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# $\mathcal{O}$-Regularly Varying Functions in Approximation Theory

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For $\mathcal{O}$-regularly varying functions a growth relation is introduced and characterized which gives an easy tool in the comparison of the rate of growth of two such functions at the limit point. In particular, methods based on this relation provide necessary and sufficient conditions in establishing chains of inequalities between functions and their geometric, harmonic, and integral means, in both directions. For periodic functions, for example, it is shown how this growth relation can be used in approximation theory in order to establish equivalence theorems between the best approximation and moduli of smoothness from prescribed inequalities of Jackson and Bernstein type.

*Keywords:* $\mathcal{O}$-regularly varying functions; growth relation; best approximation; rate of convergence; Jackson and Bernstein inequalities.

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## 1 INTRODUCTION

In 1930 J. Karamata [20] introduced regularly varying (RV) functions, namely functions $f$ having the property that there is a real number $\rho$ such that the limit

$$\lim_{x\to\infty}\frac{f(\lambda x)}{f(x)}=\lambda^{\rho}$$

exists for all $\lambda > 0$. Since then, especially after the publication of W. Feller's book [15] on probability, which contains material on this subject, the *Karamata-theory* has been developed in many directions. In the textbooks of E. Seneta [23] and N.H. Bingham, C.M. Goldie and J.L. Teugels [6], one can find a systematic treatment of this theory; in the latter there can also be found many applications to number theory, (integral) transform theory, probability theory, the theory of Tauberian theorems, etc. One of the possible extensions of RV functions, due to V.G. Avakumović [2], is to drop the requirement that the limit of the quotient $f(\lambda x)/f(x)$ has to exist. If the limes superior and the limes inferior of the term above are both finite and positive, then $f$ is said to be a $\mathcal{O}$-regularly varying function.

The aim of the paper is twofold; first, we wish to point out some algebraic properties and introduce a growth relation $\prec$ for $\mathcal{O}$-RV functions which gives an easy tool for comparing the rate of growth of such functions. Furthermore, for $\mathcal{O}$-RV functions we establish characterizations in terms of this growth relation concerning the validity of inequalities connecting functions and certain means thereof, as well as their monotonicity properties. Secondly, although the well-known paper of N.K. Bari and S.B. Stečkin [3] already contains some material on $\mathcal{O}$-RV functions, there seems to be a lack of communication in dealing with $\mathcal{O}$-RV functions in approximation theory properly. Nevertheless, the Bari-Stečkin setting was extended in some approximation theoretical papers; see e.g. P.L. Butzer and K. Scherer [11], P.L. Butzer, S. Jansche, and R.L. Stens [9], Z. Ditzian [13], as well as in S. Jansche and R.L. Stens [19], and E. van Wickeren [25]. In a brief application to best approximation by trigonometric polynomials, we will show how such functions can be used with success in approximation theory.

## 2 DEFINITIONS

We use the common convention in not distinguishing the constants in various estimates; thus the value of the constant $M > 0$ may differ in each occurrence, but always independently of the varying parameters. If necessary, we indicate dependencies by $M(C)$, etc. Furthermore, we make use of the Landau symbol $\mathcal{O}$ in the usual way, and $f \sim g$ means that $f = \mathcal{O}(g)$ as well as $g = \mathcal{O}(f)$. For our purpose it is convenient to consider the origin as limit point instead of infinity. This leads to the following definition of $\mathcal{O}$-regularly varying functions.

Definition 2.1 A (Lebesgue-) positive measurable function $\phi$: $(0, 1] \to \mathbb{R}$ is a *$\mathcal{O}$-regularly varying function ($\mathcal{O}$-RV function)*, if for each $t_0 \in (0, 1)$

$$\phi \sim 1 \qquad \text{on} \quad [t_0, 1],$$

and if

$$0 < \liminf_{t\to 0^+} \frac{\phi(Ct)}{\phi(t)}, \qquad \limsup_{t\to 0^+} \frac{\phi(Ct)}{\phi(t)} < \infty \tag{2.1}$$

for all $C \in (0, 1]$. The class of all $\mathcal{O}$-regularly varying functions is denoted by $\Phi$.

The above definition can be found in S. Aljančić and D. Arandelović [1], except, as mentioned above, that in [1] the limit $t \to \infty$ is considered instead of $t \to 0^+$. Via the relations $f(x) = \varphi(1/x)$ and $\lambda = 1/C$ one can transform the limit point to infinity.

For instance, for $\sigma, \rho \in \mathbb{R}$ the function

$$\phi(t) := t^\sigma |\log t|^\rho, \quad t \in (0, 1/2]; \qquad \phi(t) := 1, \quad t \in (1/2, 1],$$

belongs to $\Phi$, but not functions having exponential growth like $\phi(t) := e^{-1/t}$. It is convenient to define

$$\phi^*\colon (0,1] \to \mathbb{R}, \qquad \phi^*(C) := \limsup_{t\to 0^+} \frac{\phi(Ct)}{\phi(t)};$$

$$\phi_*\colon (0,1] \to \mathbb{R}, \qquad \phi_*(C) := \liminf_{t\to 0^+} \frac{\phi(Ct)}{\phi(t)}.$$

By condition (2.1) for $\phi \in \Phi$ the functions $\phi^*$ and $\phi_*$ are positive but not necessarily measurable. The following bread and butter theorem on $\mathcal{O}$-RV functions, due to S. Aljančić and D. Arandelović [1], ensures the local uniform boundedness of the quotient $\phi(Ct)/\phi(t)$. For sake of completeness, we give a slightly modified proof following [6, Theorem 2.0.1.].

