

ANTI-PERIODIC TRAVELING WAVE SOLUTIONS TO A CLASS OF HIGHER-ORDER KADOMTSEV-PETVIASHVILI-BURGERS EQUATIONS¹

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ABSTRACT

We discuss the existence, uniqueness, and continuous dependence on data, of anti-periodic traveling wave solutions to higher order two-dimensional equations of Korteweg-deVries type.

Key words: KPB equation, anti-periodic solution, traveling wave solution, existence and uniqueness.

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1. INTRODUCTION

The well-known Korteweg-deVries (referred to as KdV henceforth) equation was derived in 1895 [14]. It is a non linear evolution equation governing long one-dimensional, small amplitude, surface gravity waves propagating in a shallow channel of water. It was rediscovered in 1960, in the study of collision-free hydromagnetic waves [9]. Subsequently, the KdV equation has arisen in a number of physical problems, such as stratified internal waves, ion-acoustic waves, plasma physics, and lattice dynamics. A survey of results and applications for the KdV equation was written by Miura [16].

A two-dimensional generalization of the KdV equation is the Kadomtsev-Petviashvili (KP) equation, which was obtained in 1970 in the study of plasmas [11]. The evolution described by the KP equation is weakly nonlinear, weakly dispersive, and weakly two-dimensional, with all three effects being of the same order. The KP equation has also been proposed as a model for surface waves and

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internal waves in channels of varying depth and width [18].

A fifth-order KdV equation was considered by Nagashima [17], and Sawada and Kotera [19]. A related $(2 + 1)$ -dimensional variant appears in [13]. Higher-order KdV-like evolution equations were investigated in [4, 12]. Such equations may provide realistic models for various physical processes, including the propagation of small amplitude surface waves in straits or large channels of varying width and depth [1]. In this context, the Burgers equation appears as a one-dimensional analog of the equation governing viscous compressible flows [8]. KdV equations perturbed by a Burgers-like term were studied in [10, 15].

In a recent paper [3], two of the authors have discussed the existence and properties of anti-periodic traveling wave solutions to a nonhomogeneous generalized KP equation. It is the purpose of the present note to extend the theory of [3] to a broader class of Kadomtsev-Petviashvili-Burgers (KPB) equations. For simplicity and clarity of exposition, we specifically consider the equation

$$\{u_t + [f(u)]_x + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx}\}_x + \delta u_{yy} = \tilde{g} \quad (t \geq 0, x, y \in \mathbb{R}), \quad (1.1)$$

where $f \in C^2(\mathbb{R})$, α , β , γ and δ are real constants, while \tilde{g} is a real-valued function of x , y and t . The case when u is independent of y , $\tilde{g} = 0$, and $f(u) = \frac{1}{2}u^2$ in (1.1), corresponds to a KdV-Burgers type equation. (In particular, if also $\alpha < 0$ and $\beta = \gamma = 0$, we get the classical Burgers equation). If $\tilde{g} = 0$, $f(u) = \frac{1}{2}u^2$, $\alpha = \beta = 0$, $\gamma = -1$, and u is independent of y , (1.1) reduces to a fifth-order KdV equation. In the case when $\alpha = \gamma = 0$, we recover the equation studied in [3]. (If, in addition, $f(u) = \frac{1}{2}u^2$ and $\tilde{g} = 0$, we obtain the KP equation). Finally, we remark that (1.1) is a special case of the more general, higher-order equation

$$\{u_t + [f(u)]_x + \alpha u_{xx} + \beta u_{xxx} + \gamma (-1)^m D_x^{2m+1} u\}_x + \delta u_{yy} = \tilde{g}, \quad (1.2)$$

where m is an integer ≥ 2 .

The plan of the paper is as follows. In Section 2, we reduce the study of anti-periodic traveling wave solution to Eq. (1.1) to that of a fourth-order boundary-value problem. The main existence, uniqueness and continuous dependence results are stated in Section 3. The proofs, based on monotonicity methods and a Leray-Schauder type technique, are given in Section 4. In the last section (Section 5), we comment on the possibility of generalizing our results to

cover Eq. (1.2).

