Journal of Applied Mathematics and Stochastic Analysis
8, Number 1, 1995, 77-83

EXISTENCE OF ASYMPTOTIC SOLUTIONS OF SECOND ORDER NEUTRAL DIFFERENTIAL EQUATION WITH MULTIPLE DELAYS

S.C. PALANISWAMI
Department of Mathematics
Bharathiar University
Coimbatore - 641 046, Tamil Nadu, INDIA

(Received May, 1994; revised September, 1994)

ABSTRACT

The existence of positive solutions of second order neutral differential equation of the form

$$[x(t)-cx(t-h)]''+\sum_{i=1}^{m}p_i(t)x[g_i(t)]=0, \quad t\geq t_0, \qquad (A)$$

is investigated. Some sufficient conditions are given for the existence of positive solutions with asymptotic decay of (A). Examples are presented to illustrate the results.

Key words: Neutral Delay Differential Equation, Oscillation, Asymptotic Positive Solution.

AMS (MOS) subject classifications: 34K05, 34K10, 34C10, 34K25.

1. Introduction

Recently, several authors [1-7] have studied the oscillatory and nonoscillatory behavior of neutral differential equations. The main reason for this interest is that delay differential equations play an important role in applications. For instance, in biological applications, delay equations give better description of fluctuations in population than the ordinary ones. Also neutral delay differential equations appear as models of electrical networks which contain lossless transmission lines. Such networks arise, for example, in high speed computers where lossless transmission lines are used to interconnect switching circuits.

In [8] Zhang and Yu studied the existence of positive solutions of neutral delay differential equation (NDDE) of the form

$$[x(t)-cx(t-h)]''+p(t)x[g(t)]=0, \quad t\geq t_0. \qquad (1)$$

In this paper we present some sufficient conditions for the existence of positive solutions of second order neutral delay differential equation of the form

$$[x(t)-xc(t-h)]''+\sum_{i=1}^{m}p_i(t)x[g_i(t)]=0, \quad t\geq t_0, \qquad (2)$$

where $c,\ h \in R_+$, p_i, $g_i \in C([t_0,\infty),R)$ for $i=1,\ldots,m$ and $g_i(t)\to\infty$ as $t\to\infty$ $(i=1,\ldots,m)$.

In order to establish the results, we need the following theorem.

Krasnosel'skii's fixed point theorem: *Let X be a Banach space, Y be a bounded closed convex subset of X and let A, B be mappings of Y into X such that $Ax+By\in Y$ for every x, $y\in Y$. If A is a strict contraction mapping and B is completely continuous then the equation*

$$Ax+Bx=x$$

has a solution in Y.

2. Main Results

Theorem 1: *Assume that:*

(i) $0<c<1$, $h>0$, $p_i(t)\le 0$ *for* $i=1,\ldots,m$,

(ii) *there exists a constant* $\alpha>0$ *such that*

$$ce^{\alpha h}+\sum_{i=1}^{m}\int_t^{\infty}(t-s)p_i(s)exp[\alpha(t-g_i(s))]ds\le 1. \tag{3}$$

Then equation (2) has a positive solution $x(t)$ satisfying $x(t)\to 0$ as $t\to\infty$.

Proof: It is easy to show that if the equality in (3) holds, then equation (2) has a positive solution $x(t)=e^{-\alpha t}$.

In the rest of the proof, we assume that there exists a number $T>t_0$ such that $T-h\ge t_0$, $g_i(t)\le t_0$ for $t\ge T$ $(i=1,\ldots,m)$,

$$\beta:=ce^{\alpha h}+\sum_{i=1}^{m}\int_T^{\infty}(T-s)p_i(s)exp[\alpha(T-g_i(s))]ds<1 \tag{4}$$

and condition (3) holds for $t\ge T$.

Let X denote the Banach space of all continuous, bounded functions defined on $[t_0,\infty)$ taking values in R. The space X is endowed with the supremum norm.

Let Y be the subset of X defined by

$$Y=\{y\in X:\ 0\le y(t)\le 1 \text{ for } t\ge t_0\}.$$

Define a mapping $S:Y\to X$ by the formula

$$(Sy)(t)=(S_1y)(t)+(S_2y)(t),$$

where

$$(S_1y)(t):=\begin{cases} ce^{\alpha h}y(t-h), & t\ge T\\ (S_1y)(T)+exp[\epsilon(T-t)]-1 & t_0\le t\le T,\end{cases} \tag{5}$$

$$(S_2y)(t):=\begin{cases} \sum_{i=1}^{m}\int_t^{\infty}(t-s)p_i(s)exp[\alpha(t-g_i(s))]y[g_i(s)]ds, & t\ge T,\\ (S_2y)(T), t_0\le t\le T,\end{cases} \tag{6}$$

and $\epsilon:=[ln(2-\beta)]/(T-t_0)$.