THEOREM 2.1 *Let $\phi \in \Phi$ and $C_0 \in (0, 1)$; then there exists a constant $M = M(C_0) > 0$ such that*

$$\sup_{C\in[C_0,1]} \frac{\phi(Ct)}{\phi(t)} \le M, \qquad \sup_{C\in[C_0,1]} \frac{\phi(t)}{\phi(Ct)} \le M$$

*for all $t \in (0, 1]$.*

*Proof* The function $g(x) := \log\phi(e^{-x})$, $x \in [0, \infty)$, is measurable, and the assertion is proved if we can show that for arbitrary $a_0 > 0$,

$$\sup_{a\in[0,a_0]} |g(x+a) - g(x)| < \infty, \qquad x \ge 0.$$

Supposing the contrary, we can choose sequences $\{x_n\}_{n\in\mathbb{N}} \subset [0,\infty)$ and $\{a_n\}_{n\in\mathbb{N}} \subset [0, a_0]$ such that

$$|g(x_n + a_n) - g(x_n)| \geq 2n, \qquad n \in \mathbb{N}.$$

On the other hand, by the definition of $\Phi$, for all $a \geq 0$ we can find an upper bound $n_0 = n_0(a) > 0$ satisfying

$$|g(x) - g(x+a)| \leq n_0, \qquad x \in [0,\infty). \tag{2.2}$$

Setting $x = x_n$, we find that

$$|g(x_n + a_n) - g(x_n + a)| \geq n, \qquad n \geq n_0. \tag{2.3}$$

Now we define

$$I_j := \{a \in [a_0, 2a_0];\ |g(x_n + a_n) - g(x_n + a)| \geq n\ \forall n \geq j\}, \qquad j \in \mathbb{N}_0;$$

then by the measurability of $g$ the sets $I_j$, $j \in \mathbb{N}_0$, are also measurable, and, on using (2.3), we obtain

$$[a_0, 2a_0] = \bigcup_{j\in\mathbb{N}_0} I_j.$$

We pick one $I_{j_0}$, $j_0 \in \mathbb{N}_0$, of positive measure, i.e., $m(I_{j_0}) > 0$, to derive that the set

$$J := \bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} J_k, \qquad J_k := a_k - I_{j_0},$$

is contained in $[-2a_0, 0]$, and

$$m\Big(\bigcup_{k=i}^{\infty} J_k\Big) \geq m(a_i - I_{j_0}) = m(I_{j_0}), \qquad i \in \mathbb{N}.$$

By isotony of the measure it follows that $m(J) \geq m(I_{j_0}) > 0$. Let $b \in J$; then by construction of $J$ we find a subsequence $\{k_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$ such that $b \in J_{k_i}$ for all $i \in \mathbb{N}$. This means that $a_{k_i} - b \in I_{j_0}$, $i \in \mathbb{N}$, and by definition of $I_{j_0}$ we have

$$|g(x_{k_i} + a_{k_i}) - g(x_{k_i} + a_{k_i} - b)| \geq k_i \geq i, \qquad i \geq j_0,$$

contradicting (2.2) for $a = -b$. □

An immediate consequence is the following

COROLLARY 2.2 *If $\phi \in \Phi$, then for all $C \in (0, 1)$ we have*

$$\phi^* \sim 1, \qquad \phi_* \sim 1 \qquad \text{on} \quad [C, 1].$$

*In particular, there exists a constant $M = M(C) > 0$ such that*

$$\frac{1}{M}\phi(t) \leq \phi(h) \leq M\phi(t) \tag{2.4}$$

*for all $C \leq t/h \leq 1$, $h, t \in (0, 1]$.*

Concerning the behaviour of $\mathcal{O}$-RV functions in a neighbourhood of the origin, we have for the power function $\phi(t) = t^\sigma$, $\sigma \in \mathbb{R}$, that $\phi^*(C) = \phi_*(C) = C^\sigma$. Taking the logarithm, the order $\sigma$ is given by

$$\sigma = \frac{\log \phi^*(C)}{\log C}.$$

Thus for the power function we can extract the rate of growth $\sigma$ using this quotient of logarithms. For arbitrary functions the term may not be constant. This leads to the definition of the Matuzewska indices, see W. Matuzewska and W. Orlicz [21], and the literature of the authors cited there. For $\phi \in \Phi$ the numbers

$$\alpha(\phi) := \sup_{C \in (0,1)} \frac{\log \phi^*(C)}{\log C} \qquad \text{and} \qquad \beta(\phi) := \inf_{C \in (0,1)} \frac{\log \phi_*(C)}{\log C}$$

are called the *upper* and *lower Matuzewska index*, respectively. Some basic properties of the indices are collected in the following

LEMMA 2.3 *Let $\alpha = \alpha(\phi)$ and $\beta = \beta(\phi)$ be the Matuzewska indices of $\phi \in \Phi$.*

*(a) The indices are real numbers $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$, and there hold*

$$\alpha(\phi) = \lim_{C \to 0^+} \frac{\log \phi^*(C)}{\log C}, \qquad \beta(\phi) = \lim_{C \to 0^+} \frac{\log \phi_*(C)}{\log C}. \tag{2.5}$$

(b) For a given $\varepsilon > 0$ there exists some $C_0 \in (0, 1)$ such that

$$C^\alpha \leq \phi^*(C) < C^{\alpha - \varepsilon}, \qquad C^{\beta + \varepsilon} < \phi_*(C) \leq C^\beta, \qquad C \in (0, C_0]. \tag{2.6}$$

*In particular,*

$$\phi_*(C) \leq C^\beta \leq C^\alpha \leq \phi^*(C) \qquad \forall C \in (0, 1]. \tag{2.7}$$

Concerning the proof of (a) (which easily implies (b)) one uses the fact that $\phi^*$ is submultiplicative, i.e., $\phi^*(C_1C_2) \leq \phi^*(C_1)\phi^*(C_1)$, obtained from the definition. After taking the logarithm an application of a theorem of E. Hille and R.S. Phillips [17, Theorem 7.6.2, 7.6.3] on subadditive functions ensures the existence of the limit and implies (2.5). This was carried out in [21] for monotonic functions and in [1].