2. FORMULATION OF THE PROBLEM

We consider Eq. (1.1) with $f \in C^2(\mathbb{R})$, $\delta \neq 0$, and (for convenience) $\gamma > 0$. The case when $\delta > 0$ ($\delta < 0$), pertains to a medium with negative (positive) dispersion.

We are interested in the existence of anti-periodic traveling wave solution to (1.1), of the form

$$u(x, y, t) = U(z), \quad z = ax + by - wt, \quad (2.1)$$

where $a \neq 0$ (and for simplicity, we will suppose that $a > 0$), while b and w are real constants. Correspondingly, we make the natural assumption that \tilde{g} depends on z only, i.e.,

$$\tilde{g}(x, y, t) = g(ax + by - wt), \quad \text{with } g: \mathbb{R} \rightarrow \mathbb{R}. \quad (2.2)$$

Straightforward computations then show that (1.1) reduces to the sixth-order ordinary differential equation

$$U^{(6)}(z) + cU^{(4)}(z) + dU^{(3)}(z) + eU^{(2)}(z) + h\frac{d^2}{dz^2}f(U(z)) = g_1(z), \quad (2.3)$$

where

$$\begin{aligned} c &= a^{-2}\beta\gamma^{-1}, \quad d = a^{-3}\alpha\gamma^{-1}, \quad e = (b^2\delta - aw)a^{-6}\gamma^{-1}, \\ h &= a^{-4}\gamma^{-1}, \quad g_1(z) = a^{-6}\gamma^{-1}g(z). \end{aligned} \quad (2.4)$$

We consider (2.3) in conjunction with the anti-periodic condition

$$U(z + T) = -U(z), \quad z \in \mathbb{R}, \quad (2.5)$$

where $0 < T < \infty$ is a fixed constant.

Remark. If $\tilde{g} \equiv 0$, condition (2.5) implies that the only constant solution to (2.3), (2.5) is the trivial solution.

To further simplify our analysis, we impose additional restrictions on f and g , namely:

$$f(-r) = -f(r), \quad \forall r \in \mathbb{R} \text{ (i.e., } f \text{ is odd)}, \quad (2.6)$$

and respectively

$$g \in C(\mathbb{R}) \text{ and } g(z + T) = -g(z) \text{ (} z \in \mathbb{R}\text{)}. \quad (2.7)$$

We now confine our attention to the anti-periodic boundary value problem (on $[0, T]$)

$$\begin{aligned} (i) \quad & U^{(6)}(z) + cU^{(4)}(z) + dU^{(3)}(z) + eU^{(2)}(z) \\ & + h\frac{d^2}{dz^2}f(U(z)) = g_1(z), \quad 0 \leq z \leq T, \end{aligned} \quad (2.8)$$

$$(ii) U^{(k)}(0) = -U^{(k)}(T) \quad (k = 0, 1, \dots, 5),$$

where c, d, e, h and g_1 are given by (2.4). It is obvious that the restriction to $[0, T]$ of any C^6 -solution to (2.3), (2.5) satisfies (2.8), and conversely, if $U \in C^6[0, T]$ is a solution of (2.8), then by (2.6), (2.7), its T -anti-periodic extension to \mathbb{R} satisfies (1.1), (2.5).

