Journal of Applied Mathematics and Stochastic Analysis
8, Number 2, 1995, 139-142

CONVERGENCE OF A RANDOM ITERATION SCHEME TO A RANDOM FIXED POINT

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(Received February, 1994; revised November, 1994)

ABSTRACT

This paper discusses the convergence of random Ishikawa iteration scheme to a random fixed point for a certain class of random operators.

Key words: Random fixed point, random Ishikawa iteration, Tricomi's condition, Hilbert space, random operator.

AMS subject classification: 47H10, 47H40.

1. Introduction

In recent years the study of random fixed points have attracted a great amount of attention. Discussions on random fixed points may be found in works such as [1], [2], [6], and [7]. We review the following concepts which are essentials for the purpose of our discussion. Throughout this paper, (T, Σ) denotes a measurable space, and H denotes a separable Hilbert space.

A function $f: T \to H$ is said to be *measurable* if $f^{-1}(B) \in \Sigma$ for every Borel subset B of H.

Let C be any subset of H. A function $f: T \to C$ is said to be *measurable* if $f^{-1}(B \cap C) \in \Sigma$ for every Borel subset B of H.

A function $F: T \times H \to H$ is said to be *H-continuous* if $F(t, \cdot): H \to H$ is continuous for every $t \in T$.

A function $F: T \times H \to H$ is said to be a *random operator* if $F(\cdot, x): T \to H$ is measurable for every $x \in H$.

A measurable function $g: T \to H$ is said to be a *random fixed point of* $F: T \times H \to H$ if $F(t, g(t)) = g(t)$ for all $t \in T$.

The Ishikawa iteration scheme was obtained in [5]. We define the random Ishikawa iteration scheme in an analogous manner as follows:

Let $g_0: T \to H$ be any measurable function. The functions below are iteratively defined as follows:

$$g_{n+1}(t) = \alpha_n F(t, h_n(t)) + (1 - \alpha_n) g_n(t), \quad n \geq 0, \quad t \in T. \tag{1}$$

where

$$h_n(t) = \beta_n F(t, g_n(t)) + (1 - \beta_n) g_n(t), \quad n \geq 0, \quad t \in T. \tag{2}$$

and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers such that

$$0 < \alpha_n, \beta_n < 1 \text{ for all } n \geq 0 \tag{3}$$

$$\lim_{n \to \infty} \sup \beta_n = M < 1 \tag{4}$$

and

$$\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty. \tag{5}$$

Let $C \subset H$, $k: C \to C$ is said to satisfy Tricomi's condition in C if $p \in C$ and $k(p) = p$ imply

$$\| k(x) - p \| \leq \| x - p \| \text{ for all } x \in C. \tag{6}$$

The following lemma was proved in [5]:

Lemma: *H is a Hilbert space; therefore for any $x, y, z \in H$ and any real λ*

$$\| \lambda x + (1-\lambda)y - z \|^2 = \lambda \| x - z \|^2 + (1-\lambda) \| y - z \|^2 - \lambda(1-\lambda) \| x - y \|^2 \tag{7}$$

2. Main Result

Theorem 1: *Let H be a separable Hilbert space. $F: T \times H \to H$ is an H-continuous random operator in which case there exists $f: T \to H$ (not necessarily measurable) such that*

$$\| F(t,x) - f(t) \| \leq \| x - f(t) \| \tag{8}$$

for all $t \in T$ and $x \in H$.

Then, for any measurable function $g_0: T \to H$, the sequence of functions $\{g_n\}$ defined by the random Ishikawa iteration scheme, if convergent, converges to a random fixed point of F.

Proof: For any $t \in T$,

$$\| g_{n+1}(t) - f(t) \|^2$$

$$= \| \alpha_n F(t, h_n(t)) + (1-\alpha_n) g_n(t) - f(t) \|^2$$

$$= \alpha_n \| F(t, h_n(t)) - f(t) \|^2 + (1-\alpha_n) \| g_n(t) - f(t) \|^2$$

$$- \alpha_n (1-\alpha_n) \| F(t, h_n(t)) - g_n(t) \|^2 \text{ (by (7))}$$

$$\leq \alpha_n \| F(t, h_n(t)) - f(t) \|^2 + (1-\alpha_n) \| g_n(t) - f(t) \|^2 \text{ (by (3))}$$

$$\leq \alpha_n \| h_n(t) - f(t) \|^2 + (1-\alpha_n) \| g_n(t) - f(t) \|^2 \text{ (by (8))}$$

$$\leq \| g_n(t) - f(t) \|^2 - \alpha_n \beta_n (1-\beta_n) \| F(t, g_n(t)) - g_n(t) \|^2$$

(by using (2), (7) and (8)).

It further implies that

$$\sum_{n=0}^{N} \alpha_n \beta_n (1-\beta_n) \| F(t, g_n(t)) - g_n(t) \|^2$$

$$\leq \| g_0(t) - f(t) \|^2 - \| g_{N+1}(t) - f(t) \|^2$$

$$\leq \| g_0(t) - f(t) \|^2 < \infty. \tag{9}$$

For M' satisfying $M < M' < 1$, there exists a positive integer m_0 such that $\beta_m < M'$ for all $m \geq m_0$ (by (4)). Therefore, $1 - \beta_m > 1 - M' > 0$ for all $m \geq m_0$, or

$$\sum_{m=m_0}^{\infty} \alpha_m \beta_m (1 - \beta_m) \geq (1 - M') \sum_{m=m_0}^{\infty} \alpha_m \beta_m = \infty \tag{10}$$

(9) and (10) imply,

$$\lim_{n\to\infty} \inf \| F(t, g_n(t)) - g_n(t) \| = 0 \text{ for all } t \in T. \tag{11}$$

Hence, if $\{g_n(t)\}$ converges, for example to $g(t)$, $F(t, g_n(t))$ also converges to $g(t)$.

