Journal of Applied Mathematics and Stochastic Analysis
8, Number 2, 1995, 189-194

TRANSFORMATION FORMULAS FOR TERMINATING SAALSCHÜTZIAN HYPERGEOMETRIC SERIES OF UNIT ARGUMENT

WOLFGANG BÜHRING
Universität Heidelberg
Physikalisches Institut
D-69120 Heidelberg, Germany

(Received July, 1994; Revised November, 1994)

ABSTRACT

Transformation formulas for terminating Saalschützian hypergeometric series of unit argument ${}_{p+1}F_p(1)$ are presented. They generalize the Saalschützian summation formula for ${}_3F_2(1)$. Formulas for $p = 3, 4, 5$ are obtained explicitly, and a recurrence relation is proved by means of which the corresponding formulas can also be derived for larger p. The Gaussian summation formula can be derived from the Saalschützian formula by a limiting process, and the same is true for the corresponding generalized formulas. By comparison with generalized Gaussian summation formulas obtained earlier in a different way, two identities for finite sums involving terminating ${}_3F_2(1)$ series are found. They depend on four or six independent parameters, respectively.

Key words: Hypergeometric Series, Summation Formula of Saalschütz, Gaussian Summation Formula.

AMS (MOS) subject classifications: 33C20.

1. Introduction

This paper is concerned with hypergeometric series [1], [5], [9]

$$ {}_{p+1}F_p\left(\begin{matrix} a_1, a_2, \ldots, a_{p+1} \\ b_1, \ldots, b_p \end{matrix} \,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_{p+1})_n}{(b_1)_n \ldots (b_p)_n n!} z^n, \tag{1.1} $$

which are presented in terms of Pochhammer symbols

$$ (x)_n = x(x+1)\ldots(x+n-1) = \Gamma(x+n)/\Gamma(x). \tag{1.2} $$

The Gaussian summation formula

$$ \frac{1}{\Gamma(b_1)} {}_2F_1\left(\begin{matrix} a_1, a_2 \\ b_1 \end{matrix} \,\middle|\, 1\right) = \frac{\Gamma(b_1 - a_1 - a_2)}{\Gamma(b_1 - a_1)\Gamma(b_1 - a_2)} = \frac{\Gamma(s)}{\Gamma(a_1 + s)\Gamma(a_2 + s)}, \tag{1.3} $$

$$ (s = b_1 - a_1 - a_2, \ \mathrm{Re}(s) > 0), $$

may be viewed as a limiting case of the formula of Saalschütz

$$ \frac{1}{\Gamma(b_1)} {}_3F_2\left(\begin{matrix} a_1, a_2, -m \\ b_1, 1 - s - m \end{matrix} \,\middle|\, 1\right) = \frac{(a_1 + s)_m (a_2 + s)_m}{(s)_m \Gamma(b_1 + m)}, \tag{1.4} $$

$$(s = b_1 - a_1 - a_2 \text{ not a negative integer or zero, } m = 0,1,2,\ldots),$$

where in the series representation of ${}_3F_2(1)$, the limit $m \to \infty$ is executed term by term, which is permissible if $\mathrm{Re}(s) > 0$. We are looking for such pairs of formulas for hypergeometric series with more parameters. Well-known [1], [5] is the formula

$$\frac{1}{\Gamma(b_1)\Gamma(b_2)}{}_3F_2\left(\begin{matrix}a_1, a_2, a_3\\ b_1, b_2\end{matrix} \mid 1\right) = \frac{\Gamma(s)}{\Gamma(a_1+s)\Gamma(a_2+s)\Gamma(a_3)}\ {}_3F_2\left(\begin{matrix}b_1-a_3, b_2-a_3, s\\ a_1+s, a_2+s\end{matrix} \mid 1\right), \tag{1.5}$$

$$(s = b_1 + b_2 - a_1 - a_2 - a_3, \mathrm{Re}(s) > 0,\ \mathrm{Re}(a_3) > 0),$$

which generalizes (1.3) and reduces to (1.3) when $b_2 = a_3$ or $b_1 = a_3$. The corresponding terminating Saalschützian ${}_4F_3(1)$, which generalizes (1.4), is

$$\frac{1}{\Gamma(b_1)\Gamma(b_2)}{}_4F_3\left(\begin{matrix}a_1, a_2, a_3, -m\\ b_1, b_2, 1-s-m\end{matrix} \mid 1\right) \tag{1.6}$$

$$= \frac{(a_1+s)_m(a_2+s)_m(a_3)_m}{(s)_m\Gamma(b_1+m)\Gamma(b_2+m)}{}_4F_3\left(\begin{matrix}b_1-a_3, b_2-a_3, s, -m\\ a_1+s, a_2+s, 1-a_3-m\end{matrix} \mid 1\right),$$

$$(s = b_1 + b_2 - a_1 - a_2 - a_3 \text{ not a negative integer or zero, } m = 0,1,2,\ldots).$$