Next, introduce the function $G: [0, T] \rightarrow \mathbb{R}$ by

$$G(z) = \int_0^z \int_0^s g_1(t) dt ds + \frac{1}{4}(T-2z) \int_0^T g_1(t) dt - \frac{1}{2} \int_0^T \int_0^s g_1(t) dt ds, \quad 0 \leq z \leq T. \quad (2.9)$$

It is readily verified that $G \in C^2[0, T]$ and

$$G''(z) = g_1(z), \quad G^{(k)}(0) = -G^{(k)}(T) \quad (k = 0, 1). \quad (2.10)$$

Integrate now (2.8) (i) twice and make use of (2.8) (ii), (2.9) and (2.10), to obtain

$$\begin{aligned} (i) \quad & U^{(4)}(z) + cU''(z) + dU'(z) + eU(z) + F(U(z)) = G(z), \quad 0 \leq z \leq T, \\ (ii) \quad & U^{(k)}(0) = -U^{(k)}(T) \quad (k = 0, 1, 2, 3), \end{aligned} \quad (2.11)$$

where

$$F(r) = hf(r), \quad r \in \mathbb{R}. \quad (2.12)$$

Conversely, differentiating (2.11) (i) twice and employing (2.6), (2.10), (2.11) (ii) and (2.12), leads to (2.8). (Note that if $U \in C^4[0, T]$ is a solution of (2.11), then actually $U \in C^6[0, T]$, since $F \in C^2(\mathbb{R})$ and $G \in C^2[0, T]$).

We have thereby established the following result

Theorem 1. *Let $f \in C^2(\mathbb{R})$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ satisfy (2.6) and (2.7). Then the problem (2.8) (where c, d, e, h , and g_1 are given by (2.4)) is equivalent to (2.11) (where F and G are defined by (2.12) and (2.9), respectively).*

3. MAIN RESULTS

We are primarily concerned with the existence, uniqueness, and continuous dependence on data of solutions to Eq. (2.11). Although we view (2.11) as a boundary-value problem of independent interest, our assumptions are compatible with, and motivated by, (2.6), (2.7), (2.10) and (2.12).

We first suppose that

$$\begin{aligned} (i) \quad & c \leq 0, \quad e \geq 0, \quad d \in \mathbb{R}, \\ (ii) \quad & F \in C(\mathbb{R}), \quad F \text{ is monotonically nondecreasing,} \end{aligned} \quad (3.1)$$

(iii) $G \in C[0, T]$.

We then have:

Theorem 2. *Let conditions (3.1) be satisfied. Then the problem (2.11) has a unique solution $U \in C^4[0, T]$.*

Next, let (3.1) (i) hold, and let F_n, G_n ($n = 1, 2, \dots$) be real functions, such that

$$\begin{aligned} F_n \text{ and } G_n \text{ satisfy (3.1) (ii) (with } F_n \text{ and } G_n \\ \text{in place of } F \text{ and } G, \text{ respectively).} \end{aligned} \quad (3.2)$$

By Theorem 2, for $n = 1, 2, \dots$, the boundary-value problem

$$\begin{aligned} (i) \quad U_n^{(4)}(z) + cU_n''(z) + dU_n'(z) + eU_n(z) + F_n(U_n(z)) = G_n(z), \quad 0 \leq z \leq T, \\ (ii) \quad U_n^{(k)}(0) = -U_n^{(k)}(T) \quad (k = 0, 1, 2, 3), \end{aligned} \quad (3.3)$$

has a unique solution $U_n \in C^4[0, T]$. The following is a continuous dependence result

Theorem 3. *Let (3.1), (3.2) be satisfied, and let U and U_n denote the solutions to (2.11) and (3.3), respectively. If also, as $n \rightarrow \infty$,*

$$\begin{aligned} F_n &\rightarrow F \text{ in } C[0, l], \text{ for any } 0 < l < \infty, \\ G_n &\rightarrow G \text{ in } C[0, T], \end{aligned} \quad (3.4)$$

then

$$U_n \rightarrow U \text{ in } C^4[0, T] \quad (n \rightarrow \infty) \quad (3.5)$$

We now consider the case when the constant e is of arbitrary sign, and F is only required to be continuous and odd. The price we pay for such a generality is that $d \neq 0$ and the uniqueness of solutions of (2.11) is no longer guaranteed. The corresponding existence result is:

$$\begin{aligned} \textbf{Theorem 4.} \quad \textit{Assume that (3.1) (iii) holds, and that} \\ F \in C(\mathbb{R}); F \text{ is odd.} \end{aligned} \quad (3.6)$$

If also $c \leq 0$, $d \neq 0$ and $e \in \mathbb{R}$, then the problem (2.11) has at least one solution $U \in C^4[0, T]$.