It is easy to see that the integral in S_2 is defined whenever $y \in Y$.

Clearly, the set Y is closed, bounded and convex in X. We shall show that for every x, $y \in Y$

$$S_1 x + S_2 y \in Y. \tag{7}$$

Indeed, for any $x, y \in Y$, we have

$$\begin{aligned} S_1x)(t) + (S_2y)(t) &= ce^{\alpha h}x(t-h) \\ &+ \sum_{i=1}^{m} \int_t^{\infty} (t-s)p_i(x)exp[\alpha(t-g_i(s))]y[g_i(s)]ds \\ &\le ce^{\alpha h} + \sum_{i=1}^{m} \int_t^{\infty} (t-s)p_i(s)exp[\alpha(t-g_i(s))]ds \\ &\le 1 \text{ for } t \ge T \end{aligned}$$

and

$$\begin{aligned} (S_1x)(t) + (S_2y)(t) &= (S_1x)(T) + (S_2y)(T) + exp[\epsilon(T-t)] - 1 \\ &= \beta + exp[\epsilon(T-t)] - 1 \\ &\le \beta + exp[\epsilon(T-t_0)] - 1 = 1 \text{ for } t_0 \le t \le T. \end{aligned}$$

Obviously, $(S_1x)(t) + (S_2y)(t) \ge 0$ for $t \ge t_0$. Thus (7) is proved.

Since $0 < ce^{\alpha h} < 1$, it follows that S_1 is a strict contraction.

We shall show that S_2 is completely continuous. Indeed, from condition (3) there exists a positive constant M such that

$$\begin{aligned} \left|\frac{d}{dt}(S_2y)(t)\right| &= \left| \sum_{i=1}^{m} \int_t^{\infty} p_i(s)exp[\alpha(t-g_i(s))]y[g_i(s)]ds \right. \\ &\left. + \alpha \sum_{i=1}^{m} \int_t^{\infty} (t-s)p_i(s)exp[\alpha(t-g_i(s))]y[g_i(s)]ds \right| \\ &\le M + \alpha \text{ for } t \ge T. \end{aligned}$$

Moreover, $\frac{d}{dt}(S_2y)(t) = 0$ for $t_0 \le t \le T$.

This implies that S_2 is relatively compact. On the other hand, it is easy to see that S_2 is continuous and uniformly bounded, and so S_2 is a completely continuous mapping.

By Krasnosel'skii's fixed point theorem, S has a fixed point $y \in Y$, that is

$$y(t) = \begin{cases} ce^{\alpha h}y(t-h) + \sum_{i=1}^{m} \int_t^{\infty} (t-s)p_i(s)exp[\alpha(t-g_i(s))]y[g_i(s)]ds, & t \ge T, \\ y(T) + exp[\epsilon(T-t)] - 1, & t_0 \le t \le T. \end{cases} \tag{8}$$

Since $y(t) \geq exp[\epsilon(T-t)]-1 > 0$ for $t_0 \leq t \leq T$, it follows that $y(t) > 0$ for $t \geq t_0$. Set

$$x(t) = y(t)e^{-\alpha t}. \tag{9}$$

Then (8) becomes

$$x(t) = cx(t-h) + \sum_{i=1}^{m} \int_{t}^{\infty} (t-s)p_i(s)x[g_i(s)]ds, \quad t \geq T. \tag{10}$$

It follows that

$$[x(t) - cx(t-h)]'' + \sum_{i=1}^{m} p_i(t)x[g_i(t)] = 0 \text{ for } t \geq T.$$

It means that $x(t)$ is a positive solution of equation (2) and $x(t) \to 0$ as $t \to \infty$. The proof is complete.

Theorem 2: *Assume that:*

(*i*) $c > 0$, $p_i(t) \geq 0$ *and* $g_i(t+h) < t$, *for* $i = 1, \ldots, m$,

(*ii*) *there exists a constant* $\alpha > 0$ *such that*

$$(1/c)e^{-\alpha h} + (1/c)\sum_{i=1}^{m} \int_{t+h}^{\infty} (s-t-h)p_i(s)exp[\alpha(t-g_i(s))]ds \leq 1 \tag{11}$$

for all sufficiently large t. *Then equation* (2) *has a positive solution* $x(t)$ *and* $x(t) \to 0$ *as* $t \to \infty$.