In view of the estimates in (2.7), note that just the existence of the limit

$$\lim_{t\to 0^+} \frac{\phi(Ct)}{\phi(t)} =: g(C)$$

implies that $g(C) = C^\rho$ for a suitable $\rho \in \mathbb{R}$. Hence the subset $\Phi_R := \{\phi \in \Phi;\ \alpha(\phi) = \beta(\phi)\}$ is exactly the set of Karamata's regularly varying functions, cf. [6, Theorem 1.4.1], [23, Theorem 1.3.]. Simple calculations lead to further relations concerning the Matuzewska indices.

Corollary 2.4 *Let $\phi, \phi_1, \phi_2 \in \Phi$; then*

$$\alpha\Big(\frac{1}{\phi}\Big) = -\beta(\phi), \qquad \beta\Big(\frac{1}{\phi}\Big) = -\alpha(\phi),$$

*and for the product of $\mathcal{O}$-RV functions we have $\phi_1\phi_2 \in \Phi$, satisfying*

$$-\infty < \alpha(\phi_1) + \alpha(\phi_2) \leq \alpha(\phi_1\phi_2) \leq \beta(\phi_1\phi_2) \leq \beta(\phi_1) + \beta(\phi_2) < \infty.$$

Matuzewska indices and their modifications such as Boyd indices are frequently used in interpolation theory, cf. e.g. D.W. Boyd [8], F. Fehér [14].

## 3. ALGEBRAIC PROPERTIES OF $\Phi$

We now establish some algebraic properties of the class $\Phi$ of $\mathcal{O}$-RV functions. In particular we define a new growth relation $\prec$ in $\Phi$, which allows a comparison of the growth of $\mathcal{O}$-RV functions at the origin.

Definition 3.1 Let $\phi_1, \phi_2 \colon (0, 1] \to \mathbb{R}$ be positive. Then the *growth relation* $\phi_1 \prec \phi_2$ holds iff there exists a constant $C \in (0, 1)$ such that

$$\limsup_{t\to 0^+} \frac{\phi_1(Ct)}{\phi_1(t)} \frac{\phi_2(t)}{\phi_2(Ct)} < 1.$$

For example, let $\sigma_i, \rho_i \in \mathbb{R}$, $i = 1, 2$, and

$$\phi_i(t) := t^{\sigma_i} |\log t|^{\rho_i}, \quad t \in (0, 1/2], \qquad \phi_i(t) := 1, \quad t \in (1/2, 1].$$

Then it is obvious that $\phi_1 \prec \phi_2$, iff $\sigma_1 > \sigma_2$. In particular, for $\phi \in \Phi$ we have $\phi \prec 1$ iff $\phi^*(C) < 1$. Heuristically $\phi_1 \prec \phi_2$ means that the growth of $\phi_1$ and $\phi_2$ differ by a power $t^\varepsilon$ at the origin. Changes of growth in terms by powers of the logarithm are not effected by the growth relation $\prec$. It turns out to be useful to write $\phi_1 \preccurlyeq \phi_2$ and $\phi_2 \succcurlyeq \phi_1$, if $t^\varepsilon \phi_1(t) \prec \phi_2(t)$ for all $\varepsilon > 0$.

As mentioned above, the Matuzewska indices are measures of the behaviour of $\mathcal{O}$-RV functions; thus it seems natural to suppose that the relation $\prec$ can be characterized by the indices $\alpha$ and $\beta$. In this respect we have the following new connections between the Matuzewska indices and our growth relation.

THEOREM 3.1 *Let $\phi_1, \phi_2 \in \Phi$; then the following four assertions are equivalent:*

*(i)* $\phi_1 \prec \phi_2$;

*(ii) There exist constants $\alpha_0 > 0$ and $C_0 \in (0, 1]$ such that*

$$\frac{\phi_1(Ct)}{\phi_1(t)} \frac{\phi_2(t)}{\phi_2(Ct)} < C^{\alpha_0}, \qquad t \in (0, t_0(C)],$$

*for all $C \in (0, C_0]$ and some $t_0(C) \in (0, 1]$;*

*(iii)* $\alpha\left(\dfrac{\phi_1}{\phi_2}\right) > 0$;

*(iv)* $\beta\left(\dfrac{\phi_2}{\phi_1}\right) < 0$.

*Furthermore, the functions $\phi_1, \phi_2$ satisfy at most one of the relations $\phi_1 \prec \phi_2$ or $\phi_1 \succ \phi_2$. The relation $\phi_1 \preccurlyeq \phi_2$ is characterized in the same way, provided that $<$ and $>$ in (ii)–(iv) are replaced by $\leq$ and $\geq$, respectively.*

*Proof* The implication from (ii) to (i) follows from the definition of the growth relation. Assuming (i), then $(\phi_1/\phi_2)^*(C) < 1$ for some $C \in (0, 1)$, and on using (2.7) we obtain

$$C^{\alpha(\phi_1/\phi_2)} \leq \left(\frac{\phi_1}{\phi_2}\right)^*(C) < 1,$$

which in fact implies $\alpha(\phi_1/\phi_2) > 0$. The equivalence of (iii) and (iv) is given by Corollary 2.4. Finally, if (iii) is satisfied, we choose $\varepsilon = \alpha_0 := \alpha(\phi_1/\phi_2)/2$ in (2.6), to establish (ii). Concerning the relation $\phi_1 \preccurlyeq \phi_2$, the equivalences can be proved in the same way by using the fact that $\alpha(t^\varepsilon \phi(t)) = \varepsilon + \alpha(\phi)$, which readily follows from Lemma 2.3. $\square$

THEOREM 3.2 *The class $\Phi$ equipped with pointwise multiplication forms a multiplicative abelian group with identity element* $1(t) := 1$.