Finally, by combining our analysis of problem (2.11) (in particular, Theorems 2 and 4), with the discussion of problem (2.3), (2.5) in Section 2 (in particular, Theorem 1), we obtain:

$$\textbf{Theorem 5.} \quad \textit{Let } f \in C^2(\mathbb{R}) \text{ and } g \text{ satisfy (2.6) and (2.7), respectively,}$$

and let $c, d, e, h,$ and g_1 be given by (2.4). Then the following holds.

(i) If $c \leq 0$ and $d \neq 0$, then the problem (2.3), (2.5) has at least one solution $U \in C^6(\mathbb{R})$,

(ii) If $c \leq 0, e \geq 0$, and f is also monotonically nondecreasing, then the problem (2.3), (2.5) has a unique solution $U \in C^6[0, T]$.

4. PROOFS

We only present the proofs of Theorems 2, 3 and 4. As already mentioned in Section 3, Theorem 5 is a direct consequence of Theorems 1, 2 and 4, and the general discussion in Section 2. (Note that $a > 0$ and $\gamma > 0$ imply, by (2.4), that $h > 0$, which is essential in deriving conclusion (ii) of Theorem 5 (cf. (2.12)). For conclusion (i), we only need $a, \gamma \neq 0$).

Proof of Theorem 2. Consider the space $L^2(0, T)$ with the usual norm and inner product, denoted by $\|\cdot\|$ and (\cdot, \cdot) , respectively. For $p = 1, 2$, we define the linear operator

$$B_p U = \epsilon_p U^{(p)}, \quad \epsilon_p = \begin{cases} d, & \text{if } p = 1 \\ c, & \text{if } p = 2 \\ 1 & \text{if } p = 4 \end{cases} \quad (4.1)$$

$$D(B_p) = \{U \in W^{p,2}(0, T) : U^{(k)}(0) = -U^{(k)}(T), k = 0, 1, \dots, p-1\}.$$

By [5, Theorem 1] and (3.1) (i), it is easily verified that each B_p , given by (4.1), is maximal monotone in $L^2(0, T)$. Next recall the Poincaré type inequality (cf. e.g., [2, Proposition 1.5])

$$|U(t)| \leq \frac{1}{2} T^{1/2} \|U'\|, \quad t \in [0, T], \quad \forall U \in D(B_1). \quad (4.2)$$

(Actually, in [2], (4.2) appears with $T^{1/2}$ instead of $\frac{1}{2}T^{1/2}$, but it is immediate that $\frac{1}{2}T^{1/2}$ is the best possible constant). A repeated application of (4.2), in conjunction with [6, Theorem 2.4] then leads to the conclusion that the operator B , defined by

$$BU = (B_4 + B_2 + B_1 + eI)U, \quad D(B) = D(B_4), \quad (4.3)$$

is maximal monotone in $L^2(0, T)$. In addition, B satisfies

$$(BU, U) \geq k_1 \|U''\|^2, \quad \forall U \in D(B) \quad (4.4)$$

where k_1 denotes a positive constant, which is independent of U . Now remark that any C^4 -solution of (2.11) satisfies the equation

$$(B + \tilde{F})U = G, \quad (4.5)$$

in $L^2(0, T)$, where \tilde{F} is the L^2 -extension of F . Conversely, if $U \in C^4[0, T]$

satisfies (4.5), then it is a solution of (2.11). On account of the properties of B , and assumptions (3.1) (ii) and (iii), we can adapt the proof of Theorem (3.1) (i) in [2] (taking, in the setup of [2], $H = \mathbb{R}$ and $A = 0$), to conclude that Eq. (4.5) has a solution $U \in W^{4,2}(0, T)$. The continuity of F and G implies that $U^{(4)} \in C[0, T]$; therefore, $U^{(4)} \in C[0, T]$, and the existence of a C^4 -solution to (2.11) has been established. The uniqueness is a direct consequence of (4.2), (4.4) and (4.5)