Proof: If the equality in (11) holds, then $x(t) = e^{-\alpha t}$ is a solution. Now, we assume that there exists $T > t_0$ such that $t + h \geq t_0$ for $t \geq T$,

$$\beta: = (1/c)e^{-\alpha h} + (1/c)\sum_{i=1}^{m} \int_{T+h}^{\infty} (s-T-h)p_i(s)exp[\alpha(T-g_i(s))]ds < 1 \tag{12}$$

and (11) holds for $t \geq T$.

Define the Banach space X and its subset Y as in the proof of Theorem 1.

Next, define a mapping $S: Y \to X$ by the formula

$$(Sy)(t): = (S_1y)(t) + (S_2y)(t), \tag{13}$$

where

$$(S_1y)(t): = \begin{cases} (1/c)e^{\alpha h}y(t+h), & t \geq T \\ (S_1y)(T) + exp[\epsilon(T-t)] - 1 & t_0 \leq t \leq T \end{cases}$$

$$(S_2y)(t): = \begin{cases} (1/c)\sum_{i=1}^{m} \int_{t+h}^{\infty} (s-t-h)p_i(s)exp[\alpha(t-g_i(s))]y[g_i(s)]ds, & t \geq T, \\ (S_2y)(T), & t_0 \leq t \leq T \end{cases}$$

and $\epsilon: = [ln(2-\beta)]/(T-t_0)$.

We can easily show that the mapping S satisfied all the conditions of Krasnosel'skii's fixed point theorem, and so S has a fixed point y in Y. Clearly, $y(t) > 0$ for $t \geq t_0$ and $x(t) = y(t)e^{-\alpha t}$ is a solution of equation (2), and so the proof is complete.

Theorem 3: *Assume that* $c > 1$ *and*

$$\int_{t_0}^{\infty} s \mid p_i(s) \mid ds < \infty. \tag{14}$$

Then equation (2) *has a bounded, positive solution.*

Proof: Let $T > t_0$ be a sufficiently large number such that $T + h \geq t_0$, $g_i(t+h) \geq t_0$ $(i = 1,\ldots,m)$ for $t \geq T$ and

$$\sum_{i=1}^{m} \int_{t+h}^{\infty} s \mid p_i(s) \mid ds \leq (c-1)/4, \qquad t \geq T. \tag{15}$$

Let X be the Banach space of all continuous, bounded functions defined on $[t_0, \infty)$ with the norm

$$\| x \| = \sup_{t \geq t_0} \mid x(t) \mid .$$

Set

$$Y: = \{x \in X: \ (1/2)c \leq x(t) \leq 2c \text{ for } t \geq t_0\}.$$

Clearly, Y is a bounded, closed, convex subset of X. Define a mapping $S: Y \rightarrow X$ by the formula

$$(Sx)(t): = \begin{cases} (c-1) + (1/c)x(t+h) + (1/c)\sum_{i=1}^{m} \int_{t+h}^{\infty} (s-t-h)p_i(s)x[g_i(s)]ds, & t \geq T, \\ (Sx)(T), & t_0 \leq t \leq T \end{cases} \tag{16}$$

It is easy to show that $SY \subseteq Y$. Indeed, for any $x \in Y$ we have

$$(Sx)(t) \leq (c-1) + (1/c)2c + (1/c)\sum_{i=1}^{m} \int_{t+h}^{\infty} (s-t-h) \mid p_i(s) \mid 2cds$$

$$\leq (2c+1)/2 < 2c \text{ for } t \geq T$$

and
$$(Sx)(t) \geq (c-1) + (1/2) - 2(c-1)/4 = (1/2)c \text{ for } t \geq T.$$

Consequently, $(1/2)c \leq (Sx)(T) \leq 2c$ for $t_0 \leq t \leq T$. Therefore, $SY \subseteq Y$.

We shall show that S is a contraction. For any $x_1, x_2 \in Y$, we have

$$\mid (Sx_1)(t) - (Sx_2)(t) \mid \ \leq (1/c) \mid x_1(t+h) - x_2(t+h) \mid$$

$$+ (1/c)\sum_{i=1}^{m} \int_{t+h}^{\infty} (s-t-h) \mid p_i(s) \mid \mid x_1[g_i(s)] - x_2[g_i(s)] \mid ds$$

$$\leq (1/c) \| x_1 - x_2 \| + (1/c) \| x_1 - x_2 \| \sum_{i=1}^{m} \int_{t+h}^{\infty} (s-t-h) \mid p_i(s) \mid ds$$

$$\leq \| x_1 - x_2 \| [(1/c) + (1/c)(c-1)/4]$$

$$= (1/4)(1 + (3/c)) \| x_1 - x_2 \| \text{ for } t \geq T,$$

which implies that

$$\| Sx_1 - Sx_2 \| = \sup_{t \geq t_0} | (Sx_1)(t) - (Sx_2)(t) |$$

$$= \sup_{t \geq T} | (Sx_1)(t) - (Sx_2)(t) |$$

$$\leq (1/4)(1 + (3/c)) \| x_1 - x_2 \| .$$

Since $(1/4)(1 + (3/c)) < 1$, then S is a contraction. Hence, there exists a fixed point $x \in Y$. It is easy to see that $x(t)$ is a bounded solution of equation (2). The proof is complete. □