*The relation $\prec$ in $\Phi$ is transitive, and $\Phi$ forms with $\prec$ a (partial) ordered group. In particular, for $\phi_1, \phi_2, \phi_3 \in \Phi$ satisfying $\phi_1 \prec \phi_2$ and $\phi_2 \prec \phi_3$, we have $\phi_1 \prec \phi_3$, and $\phi_1 \prec \phi_2$ implies $\phi_1\phi_3 \prec \phi_2\phi_3$.*

*Proof* Obviously, $\Phi$ is an abelian group, the inverse element of $\phi \in \Phi$ is given by $1/\phi$. By Corollary 2.4 we have

$$\alpha\left(\frac{\phi_1}{\phi_3}\right) = \alpha\left(\frac{\phi_1}{\phi_2}\frac{\phi_2}{\phi_3}\right) \geq \alpha\left(\frac{\phi_1}{\phi_2}\right) + \alpha\left(\frac{\phi_2}{\phi_3}\right), \qquad \alpha\left(\frac{\phi_1}{\phi_2}\right) = \alpha\left(\frac{\phi_1\phi_3}{\phi_2\phi_3}\right);$$

thus the remaining assertions hold true by using Theorem 3.1. □

*Remark 3.2* The class $\Phi_R$ of regularly varying functions forms a subgroup of $\Phi$, and it is related to the power functions $\Phi_P := \{\phi \in \Phi;\ \phi(t) = t^\gamma, \gamma \in \mathbb{R}\}$ by the isomorphism

$$\Phi_R/\Phi_0 \cong \Phi_P \cong (\mathbb{R}, +),$$

where $\Phi_0 := \{\phi \in \Phi;\ \alpha(\phi) = \beta(\phi) = 0\}$. Additionally, the Matuzewska indices are homomorphisms

$$\alpha\colon \Phi_R \to \mathbb{R}, \quad \phi \mapsto \alpha(\phi), \qquad \beta\colon \Phi_R \to \mathbb{R}, \quad \phi \mapsto \beta(\phi),$$

mapping the partial ordered group $(\Phi_R, \cdot, \prec)$ onto the ordered group $(\mathbb{R}, +, >)$, preserving the relations in the sense that $\phi_1 \prec \phi_2$ implies $\alpha(\phi_1) > \alpha(\phi_2)$ and $\beta(\phi_1) > \beta(\phi_2)$.

The identity element 1 separates the convergent and divergent elements of $\Phi$. To see this, consider (cf. [1])

LEMMA 3.3 *Let $\phi \in \Phi$. If $\phi \prec 1$, then*

$$\lim_{t \to 0^+} \phi(t) = 0,$$

*and if $\phi \succ 1$, we have*

$$\lim_{t \to 0^+} \phi(t) = \infty.$$

*Proof* Assuming $\phi \prec 1$, by Theorem 3.1 we can choose $t_0, C, \rho \in (0, 1)$ such that

$$\frac{\phi(Ct)}{\phi(t)} < \rho, \qquad 0 < t \leq t_0.$$

Now for a given $t \in (0, t_0]$ there exists an integer $m = m(t) \in \mathbb{N}_0$, satisfying $C^{m+1} t_0 \leq t \leq C^m t_0$, and we obtain by iterating the estimate above

$$\phi(t) \leq \rho\phi(C^{-1}t) \leq \cdots \leq \rho^m \phi(C^{-m}t) \leq \rho^m M,$$

noting that $C^{-m}t \in [Ct_0, 1]$ and $\phi \sim 1$ on $[Ct_0, 1]$. This implies

$$\lim_{t \to 0^+} \phi(t) \leq \lim_{m \to \infty} \rho^m M = 0,$$

as well as

$$\lim_{t \to 0^+} \frac{1}{\phi(t)} = \infty.$$

By applying Theorem 3.2 the assertion is shown.

## 4 CHARACTERIZATION OF $\mathcal{O}$-RV FUNCTIONS

Now we want to give some characterizations for $\mathcal{O}$-RV functions, combined with our growth relation. Owing to the group property it is sufficient to compare only one member of $\Phi$ with the identity element. Results related to the following lemmas can be found in [1], [6, Chapter 2], and [23, Appendix]. A function $\phi$ on $(0, 1]$ is said to be *almost increasing* or *almost decreasing*, if there is some constant $M > 0$ such that

$$\phi(t) \leq M\phi(h) \qquad \text{or} \qquad \phi(h) \leq M\phi(t)$$

for all $0 < t \leq h \leq 1$.