Proof of Theorem 3. Let $(\mathcal{C}, |\cdot|_{\mathcal{C}})$ denote the space $C[0, T]$ with the usual sup-norm. In view of (4.1), (4.3) and (4.5), it is obvious that (3.3) can be rewritten as

$$(B + F_n)U_n = G_n, \quad (4.6)$$

in $L^2(0, T)$. Form the L^2 -inner product of (4.6) with U_n and use (3.2), (3.4), (4.2), (4.4), and Hölder's inequality, to obtain

$$\{U_n''\} \text{ is bounded in } L^2(0, T). \quad (4.7)$$

Applying (4.2) successively, with U_n' and U_n'' in place of U , it follows from (4.7) that

$$\{U_n\} \text{ is bounded in } \mathcal{C}. \quad (4.8)$$

Next, by (4.6), we have

$$B(U_n - U_m) + F_n(U_n) - F_m(U_m) = G_n - G_m. \quad (4.9)$$

Rewrite $F_n(U_n) - F_m(U_m)$ as $(F_n(U_n) - F(U_n)) + (F(U_n) - F(U_m)) + (F(U_m) - F_m(U_m))$, and take the inner-product of (4.9) with $U_n - U_m$. Invoking (3.1) (ii), (4.2) and (4.4), we arrive at

$$\|U_n'' - U_m''\| \leq k_2[|F_n(U_n) - F(U_n)|_{\mathcal{C}} + |F(U_m) - F_m(U_m)|_{\mathcal{C}} + |G_n - G_m|_{\mathcal{C}}],$$

for some $k_2 > 0$ (which is independent of n and m). This, in conjunction with (3.4), (4.2) and (4.8), implies that $\{U_n\}$ is a Cauchy sequence in $W^{2,2}(0, T)$; consequently

$$U_n \rightarrow U_* \text{ in } W^{2,2}(0, T), \text{ as } n \rightarrow \infty. \quad (4.10)$$

Going back to (3.3) and using (3.4) and (4.10) yields

$$U_n^{(4)} \rightarrow U_*^{(4)} \text{ in } L^2(0, T), \text{ as } n \rightarrow \infty. \quad (4.11)$$

Applying again (4.2) with $(U_n - U_m)^{(3)}$ in place of U , we deduce (by (4.10) and (4.11)) that

$$U_n \rightarrow U_* \text{ in } C^3[0, T]. \quad (4.12)$$

Passing to the limit as $n \rightarrow \infty$ in (3.3) and making use of (3.4) and (4.12) we see that actually $U_n \rightarrow U_*$ in $C^4[0, T]$, and that U_* satisfies (2.11). Since the solution

of (2.11) is unique (cf. Theorem 2), U_* must coincide with U , and (3.5) follows.

Proof of Theorem 4. For each $w \in C[0, T]$, let $U_w \in C^4[0, T]$ be the unique solution of

$$\begin{aligned} (i) \quad & U_w^{(4)}(z) + cU_w''(z) + dU_w'(z) = G(z) - ew(z) - F(w(z)), \quad z \in [0, T], \\ (ii) \quad & U_w^{(k)}(0) = -U_w^{(k)}(T) \quad (k = 0, 1, 2, 3). \end{aligned} \quad (4.13)$$