When $b_2 = a_3$ or $b_1 = a_3$, (1.6) reduces to (1.4). The limit $m \to \infty$ of (1.6) yields (1.5), provided that $\mathrm{Re}(a_3) > 0$, which is the condition for convergence of the resulting ${}_3F_2(1)$ series on the right-hand side, and provided that $\mathrm{Re}(s) > 0$, which ensures that the left-hand side remains bounded. Formula (1.6) is essentially a known formula ([1] or (3.13.56) of [5]) if the finite series on the right is turned around.

It is the main purpose of the present work to obtain such pairs of formulas as (1.5) and (1.6) for hypergeometric series with even more parameters. A motivation was that they are useful for the analytical continuation near $z = 1$ of the hypergeometric functions ${}_{p+1}F_p(z)$. The case of the ${}_3F_2(z)$, in fact, could be solved [2], [4] by means of (1.5), although some authors [3], [7], [8] do not need such formulas.

2. A Recurrence Relation for Terminating Saalschützian Hypergeometric Series of Unit Argument

In order to generalize (1.6) let us first consider the product of two hypergeometric functions

$$L := {}_2F_1\left(\begin{matrix}\alpha, \beta\\ \gamma\end{matrix} \mid z\right){}_pF_{p-1}\left(\begin{matrix}A_1, A_2, \ldots, A_p\\ B_1, B_2, \ldots, B_{p-1}\end{matrix} \mid z\right). \tag{2.1}$$

It is convenient to introduce the non-trivial characteristic exponent at $z = 1$ for each of the hypergeometric functions,

$$\sigma = \gamma - \alpha - \beta \tag{2.2}$$

or

$$S = \sum_{j=1}^{p-1} B_j - \sum_{j=1}^{p} A_j, \tag{2.3}$$

respectively, and to consider in addition

$$R := {}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\,|\,z\right)(1-z)^{-\sigma}(1-z)^{-S}\,{}_pF_{p-1}\left(\begin{matrix}A_1,A_2,\dots,A_p\\ B_1,B_2,\dots,B_{p-1}\end{matrix}\,|\,z\right), \tag{2.4}$$

which is equal to

$$R = {}_2F_1\left(\begin{matrix}\gamma-\alpha,\gamma-\beta\\ \gamma\end{matrix}\,|\,z\right)\sum_{m=0}^{\infty}{}_{p+1}F_p\left(\begin{matrix}A_1,A_2,\dots,A_p,-m\\ B_1,B_2,\dots,B_{p-1},1-S-m\end{matrix}\,|\,1\right)\frac{(S)_m}{m!}z^m. \tag{2.5}$$

We now assume that the parameters of the hypergeometric functions in (2.1) are not all independent of each other but satisfy

$$\sigma + S = 0, \tag{2.6}$$

so that we have

$$L = R. \tag{2.7}$$

Expanding both sides in powers of z and equating the coefficients of z^m we obtain

$$\frac{\prod_{j=1}^{p}(A_j)_m}{m!\prod_{j=1}^{p-1}(B_j)_m}\,{}_{p+2}F_{p+1}\left(\begin{matrix}\alpha,\beta,1-B_1-m,1-B_2-m,\dots,1-B_{p-1}-m,-m\\ \gamma,1-A_1-m,1-A_2-m,\dots,1-A_p-m\end{matrix}\,|\,1\right) \tag{2.8}$$

$$= \frac{(\gamma-\alpha)_m(\gamma-\beta)_m}{(\gamma)_m m!}\sum_{k=0}^{m}\frac{(1-\gamma-m)_k(-m)_k(S)_k}{(1-\gamma+\alpha-m)_k(1-\gamma+\beta-m)_k k!}$$

$$\times\,{}_{p+1}F_p\left(\begin{matrix}A_1,A_2,\dots,A_p,-k\\ B_1,B_2,\dots,B_{p-1},1-S-k\end{matrix}\,|\,1\right).$$

Identifying the parameters as

$$A_1 = 1-b_1-m,\quad A_2 = 1-b_2-m,\quad \dots,\quad A_p = 1-b_p-m,$$

$$B_1 = 1-a_3-m,\quad B_2 = 1-a_4-m,\quad \dots,\quad B_{p-1} = 1-a_{p+1}-m,$$

$$\alpha = a_1,\quad \beta = a_2,\quad \gamma = 1-s-m,$$

where s is determined by (2.6) and is explicitly given below in (2.10), we obtain the following recurrence relation.