LEMMA 4.1 *Let* $\phi \in \Phi$.
*(a) If* $\phi$ *is almost increasing, then* $\phi \preccurlyeq 1$.
*(b) If* $\phi \prec 1$, *then* $\phi$ *is almost increasing.*
*(c) If* $\phi$ *is almost decreasing, then* $\phi \succcurlyeq 1$.
*(d) If* $\phi \succ 1$, *then* $\phi$ *is almost decreasing.*

*Proof* Let $\phi$ be almost increasing, i.e., $\phi(Ct)/\phi(t) \leq M$, $t, C \in (0, 1]$, for some $M > 0$. Thus we have $\phi^*(C) \leq M$, which implies

$$\alpha(\phi) = \lim_{C\to 0^+} \frac{\log \phi^*(C)}{\log C} \geq \lim_{C\to 0^+} \frac{\log M}{\log C} = 0.$$

This gives $\phi \preccurlyeq 1$ by Theorem 3.1. Conversely, suppose that $\phi \prec 1$; then on using Theorem 3.1 again, we find some $C_0, t_0 \in (0, 1)$ such that

$$\frac{\phi(C_0 t)}{\phi(t)} < 1, \qquad t \in (0, t_0]. \tag{4.1}$$

Additionally, by Theorem 2.1 we have

$$\sup_{C\in[C_0,1]} \frac{\phi(Ct)}{\phi(t)} \leq M, \qquad t \in (0, 1]. \tag{4.2}$$

Now let $0 < t_1 \leq t_2 \leq 1$ be chosen arbitrarily. Noting that $\phi \sim 1$ on $[t_0, 1]$, without loss of generality we can further assume that $t_2 \leq t_0$. We pick $n \in \mathbb{N}_0$ satisfying $C_0^{n+1} < t_1/t_2 \leq C_0^n$; then on using (4.1) and (4.2) we obtain

$$\frac{\phi(t_1)}{\phi(t_2)} = \frac{\phi(t_1)}{\phi(C_0^n t_2)} \frac{\phi(C_0^n t_2)}{\phi(C_0^{n-1} t_2)} \cdots \frac{\phi(C_0 t_2)}{\phi(t_2)} \leq \frac{\phi(t_1)}{\phi(C_0^n t_2)} = \frac{\phi(\frac{t_1}{C_0^n t_2} C_0^n t_2)}{\phi(C_0^n t_2)} \leq M,$$

which proves (b). The remaining assertions can be shown along the same lines. □

The most important characterization for $\mathcal{O}$-RV functions is the fact that the growth of integrals over $\phi$ can be estimated by $\phi$ itself. First we need some elementary estimates between sums and integrals of $\mathcal{O}$-RV functions.

LEMMA 4.2 *Let $\phi \in \Phi$ and $C \in (0, 1)$. Then there exists a constant $M = M(C) > 0$ such that*

$$\frac{1}{M}\phi(t) \leq \int_{Ct}^{t} \phi(u)\frac{du}{u} \leq M\phi(t), \qquad t \in (0, 1], \tag{4.3}$$

*and for a suitable $t_0 = t_0(C) \in (0, 1)$ we have*

$$\frac{1}{M}\phi(t) \leq \sum_{Ct<1/k\leq t} \frac{1}{k}\phi(k^{-1}) \leq M\phi(t), \qquad t \in (0, t_0]. \tag{4.4}$$

*Proof* Noting that $\int_{Ct}^{t} \frac{du}{u} = -\log C > 0$, Corollary 2.2 immediately implies

$$\phi(t) \le M \int_{Ct}^{t} \phi(t)\frac{du}{u} \le M \int_{Ct}^{t} \phi(u)\frac{du}{u} \le M\phi(t).$$

Now we consider the integer $K_t := \#\{k \in \mathbb{N};\ Ct < 1/k \le t\}$, $\#A$ denoting the cardinality of a set $A$; then

$$\frac{1}{Ct} - \frac{1}{t} - 1 \le K_t \le \frac{1}{Ct} - \frac{1}{t} + 1.$$

Setting $t_0 := \min\{1, (1-C)/(2C)\}$, then for $t \in (0, t_0]$ we obtain

$$K_t \ge \frac{2C}{1-C}\left(\frac{1}{C} - 1\right) - 1 = 1,$$

i.e., $K_t \ge 1$, which ensures that the sum in (4.4) is not empty. This gives also

$$\sum_{Ct<1/k\le t} \frac{1}{k} \le tK_t \le \frac{1}{C} - 1 + t \le \frac{1}{C},$$

$$\sum_{Ct<1/k\le t} \frac{1}{k} \ge CtK_t \ge 1 - C - Ct \ge 1 - C - \frac{1-C}{2} = \frac{1-C}{2}.$$

Thus there follows the inequality

$$\frac{1}{M} \le \sum_{Ct<1/k\le t} \frac{1}{k} \le M,$$

yielding (4.4) by using Corollary 2.2 again. □

With the help of the above lemma we can prove

LEMMA 4.3 *Let $\phi \in \Phi$ and $C \in (0, 1)$; then for $t \in (0, 1)$ we have*

$$\int_0^t \phi(u)\frac{du}{u} \sim \sum_{k\ge 1/t} \frac{1}{k}\phi(k^{-1}) \sim \sum_{j=0}^{\infty} \phi(C^j t), \tag{4.5}$$

$$\int_t^1 \phi(u)\frac{du}{u} \sim \sum_{1\le k\le 1/t} \frac{1}{k}\phi(k^{-1}) \sim \sum_{j;\, t\le C^j\le 1} \phi(C^j), \tag{4.6}$$

*provided that the integral exists or one of the series converges. The constants induced by $\sim$ depend on $C$.*

*Proof* The proof follows by repeated application of Lemma 4.2 and Corollary 2.2; for instance there holds

$$\int_{C^{n+1}t}^{t} \phi(u)\frac{du}{u} \sim \sum_{j=0}^{n} \phi(C^j t).$$

Taking the limit $n \to \infty$, we obtain one of the equivalences in (4.5). The remaining assertions follow by the same way. □

*Remark 4.1* A careful examination of the lemma above shows that the constants involved only depend on $M > 0$ and $C > 0$ of (2.4). In particular, if a family $\{\phi_x\}_{x\in X} \subset \Phi$, $X$ an arbitrary index set, satisfies

$$\frac{1}{M}\phi_x(t) \le \phi_x(h) \le M\phi_x(t)$$

for all $x \in X$ and $h, t \in (0, 1]$, $C \le t/h \le 1$, then the estimates (4.5) and (4.6) hold uniformly in $x \in X$.