The existence and uniqueness of U_w follows from Theorem 2 (where we take $e = 0$, $F = 0$, and replace G by $G - ew - F \circ w$). Define the map $\mathfrak{F}: \mathcal{C} \rightarrow \mathcal{C}$ by $\mathfrak{F}w = U_w$, and invoke Theorem 3 (with $F_n = F = 0$, and $e = 0$) to conclude that \mathfrak{F} is continuous. Moreover, \mathfrak{F} is compact, in the sense that it maps bounded subsets into precompact subsets of \mathcal{C} . To see this, let w belong to a bounded subset of \mathcal{C} . Since, by assumptions (3.1) (iii) and (3.6), F and G are continuous, it is clear that the right-hand side of (4.13) will then lie in a bounded subset of \mathcal{C} , as well. Multiplying (4.13) (i) by $U_w(t)$ and integrating the result over $(0, T)$ yields (on account of (4.2), (4.13) (ii), $c \leq 0$, and Hölder's inequality) that $\{U_w\}$ is uniformly bounded and equicontinuous. Therefore, by the Ascoli-Arzelà theorem, $\{U_w\}$ is precompact in \mathcal{C} , as needed. (Note that the condition $d \neq 0$ has not yet been used).

We next employ a Leray-Schauder type argument, comparable to the one of [7, p.244]. (One can also rely on the result of [20]). Specifically, we show that there exists a sufficient large $r > 0$, such that if $v \in \mathcal{C}$ satisfies

$$\mathfrak{F}v = \lambda v \quad (4.14)$$

for some $\lambda \geq 1$, then

$$|v(z)| < r, \quad z \in [0, T]. \quad (4.15)$$

By the definition of \mathfrak{F} , it follows from (4.14) that $v \in C^4[0, T]$, and

$$\begin{aligned} (i) \quad & \lambda v^{(4)}(z) + c\lambda v''(z) + d\lambda v'(z) = G(z) - ev(z) - F(v(z)), \quad z \in [0, T], \\ (ii) \quad & v^{(k)}(0) = -v^{(k)}(T) \quad (k = 0, 1, 2, 3). \end{aligned} \quad (4.16)$$

Now recall (cf. (3.6)) that F is odd. This implies that the function $F_1: \mathbb{R} \rightarrow \mathbb{R}$, defined by $F_1(x) = \int_0^x F(t)dt$, is even (i.e., $F_1(-x) = F_1(x)$). Moreover, $F_1 \in C^1(\mathbb{R})$, and

$$\frac{d}{dz}F_1(v(z)) = v'(z)F(v(z)), \quad z \in [0, T], \quad (4.17)$$

for all $v \in C^1[0, T]$.

Multiply (4.16) (i) by $v'(z)$ and integrate over $(0, T)$. Taking into account (4.16) (ii), (4.17) and the evenness of F_1 , we obtain

$$|d|\|v'\|^2 \leq \|G\| \|v'\|. \quad (4.18)$$

Inasmuch as $d \neq 0$ (by assumption) and $\lambda \geq 1$, (4.18) leads to

$$\|v'\| \leq |d|^{-1} \|G\|.$$

Recalling (4.2), we finally have

$$|v(z)| \leq \frac{1}{2} |d|^{-1} T^{1/2} \|G\|, \quad z \in [0, T].$$

This shows that (4.15) holds as soon as r satisfies

$$r > \frac{1}{2} |d|^{-1} T^{1/2} \|G\|. \quad (4.19)$$

Let next P_r denote the r -radial retraction in \mathbb{C} , i.e.,

$$P_r u = \begin{cases} u, & \text{if } |u|_{\mathbb{C}} \leq r, \\ r \frac{u}{|u|_{\mathbb{C}}}, & \text{if } |u|_{\mathbb{C}} \geq r. \end{cases} \quad (4.20)$$

In (4.20), it is assumed that r satisfies (4.19). Remark that P_r is continuous on \mathbb{C} , while $P_r \circ \mathfrak{F}$ is continuous and compact. In addition, by (4.20), $P_r \circ \mathfrak{F}$ maps $B(0, r)$ (the closed ball of radius r , centered at the origin, in \mathbb{C}) into itself. Apply Schauder's fixed point theorem to conclude that there exists $v \in B(0, r)$, such that

$$P_r \mathfrak{F}v = v. \quad (4.21)$$

We claim that

$$|\mathfrak{F}v|_{\mathbb{C}} < r. \quad (4.22)$$

Assume the contrary; that is, $|\mathfrak{F}v|_{\mathbb{C}} > r$. Then, in view of (4.20) and (4.21), we have

$$\mathfrak{F}v = \lambda v, \quad \text{with } \lambda = \frac{|\mathfrak{F}v|_{\mathbb{C}}}{r} \geq 1. \quad (4.23)$$