Since $F: T \times H \to H$ is an H-continuous random operator and H is separable, $\{g_n\}$ is a sequence of measurable functions [4]. Therefore, $g = \lim_{n\to\infty} g_n$ is measurable. Furthermore, F is H-continuous; thus, for all $t \in T$,

$$g(t) = \lim_{n\to\infty} g_n(t) = \lim_{n\to\infty} F(t, g_n(t))$$

$$= F(t, \lim_{n\to\infty} g_n(t)) = F(t, g(t)).$$

Hence, g is a random fixed point of F.

Therefore, $\{g_n\}$ (if convergent) converges to a random fixed point of F.

Theorem 2: *Let $C \subset H$ be a convex and compact subset and $F: T \times C \to C$ satisfies*

a) *F is H-continuous;*

b) *there exists $f: T \to C$ such that $\| F(t,x) - f(t) \| \leq \| x - f(t) \|$ for all $t \in T$ and $x \in C$; and*

c) *$F(t, \cdot): C \to C$ satisfies Tricomi's condition in C for every $t \in T$.*

Then, for any measurable function $g_0: T \to C$, the sequence $\{g_n\}$ of measurable functions constructed by the random Ishikawa iteration scheme converges to a random fixed point of F.

Proof: Since C is convex and compact, H is a separable Hilbert space and F is an H-continuous random operator and $\{g_n\}$ is a sequence of measurable functions from C to C. Proceeding exactly as in Theorem 1, we obtain (as in (11))

$$\lim_{n\to\infty} \inf \| F(t, g_n(t)) - g_n(t) \| = 0.$$

Therefore, for fixed $t \in T$, there exists $\{g_{n_i}(t)\} \subset \{g_n(t)\}$ such that

$$\lim_{i\to\infty} \| F(t, g_{n_i}(t)) - g_{n_i}(t) \| = 0. \tag{12}$$

Since C is compact, there exists $\{g_{n_{i_k}}(t) \subset \{g_{n_i}(t)\}$, such that $\{g_{n_{i_k}}(t)\}$ is convergent.

Let

$$\lim_{k\to\infty} g_{n_{i_k}}(t) = g(t). \tag{13}$$

From (12) and (13), and since $\{g_{n_{i_k}}(t)\} \subset \{g_{n_i}(t)\}$, we have

$$\lim_{k\to\infty} F(t, g_{n_{i_k}}(t)) = g(t),$$

or

$$F(t, g(t)) = g(t) \text{ (since } F \text{ is } H\text{-continuous)}. \tag{14}$$

Hence, for fixed $t \in T$, $g(t)$ is a fixed point of $F(t, \cdot)$.

For any fixed $t \in T$,

$$\| g_{n+1}(t) - g(t) \|^2 = \| \alpha_n F(t, h_n(t)) + (1 - \alpha_n) g_n(t) - g(t) \|^2$$

$$= \alpha_n \| F(t, h_n(t)) - g(t) \|^2 + (1 - \alpha_n) \| g_n(t) - g(t) \|^2$$

$$- \alpha_n(1-\alpha_n) \| F(t,h_n(t)) - g_n(t) \|^2 \text{ (by (7))}$$

$$\leq \alpha_n \| h_n(t) - g(t) \|^2 + (1-\alpha_n) \| g_n(t) - g(t) \|^2 \text{ (by (3) and Tricomi's condition (6))}$$

$$= \alpha_n \| \beta_n F(t,g_n(t)) + (1-\beta_n) g_n(t) - g(t) \|^2 + (1-\alpha_n) \| g_n(t) - g(t) \|^2$$

$$= \alpha_n(\beta_n \| F(t,g_n(t)) - g(t) \|^2 + (1-\beta_n) \| g_n(t) - g(t) \|^2$$

$$- \beta_n(1-\beta_n) \| F(t,g_n(t)) - g_n(t) \|^2) + (1-\alpha_n) \| g_n(t) - g(t) \|^2 \text{ (by (7))}$$

$$\leq \alpha_n(\beta_n \| g_n(t) - g(t) \|^2 + (1-\beta_n) \| g_n(t) - g(t) \|^2)$$

$$+ (1-\alpha_n) \| g_n(t) - g(t) \|^2 \text{ (by (3) and Tricomi's condition (6))}$$

$$= \alpha_n \| g_n(t) - g(t) \|^2 + (1-\alpha_n) \| g_n(t) - g(t) \|^2$$

$$= \| g_n(t) - g(t) \|^2.$$

Therefore, for $t \in T$,

$$\| g_{n+1}(t) - g(t) \| \leq \| g_n(t) - g(t) \|. \tag{15}$$

Since $\{g_{n_{i_k}}(t)\} \to g(t)$, given $\epsilon > 0$, there exists $n_{i_{k_0}}$ such that

$$\| g_{n_{i_{k_0}}}(t) - g(t) \| < \epsilon.$$

By virtue of (15),

$$\| g_n(t) - g(t) \| < \epsilon \text{ for all } n \geq n_{i_{k_0}}.$$

Therefore, for $t \in T$, $\lim_{n\to\infty} g_n(t) = g(t)$. Since $\{g_n\}$ is a sequence of measurable functions, g is also measurable. Thus, $g: T \to C$ is a random fixed point of $F: T \times C \to C$.

Acknowledgement

I acknowledge the kind help extended to me by Professor J.N. Das, Department of Applied Mathematics, University of Calcutta, during the preparation of this paper.

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