3. Examples

Example 1: Consider

$$[x(t) - cx(t-h)]'' + p_1(t)x[g_1(t)] + p_2(t)x[g_2(t)] = 0, \quad t \geq t_0, \tag{17}$$

where $c: = (1/2e)$, $h: = 1$, $p_1(t) = p_2(t): = (-1/4)e^{-t}(1+t)$ and $g_1(t) = g_2(t): = log(1+t)$. Then $x(t) = e^{-t-1}$ and choosing $\alpha = 1$, it is easy to see that

$$ce^{\alpha h} + \sum_{i=1}^{2} \int_t^{\infty} (t-s)p_i(s)exp[\alpha(t - g_i(s))]ds \leq 1. \tag{18}$$

Hence equation (17) satisfies the hypotheses of Theorem 1. Thus this equation has a solution $x(t) = e^{-t-1}$ which tends to zero as $t \to \infty$.

Example 2: Assume that

$$c: = (e^2 - 4)/(e), h: = 1, p_1(t) = p_2(t): = e^{-t/2} \text{ and } g_1(t) = g_2(t) = (t/2). \tag{19}$$

Then $x(t) = e^{-t-1}$ and choosing $\alpha = 1$, it is easy to see that

$$(1/c)e^{\alpha h} + (1/c)\sum_{i=1}^{2} \int_{t+1}^{\infty} (t-s)p_i(s)exp[\alpha(t - g_i(s))]ds \leq 1. \tag{20}$$

Hence equation (17) with conditions (19) satisfies the hypotheses of Theorem 2. Thus equation (17) with conditions (19) has a solution $x(t) = e^{-t-1}$ which tends to zero as $t \to \infty$.

Example 3: Let

$$p_1(t) = p_2(t) = (sin\, t/t)e^{-t}, \quad t > t_0 > 0,$$

$$c: = 1 + 4/e^2, \; h: = 1, \tag{21}$$

$$g_1(t) = g_2(t): = (1/2)\{t + log[2\, sin\, t/(c-1)]$$

in equation (17). Then

$$\sum_{i=1}^{2} \int_{t+1}^{\infty} s \mid p(s) \mid ds \leq (c-1)/4. \tag{22}$$

Therefore, equation (17) with conditions (21) satisfies the hypotheses of Theorem 3. Hence equation (17) with conditions (21) and (22) has a solution $x(t) = e^{-t-1}$ which is a bounded positive solution.

Acknowledgement

The author is thankful to Dr. K. Balachandran and Dr. E.K. Ramasami for their help in preparation of this paper.

References

[1] Chuanxi, Q., Ladas, G., Zhang, B.G., and Zhao, T., Sufficient conditions for oscillation and existence of positive solutions, *Appl. Anal.* **35** (1990), 187-194.

[2] Chuanxi, Q. and Ladas, G., Oscillations of first order neutral equation with variable coefficients, *Mh. Math.* **109** (1990), 103-111.

[3] Gopalsamy, K., Oscillations in neutral delay differential equations, *J. Math. Phys. Sci.* **21:1** (1990), 23-34.

[4] Grove, E.A., Kulenovic, M.R.S. and Ladas, G., Sufficient conditions for oscillations and nonoscillations of neutral equations, *J. Differential Equations* **68** (1987), 673-682.

[5] Ladas, G., Partheniadis, E.C. and Sficas, Y.G., Necessary and sufficient conditions for oscillations of second order neutral equations, *J. Math. Anal. Appl.* **138** (1989), 214-231.

[6] Ladde, G.S., Lakshmikantham, V. and Zhang, B.G., *Oscillation Theory of Differential Equations with Deviating Arguments*, Marcel Dekker, New York 1987.

[7] Zhang, B.G. and Gopalsamy, K., Oscillation and nonoscillation in higher order neutral equation, *J. Math. Phys. Sci.* **25:2** (1991), 152-165.

[8] Zhang, B.G. and Yu, J.S., On the existence of asymptotically decaying positive solutions of second order neutral differential equations, *J. Math. Anal. Appl.* **166** (1992), 1-11.