_n = \infty$ and if*

$$\frac{1}{a_n} \sum_{i=1}^{n} E_\zeta X_{ni} \to 0 \text{ completely,} \tag{2.3}$$

*and if for some positive integer k*

$$\sum_{n=1}^{\infty} E\left( \sum_{i=1}^{n} E_\zeta \| \frac{X_{ni}}{a_n} \|^p \right)^{pk} < \infty, \tag{2.4}$$

*then Condition (1.17) implies that*

$$\frac{1}{a_n} \sum_{i=1}^{n} X_{ni} \to 0 \quad a.s.$$

**Proof:** Let $Y_{ni} = X_{ni} I_{[\|X_{ni}\| \le a_n]}$ and $Z_{ni} = X_{ni} I_{[\|X_{ni}\| > a_n]}$. Using Markov's inequality and Condition (1.1.7), it follows that the two sequences

$$\left\{ \sum_{i=1}^{n} \left( \frac{X_{ni}}{a_n} \right) \right\} \text{ and } \left\{ \sum_{i=1}^{n} \left( \frac{Y_{ni}}{a_n} \right) \right\}$$

are equivalent. Conditions (1.15), (1.16) and (1.17) imply that

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} \| E\left( \frac{Y_{ni}}{a_n} \right) \| = \sum_{n=1}^{\infty} \sum_{i=1}^{n} \| E\left( \frac{Z_{ni}}{a_n} \right) \|$$

$$\le \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E(\Psi(\|X_{ni}\|))}{\Psi(a_n)} < \infty. \tag{2.5}$$

Next,

$$\| \frac{1}{a_n} \sum_{i=1}^{n} (Y_{ni} - EY_{ni}) \| \le \| \frac{1}{a_n} \sum_{i=1}^{n} (Y_{ni} - E_\zeta Y_{ni}) \|$$

$$+ \| \frac{1}{a_n} \sum_{i=1}^{n} E_\zeta Y_{ni} \| + \| \frac{1}{a_n} \sum_{i=1}^{n} EY_{ni} \| . \tag{2.6}$$

The last term of (2.6) converges to 0 by (2.5). By Condition (2.3)

$$\| \frac{1}{a_n} \sum_{i=1}^{n} E_\zeta Y_{ni} \| \to 0 \text{ completely}$$

if and only if

$$\| \frac{1}{a_n} \sum_{i=1}^{n} E_\zeta Z_{ni} \| \to 0 \quad \text{completely.} \tag{2.7}$$

But, (2.7) follows from (1.17) since

$$E\,\|\frac{1}{a_n}\sum_{i=1}^{n}E_\zeta Z_{ni}\| \le E\left(E_\zeta\,\|\sum_{i=1}^{n}\frac{Z_{ni}}{a_n}\|\right)$$
$$\le \sum_{i=1}^{n}E\,\|\frac{Z_{ni}}{a_n}\|$$
$$\le \sum_{i=1}^{n}\frac{E(\Psi(\|Z_{ni}\|))}{\Psi(a_n)}$$
$$\le \sum_{i=1}^{n}\frac{E(\Psi(\|X_{ni}\|))}{\Psi(a_n)}.$$

Thus, it suffices to show that

$$\sum_{i=1}^{n}\frac{Y_{ni}-E_\zeta Y_{ni}}{a_n}\longrightarrow 0 \ \ a.s. \tag{2.8}$$

Let $W_{ni}=\frac{Y_{ni}}{a_n} - \frac{E_\zeta Y_{ni}}{a_n}$ for all $n$ and $i$. Then, $\|W_{ni}\| \le 2$ and $E_\zeta W_{ni}=0$. Now by Lemma 2.1,

$$E\,\|\sum_{i=1}^{n}W_{ni}\|^{pk(r+1)} = E\left(E_\zeta\,\|\sum_{i=1}^{n}W_{ni}\|^{pk(r+1)}\right)$$
$$\le CE\left(E_\zeta\left(\sum_{i=1}^{n}\|W_{ni}\|^{p}\right)^{k(r+1)}\right)$$
$$= CE\sum{}^{*}\binom{k(r+1)}{s_1,\ldots,s_n}E_\zeta\,\|W_{n1}\|^{ps_1}\cdots E_\zeta\,\|W_{nn}\|^{ps_n} \tag{2.9}$$

where the sum $\sum^*$ is over all choices of nonnegative integers $s_1,\ldots,s_n$ such that $\sum_{i=1}^{n} s_i = k(r+1)$. Now (2.9) can be shown to be summable with respect to $n$ following the same steps as in the proof of Theorem 2.2 of Bozorngia et. al [2] for the case $s_i p \ge r+1$ for at least one $s_i$. The case $s_i p < r+1$ for all $i$ is accomplished by using (2.4) instead of (1.18). Hence, the result follows. ///

**Remark 4:** Theorem 2.2 extends the random variable result in Hu, Taylor and Wu [6] for $p=2$ and the random element results in Bozorgnia, Patterson and Taylor [2] to the class of conditionally independent random variables and random elements. Again, if the r.e.'s are rowwise independent with zero means, then Condition (2.3) is equal to zero via the trivial $\sigma$-field, and (2.4) becomes (1.18) with the trivial $\sigma$-field.

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