Theorem 1: *For $p = 2,3,4,\dots$ and for $m = 0,1,2,\dots$, it holds true that*

$$\frac{1}{\prod_{j=1}^{p}\Gamma(b_j)}\,{}_{p+2}F_{p+1}\left(\begin{matrix}a_1,a_2,\dots,a_{p+1},-m\\ b_1,b_2,\dots,b_p,1-s-m\end{matrix}\,|\,1\right) \tag{2.9}$$

$$= \frac{(a_1+s)_m(a_2+s)_m\prod_{j=3}^{p+1}(a_j)_m}{(s)_m\prod_{j=1}^{p}\Gamma(b_j+m)}\sum_{k=0}^{m}\frac{(s)_k(-m)_k}{(a_1+s)_k(a_2+s)_k k!}$$

$$\times (S)_k\,{}_{p+1}F_p\left(\begin{matrix}1-b_1-m,1-b_2-m,\dots,1-b_p-m,-k\\ 1-a_3-m,1-a_4-m,\dots,1-a_{p+1}-m,1-S-k\end{matrix}\,|\,1\right)$$

where

$$s = \sum_{j=1}^{p} b_j - \sum_{j=1}^{p+1} a_j \tag{2.10}$$

and

$$S = a_1 + a_2 + s + m - 1 \tag{2.11}$$

(*s not a negative integer or zero*).

With $p = 2$, ${}_3F_2(1)$ on the right-hand side of (2.9) evaluated by means of (1.4) yields (1.6). With $p = 3$, ${}_4F_3(1)$ on the right-hand side of (2.9) rewritten by means of (1.6) yields the following new formula which generalizes (1.6).

Corollary 1: *For* $m = 0, 1, 2, \ldots$, *and*

$$s = b_1 + b_2 + b_3 - a_1 - a_2 - a_3 - a_4$$

not a negative integer or zero, it holds true that

$$\frac{1}{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}\,{}_5F_4\left(\begin{matrix} a_1, a_2, a_3, a_4, -m \\ b_1, b_2, b_3, 1-s-m \end{matrix} \,\Big|\, 1\right) = \frac{(a_1+s)_m (a_2+s)_m (a_3)_m (a_4)_m}{(s)_m \Gamma(b_1+m)\Gamma(b_2+m)\Gamma(b_3+m)} \tag{2.12}$$

$$\times \sum_{k=0}^{m} \frac{(b_1+b_3-a_3-a_4)_k (b_2+b_3-a_3-a_4)_k (1-b_3-m)_k (s)_k (-m)_k}{(a_1+s)_k (a_2+s)_k (1-a_3-m)_k (1-a_4-m)_k k!}$$

$$\times\,{}_4F_3\left(\begin{matrix} b_3-a_3, b_3-a_4, a_1+a_2+s+m-1, -k \\ b_1+b_3-a_3-a_4, b_2+b_3-a_3-a_4, b_3+m-k \end{matrix} \,\Big|\, 1\right).$$

When a_3 or a_4 are equal to any of b_1, b_2, b_3, the new formula (2.12) reduces to (1.6).

Theorem 1, for $p = 4$ and the right-hand side rewritten by means of Corollary 1, yields the following new formula.