If only $\chi\colon (0, 1] \to \mathbb{R}$ is a non negative almost increasing function, and $\phi \in \Phi$, then by similar arguments one can show that for $C \in (0, 1)$ there exists a $M = M(C) > 0$ such that

$$\sum_{j=0}^{\infty} \frac{\chi(C^j t)}{\phi(C^j t)} \le M \sum_{k\ge[1/t]} \frac{1}{k}\frac{\chi(k^{-1})}{\phi(k^{-1})}, \qquad t \in (0, 1], \tag{4.7}$$

$$\sum_{j;\, t\le C^j\le 1} \frac{\chi(C^j)}{\phi(C^j)} \le M \sum_{1\le k\le 1/t} \frac{1}{k}\frac{\chi(k^{-1})}{\phi(k^{-1})}, \qquad t \in (0, 1]. \tag{4.8}$$

The middle or *harmonic* part of the two equivalences (4.5) and (4.6), common in approximation theory, seems to be new in the frame of $\mathcal{O}$-RV functions. However, for regularly varying sequences see R. Bojanic and E. Seneta [7].

LEMMA 4.4 *Let* $\phi \in \Phi$.
*(a) If*

$$\phi(t) \sim \int_0^t \phi(u)\frac{du}{u}, \qquad t \to 0^+, \tag{4.9}$$

*then there holds* $\phi \preccurlyeq 1$.
*(b) Conversely, for each* $\phi \prec 1$ *we have (4.9).*

*(c) If $\phi$ satisfies*

$$\phi(t) \sim \int_t^1 \phi(u)\frac{du}{u}, \qquad t \to 0^+, \tag{4.10}$$

*then $\phi \succcurlyeq 1$.*

*(d) $\phi \succ 1$ implies (4.10).*

*Proof* (a) The positivity of $\phi$ implies that the integral in (4.9) is increasing. Hence, by assumption, $\phi$ is almost increasing, and on using Lemma 4.1 we obtain $\phi \preccurlyeq 1$. (c) can be shown in the same way. (b) Lemma 4.3 yields for $t, C \in (0, 1)$,

$$\phi(t) \le \sum_{j=0}^{\infty} \phi(C^j t) \le M \int_0^t \phi(u)\frac{du}{u}.$$

Now let $\phi \prec 1$; then by Theorem 3.1 there exist constants $t_0, C \in (0, 1)$ and an $\alpha_0 > 0$ such that $\phi(Ct) < C^{\alpha_0}\phi(t)$, $0 < t \le t_0$. Iteration gives for $j \in \mathbb{N}$

$$\phi(C^j t) < C^{j\alpha_0}\phi(t), \qquad 0 < t \le t_0.$$

Thus, on using Lemma 4.3 we obtain for $0 < t \le t_0$

$$\int_0^t \phi(u)\frac{du}{u} \le M\sum_{j=0}^{\infty}\phi(C^j t) \le M\phi(t)\sum_{j=0}^{\infty} C^{j\alpha_0} = M\phi(t)\frac{1}{1 - C^{\alpha_0}},$$

which proves (b).

(d) We choose $t, C \in (0, 1)$, and $n \in \mathbb{N}_0$ such that $C^{n+1} < t \le C^n$; then on applying Corollary 2.2 and Lemma 4.3 again we have

$$\phi(t) \le M\phi(C^n) \le M \sum_{j;\, t \le C^j \le 1} \phi(C^j) \le M \int_t^1 \phi(u)\frac{du}{u}.$$

To verify the converse estimate, for $\phi \succ 1$ on using Theorem 3.1, there exist constants $t_0, C \in (0, 1)$ and some $\beta_0 > 0$ such that $\phi(t) < C^{\beta_0}\phi(Ct)$, $0 < t \le t_0$. This gives for $j \in \mathbb{N}$

$$\phi(t) < C^{j\beta_0}\phi(C^j t), \qquad 0 < t \le t_0.$$

For arbitrary $t \in (0, 1)$ and $n \in \mathbb{N}_0$ satisfying $C^{n+1} < t \le C^n$, we obtain on using $\phi \sim 1$ on $[Ct_0, 1]$ and Corollary 2.2,

$$\phi(C^j) \le MC^{(n-j)\beta_0}\phi(C^n) \le MC^{(n-j)\beta_0}\phi(t), \qquad 0 \le j \le n.$$

Applying Lemma 4.3, we conclude

$$\int_t^1 \phi(u)\frac{du}{u} \leq M\sum_{j=0}^{n}\phi(C^j) \leq M\phi(t)\sum_{j=0}^{n} C^{(n-j)\beta_0}$$
$$\leq M\phi(t)\frac{1}{1-C^{\beta_0}} \leq M\phi(t).$$

Thus the Lemma is shown. □

It should be remarked that the estimates

$$\phi(t) = \mathcal{O}\Big(\int_0^t \phi(u)\frac{du}{u}\Big), \qquad \phi(t) = \mathcal{O}\Big(\int_t^1 \phi(u)\frac{du}{u}\Big)$$

hold for all $\mathcal{O}$-RV functions $\phi$. By virtue of Lemma 4.3 we immediately obtain the following Corollary dealing with two chains of inequalities (in both directions) between $\phi$ and its integral, harmonic, and geometric means.