It also follows that $|v|_{\mathbb{C}} = r$. By the preceding discussion (recall (4.14), (4.15) and (4.19)), we see that (4.23) contradict (4.15); therefore (4.22) must hold.

Finally combining (4.20), (4.21) and (4.22), we deduce that $P_r \mathfrak{F}v = \mathfrak{F}v = v$. By the definition of \mathfrak{F} , this implies that $v = U_v$, and U_v is the desired solution of (2.11). The proof is complete.

5. CONCLUDING REMARKS

The purpose of this section is to outline an extension of our theory to Eq. (1.2).

We consider (1.2) with $\gamma > 0$, $\delta \neq 0$, $f \in C^2(\mathbb{R})$ and $\tilde{g} \in C(\mathbb{R})$, and look for traveling wave solutions of the form (2.1) (where $a > 0$). Letting again \tilde{g} be

given by (2.2), we see that (1.2) reduces to

$$(-1)^m U^{(2m+2)}(z) + cU^{(4)}(z) + dU^{(3)}(z) + eU^{(2)}(z) + h \frac{d^2}{dz^2} f(U(z)) = g_1(z) \quad (5.1)$$

where

$$\begin{aligned} c &= a^{-2m+2} \beta \gamma^{-1}, & d &= a^{-2m+1} \alpha \gamma^{-1}, & e &= (b^2 \delta - aw) a^{-2m-2} \gamma^{-1}, \\ h &= a^{-2m} \gamma^{-1}, & g_1(z) &= a^{-2m-2} \gamma^{-1} g(z). \end{aligned} \quad (5.2)$$

If we associate condition (2.5) to Eq. (5.1), assume that (2.6) and (2.7) hold, and define G and F by (2.9) and (2.12), respectively, we arrive at

$$\begin{aligned} (-1)^m U^{(2m)}(z) + cU''(z) + dU'(z) + eU(z) + F(U(z)) &= G(z), \quad 0 \leq z \leq T, \\ U^{(k)}(0) &= -U^{(k)}(T) \quad (k = 0, 1, \dots, 2m-1). \end{aligned} \quad (5.3)$$

By combining the methods of Section 4 with the general discussion in [2], we conclude that analogs of Theorems 2, 3 and 4 (with essentially similar proofs) hold for the problem (5.3). (The only change is that the space $C^4[0, T]$ is replaced by $C^{2m}[0, T]$ in the conclusions of the theorems). Going back to Eq. (5.1), and recalling Theorems 1 and 5, we obtain:

Theorem 6. *Let $f \in C^2(\mathbb{R})$ and g satisfy (2.6) and (2.7) respectively, and let c, d, e, h and g_1 be given by (5.2). Then the following conclusions hold.*

(i) *If $c \leq 0$ and $d \neq 0$, the problem (5.1), (2.5) has at least one solution $U \in C^{2m+2}(\mathbb{R})$.*

(ii) *If $c \leq 0$, $e \geq 0$, and f is also monotonically nondecreasing, the problem (5.1), (2.5) has a unique solution $U \in C^{2m+2}(\mathbb{R})$.*

Remark. The case when the term $(-1)^m D_x^{2m+1} u$, in (1.2), is replaced by a general differential expression of the type $\sum_{k=2}^m \mu_k (-1)^m D_x^{2k+1} u$, where m is an integer ≥ 2 and μ_k ($k = 2, \dots, m$) are nonnegative constants (with $\mu_m > 0$) can be treated in a similar way.

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