Corollary 2: *For* $m = 0, 1, 2, \ldots$, *and*

$$s = b_1 + b_2 + b_3 + b_4 - a_1 - a_2 - a_3 - a_4 - a_5$$

not a negative integer or zero, it holds true that

$$\frac{1}{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\Gamma(b_4)}\,{}_6F_5\left(\begin{matrix} a_1, a_2, a_3, a_4, a_5, -m \\ b_1, b_2, b_3, b_4, 1-s-m \end{matrix} \,\Big|\, 1\right) \tag{2.13}$$

$$= \frac{(a_1+s)_m (a_2+s)_m (a_3)_m (a_4)_m (a_5)_m}{(s)_m \Gamma(b_1+m)\Gamma(b_2+m)\Gamma(b_3+m)\Gamma(b_4+m)} \sum_{k=0}^{m}$$

$$\times \frac{(b_1+b_3+b_4-a_3-a_4-a_5)_k (b_2+b_3+b_4-a_3-a_4-a_5)_k (1-b_3-m)_k (1-b_4-m)_k (s)_k (-m)_k}{(a_1+s)_k (a_2+s)_k (1-a_3-m)_k (1-a_4-m)_k (1-a_5-m)_k k!}$$

$$\times \sum_{l=0}^{k} \frac{(b_3+b_4-a_3-a_5)_l (b_3+b_4-a_4-a_5)_l (a_5+m-k)_l (a_1+a_2+s+m-1)_l (-k)_l}{(b_1+b_3+b_4-a_3-a_4-a_5)_l (b_2+b_3+b_4-a_3-a_4-a_5)_l (b_3+m-k)_l (b_4+m-k)_l l!}$$

$$\times\,{}_4F_3\left(\begin{matrix} b_3-a_5, b_4-a_5, b_3+b_4-a_3-a_4-a_5-m+k, -l \\ b_3+b_4-a_3-a_5, b_3+b_4-a_4-a_5, 1-a_5-m+k-l \end{matrix} \,\Big|\, 1\right).$$

We might proceed further this way, but at each step, one more sum appears, and so, the formulas become more and more complicated and lengthy as the number of parameters increases.

3. The Limit $m \to \infty$

It is straightforward to let $m \to \infty$ in the above corollaries, but it seems to be difficult to establish the conditions for convergence of the resulting series on the right-hand sides. If $\mathrm{Re}(s) > 0$, we get from (2.12):

$$\frac{1}{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}\,{}_4F_3\left(\begin{matrix}a_1, a_2, a_3, a_4\\ b_1, b_2, b_3\end{matrix} \,\Big|\, 1\right) = \frac{\Gamma(s)}{\Gamma(a_1+s)\Gamma(a_2+s)\Gamma(a_3)\Gamma(a_4)} \tag{3.1}$$

$$\times \sum_{k=0}^{\infty} \frac{(b_1+b_3-a_3-a_4)_k(b_2+b_3-a_3-a_4)_k(s)_k}{(a_1+s)_k(a_2+s)_k k!}\,{}_3F_2\left(\begin{matrix}b_3-a_3, b_3-a_4, -k\\ b_1+b_3-a_3-a_4, b_2+b_3-a_3-a_4\end{matrix} \,\Big|\, 1\right).$$

When a_4 or a_3 are equal to any of b_1, b_2, b_3, then the new formula (3.1) reduces to (1.5). The condition for convergence of ${}_3F_2(1)$ on the right-hand side then is $\mathrm{Re}(a_3) > 0$ or $\mathrm{Re}(a_4) > 0$, respectively. Based on this information, we conjecture that $\mathrm{Re}(a_3) > 0 \wedge \mathrm{Re}(a_4) > 0$ is the condition for the convergence of the series on the right-hand side of (3.1). The occurrence of the gamma functions $\Gamma(a_3)\Gamma(a_4)$ in the denominator in front of the series supports this hypothesis, for if a_3 or a_4 are equal to zero (or a negative integer), the right-hand side of (3.1) can be different from zero only if the sum of the series is infinite.

Letting $m \to \infty$ in (2.13) for $\mathrm{Re}(s) > 0$ yields

$$\frac{1}{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\Gamma(b_4)}\,{}_5F_4\left(\begin{matrix}a_1, a_2, a_3, a_4, a_5\\ b_1, b_2, b_3, b_4\end{matrix} \,\Big|\, 1\right) = \frac{\Gamma(s)}{\Gamma(a_1+s)\Gamma(a_2+s)\Gamma(a_3)\Gamma(a_4)\Gamma(a_5)} \tag{3.2}$$

$$\times \sum_{k=0}^{\infty} \frac{(b_1+b_3+b_4-a_3-a_4-a_5)_k(b_2+b_3+b_4-a_3-a_4-a_5)_k(s)_k}{(a_1+s)_k(a_2+s)_k k!}$$

$$\times \sum_{l=0}^{k} \frac{(b_3+b_4-a_3-a_5)_l(b_3+b_4-a_4-a_5)_l(-k)_l}{(b_1+b_3+b_4-a_3-a_4-a_5)_l(b_2+b_3+b_4-a_3-a_4-a_5)_l l!}$$

$$\times\, {}_3F_2\left(\begin{matrix}b_3-a_5, b_4-a_5, -l\\ b_3+b_4-a_3-a_5, b_3+b_4-a_4-a_5\end{matrix} \,\Big|\, 1\right).$$