COROLLARY 4.5 *Let $\phi \in \Phi$ and $C \in (0, 1)$ be chosen arbitrarily.*
*(a)* If $\phi \prec 1$, then $\phi$ is bounded, almost increasing, and we have

$$\phi(t) \sim \int_0^t \phi(u)\frac{du}{u} \sim \sum_{1/t\leq k}\frac{1}{k}\phi(k^{-1}) \sim \sum_{j=0}^{\infty}\phi(C^j t), \qquad t \to 0^+.$$

*(b) If $\phi \succ 1$, then $\phi$ is almost decreasing, and*

$$\phi(t) \sim \int_t^1 \phi(u)\frac{du}{u} \sim \sum_{1\leq k\leq 1/t}\frac{1}{k}\phi(k^{-1}) \sim \sum_{j;\, t\leq C^j\leq 1}\phi(C^j), \qquad t \to 0^+.$$

*In the last equivalence of (a) and (b), the constants, induced by $\sim$, depend on $C$.*

For (almost) monotonic functions the requirements being a $\mathcal{O}$-RV function can be weakened (cf. W. Feller [16], or [6, Corollary 2.0.6] for monotonic functions).

LEMMA 4.6 *If a positive measurable function $\phi$: $(0, 1] \to \mathbb{R}$ is almost increasing, such that $\phi_*(C_0) > 0$ for one $C_0 \in (0, 1)$, then $\phi \in \Phi$ satisfying $\phi \preccurlyeq 1$.*

*Proof* By assumption we have $\phi(Ct)/\phi(t) \leq M$ for all $C, t \in (0, 1]$, i.e., $\phi^* < \infty$ on $(0, 1]$. Thus we have only to show that $\phi_* > 0$ on $(0, 1]$. Since $\phi$ is almost increasing, it follows $0 < \phi_*(C_0) \leq M\phi_*(C)$ for all $C \in [C_0, 1]$. If $C \in (0, C_0)$ we choose $n \in \mathbb{N}_0$ such that $C_0^{n+1} < C \leq C_0^n$ to obtain

$$\begin{aligned} 0 < \phi_*(C_0)^{n+1} &\leq \liminf_{t\to 0^+}\left\{\frac{\phi(C_0^{n+1}t)}{\phi(C_0^n t)}\cdots\frac{\phi(C_0 t)}{\phi(t)}\right\} = \liminf_{t\to 0^+}\frac{\phi(C_0^{n+1}t)}{\phi(t)} \\ &\leq M\liminf_{t\to 0^+}\frac{\phi(Ct)}{\phi(t)} = M\phi_*(C). \end{aligned}$$

Hence $\phi_*$ is positive and $\phi \preccurlyeq 1$, by Lemma 4.1. □

The following theorem originates from a proof of an inverse theorem for Bernstein polynomials due to Berens and Lorentz [5]. Some extensions are in use in approximation theory, see e.g. M. Becker and R.J. Nessel [4], E. van Wickeren [25] or X.L. Zhou [26].

THEOREM 4.7 *Let $\phi, \psi \in \Phi$ such that $\psi \prec \phi$, and let $\chi: (0, 1] \to \mathbb{R}$ be a positive measurable function which is almost increasing. If there exists a constant $K > 0$ such that*

$$\phi(t) \leq K\left\{\chi(h) + \frac{\psi(t)}{\psi(h)}\phi(h) + \psi(t)\right\}, \qquad 0 < t \leq h \leq 1, \tag{4.11}$$

*and if*

$$\chi(t) \leq M\phi(t), \qquad 0 < t \leq 1,$$

*then $\chi \in \Phi$, and*

$$\chi(t) \sim \phi(t), \qquad t \to 0^+.$$

*Proof* We have to show that $\phi$ can be estimated by $\chi$. First, by Theorem 3.1 we find some $\varepsilon > 0$ and $C_0 \in (0, 1]$ such that

$$\frac{\psi(Ch)}{\psi(h)}\frac{\phi(h)}{\phi(Ch)} < C^\varepsilon, \qquad 0 < h \leq t_0(C). \tag{4.12}$$

for all $C \in (0, C_0]$ and a suitable $t_0(C) \in (0, 1]$. We now fix one $C \in (0, C_0]$ such that

$$C^\varepsilon < 1/(3K). \tag{4.13}$$

By virtue of Lemma 3.3 applied to $\phi/\psi \succ 1$, there is some $0 < t_1 \leq t_0(C)$ such that

$$3K \leq \frac{\phi(h)}{\psi(h)}, \qquad 0 < h \leq t_1. \tag{4.14}$$

Inserting the inequalities (4.12), (4.13), and (4.14) into the assumption (4.11) we obtain by setting $t = Ch$

$$\phi(Ch) \leq K\chi(h) + \frac{2}{3}\phi(Ch), \qquad 0 < h \leq t_1.$$

A final application of Corollary 2.2 gives the desired estimate $\phi(t) \leq M\chi(t)$ for $0 < t \leq t_1$. □