For similar reasons as above, the condition for convergence of the series on the right-hand side is expected to be $\mathrm{Re}(a_3) > 0 \wedge \mathrm{Re}(a_4) > 0 \wedge \mathrm{Re}(a_5) > 0$.

Generalizations of the Gaussian summation formula like (3.1) or (3.2) were also obtained in a different way [3]. This includes the proof that the conditions for convergence of the series (3.1) and (3.2) conjectured above are really true. While that way of proof again yields (1.5) as expected, it leads to formulas which look significantly different from (3.1) or (3.2).

4. Identities

By comparison, formulas (3.1) and (3.2) proved here together with the corresponding formulas proved in [3] imply, for $k = 0, 1, 2, \ldots$, the following identities:

$$(B_2)_k\,{}_3F_2\left(\begin{matrix}A_1, A_2, -k\\ B_1, B_2\end{matrix} \,\Big|\, 1\right) = (B_2-A_1)_k\,{}_3F_2\left(\begin{matrix}A_1, B_1-A_2, -k\\ B_1, 1+A_1-B_2-k\end{matrix} \,\Big|\, 1\right) \tag{4.1}$$

and

$$(B_2)_k \sum_{l=0}^{k} \frac{(A_1)_l (A_2)_l (-k)_l}{(B_1)_l (B_2)_l l!} \, {}_3F_2\left(\begin{matrix} C_1, C_2, -l \\ A_1, A_2 \end{matrix} \,\Big|\, 1 \right) \tag{4.2}$$

$$= (B_2 - A_1)_k \sum_{l=0}^{k} \frac{(A_1)_l (B_1 - A_2)_l (-k)_l}{(B_1)_l (1 + A_1 - B_2 - k)_l l!} \, {}_3F_2\left(\begin{matrix} C_1, C_2, -l \\ A_1, 1 + A_2 - B_1 - l \end{matrix} \,\Big|\, 1 \right).$$

Here terminating hypergeometric series appear, and thus the identities consist of finite sums only.

5. Other Work

A generalization of Saalschütz's formula was considered earlier by MacRobert [6]. The special cases following for $p = 3$ or $p = 4$ from his general formula look significantly different form our formulas. In particular, they are of such a type that for $m \to \infty$ all the sums become infinite sums. In contrast, as an important feature of our formulas (2.12) and (2.13), for $m \to \infty$, (3.1) and (3.2) contain just one infinite series with terms consisting of finite sums only.

References

[1] Bailey, W.N., *Generalized Hypergeometric Series*, Stechert-Hafner, New York 1964.

[2] Bühring, W., The behavior at unit argument of the hypergeometric function ${}_3F_2$, *SIAM J. Math. Anal.* **18** (1987), 1227-1234.

[3] Bühring, W., Generalized hypergeometric functions at unit argument, *Proc. Amer. Math. Soc.* **114** (1992), 145-153.

[4] Evans, R.J. and Stanton, D., Asymptotic formulas for zero-balanced hypergeometric series, *SIAM J. Math. Anal.* **15** (1984), 1010-1020.

[5] Luke, Y.L, *The Special Functions and their Approximations*, Vol. 1, Academic Press, New York 1969.

[6] MacRobert, Th.M., *Functions of a Complex Variable*, Fifth edition, MacMillan, London 1962.

[7] Saigo, M., On properties of the Appell hypergeometric functions F_2 and F_3 and the generalized Gauss function ${}_3F_2$, *Bull. Central Research Inst. Fukuoka Univ.* **66** (1983), 27-32.

[8] Saigo, M. and Srivastava, H.M., The behavior of the zero-balanced hypergeometric series ${}_pF_{p-1}$ near the boundary of its convergence region, *Proc. Amer. Math. Soc.* **110** (1990), 71-76.

[9] Slater, L.J., *Generalized Hypergeometric Functions*, Cambridge Univ. Press, Cambridge 1966.