## 5 APPLICATION TO APPROXIMATION THEORY

In this section we give some brief applications of the theory of $\mathcal{O}$-RV functions to approximation theory. Let $L^p_{2\pi}$, $1 \leq p < \infty$, be the Banach space of the Lebesgue measurable $2\pi$-periodic functions $f : \mathbb{R} \to \mathbb{C}$ endowed with the norm

$$\|f\|_p := \left\{ \int_{-\pi}^{\pi} |f(x)|^p \, dx \right\}^{1/p}.$$

For simplicity we identify $L^\infty_{2\pi}$ with the space of all continuous $2\pi$-periodic functions on $\mathbb{R}$ equipped with the usual supremum norm $\|f\|_\infty := \sup_{x\in\mathbb{R}} |f(x)|$. In the following let $1 \leq p \leq \infty$ and $f \in L^p_{2\pi}$. In these spaces one can ask for a characterization of the behaviour of the (error of) best approximation by trigonometric polynomials. If we denote by $\Pi_n$ the set of trigonometric polynomials $t_n(x) = \sum_{k=-n}^{n} a_k e^{ikx}$, $a_k \in \mathbb{C}$, of degree not exceeding $n \in \mathbb{N}_0$, then the *best approximation* of $f \in L^p_{2\pi}$ is given by

$$E_n[f] := \inf_{t_n \in \Pi_n} \|f - t_n\|_p.$$

Since the subspaces $\Pi_n \subset L^p_{2\pi}$ are finite dimensional, the infimum above is achieved by a *polynomial of best approximation* $t_n^* = t_n^*(f)$. The growth of $E_n[f]$ depends on the smoothness of the given function $f$, which can be measured by moduli of smoothness. Defining for $r \in \mathbb{N}$ the $r$-th difference of $f$ with increment $h > 0$ by $\Delta_h^1 f(x) := f(x+h) - f(x)$, $\Delta_h^{r+1} := \Delta_h^1 \Delta_h^r$, then the *$r$-th modulus of smoothness* is given by

$$\omega_r(f, t) := \sup_{0<h\leq t} \|\Delta_h^r f\|_p, \qquad t > 0.$$

Moduli of smoothness are strongly related to $K$-functionals. We denote by $W^{p,r}_{2\pi}$ the Sobolev space of all functions $f \in L^p_{2\pi}$ which coincide almost everywhere with an $(r-1)$-times continuously differentiable function $g$, $g^{(r-1)}$ being absolutely continuous with derivative in $L^p_{2\pi}$, i.e., $f \in W^{p,r}_{2\pi}$ iff $f^{(r)}$ exists almost everywhere and belongs to $L^p_{2\pi}$. Of course, if $p = \infty$, $W^{\infty,r}_{2\pi}$ is the space of all $r$-times continuously differentiable $2\pi$-periodic functions on $\mathbb{R}$. The *K-functional* is then defined by

$$K_r(f,t) := \sup_{g \in W^{p,r}_{2\pi}} \{\|f-g\|_p + t\|g^{(r)}\|_p\}, \qquad t > 0.$$

For each fixed $t > 0$ the $K$-functional $K_r(f,t)$ is a sublinear functional, bounded by $\|f\|_p$. The modulus of smoothness can be replaced by the $K$-functional, using the following well known equivalence (see e.g. R.A. DeVore and G.G. Lorentz [12, Chapter 7., §2]).

Proposition 5.1 *For $r \in \mathbb{N}$ there holds*

$$\omega_r(f,t) \sim K_r(f,t^r), \qquad t \in (0,1],$$

*where the constants induced are independent of $f$ and $t$.*

Our main connection with $\mathcal{O}$-RV functions is given by

Theorem 5.2 *Let $\psi \in \Phi$ such that $\psi \prec 1$. Then either $K_r(f,\psi(\cdot)) \equiv 0$ on $(0,1]$ or $K_r(f,\psi(\cdot)) \in \Phi$. In the latter case we have*

$$\psi \preccurlyeq K_r(f,\psi(\cdot)) \preccurlyeq 1.$$

*Proof* Applying (2.4) of Corollary (2.2) we find for each $C \in (0,1]$ some constant $M = M(C) \geq 1$ such that for arbitrary $g \in W^{p,r}_{2\pi}$,

$$\begin{aligned}\|f-g\|_p + \frac{1}{M}\psi(t)\|g^{(r)}\|_p &\leq \|f-g\|_p + \psi(h)\|g^{(r)}\|_p \\ &\leq \|f-g\|_p + M\psi(t)\|g^{(r)}\|_p,\end{aligned}$$

and hence

$$\frac{1}{M}K_r(f,\psi(t)) \leq K_r(f,\psi(h)) \leq MK_r(f,\psi(t))$$

for all $t, h \in (0,1]$ satisfying $C \leq t/h \leq 1$. This proves the first statement. Since $\psi$ is almost increasing, it follows for $0 < t \leq h \leq 1$ and arbitrary $g \in W^{p,r}_{2\pi}$,

$$K_r(f,\psi(h)) \le \frac{\psi(h)}{\psi(t)}\left\{\frac{\psi(t)}{\psi(h)}\|f-g\|_p + \psi(t)\|g^{(r)}\|_p\right\}$$
$$\le M\frac{\psi(h)}{\psi(t)}\{\|f-g\|_p + \psi(t)\|g^{(r)}\|_p\}.$$

Taking the infimum over all $g \in W_{2\pi}^{p,r}$ we deduce that $K_r(f,\psi(t))/\psi(t)$ is almost decreasing, i.e., by Lemma 4.1 $\psi \preccurlyeq K_r(f,\psi(\cdot))$. On the other hand, recalling again that $\psi$ is almost increasing, we find $K_r(f,\psi(\cdot)) \preccurlyeq 1$, after using the estimate

$$K_r(f,\psi(t)) \le M\{\|f-g\|_p + \psi(h)\|g^{(r)}\|_p\}, \qquad 0 < t \le h \le 1,$$

and taking the infimum afterwards. □

The corresponding result for the modulus of smoothness now reads

$$t^r \preccurlyeq \omega_r(f,t) \preccurlyeq 1.$$

The relations between the best approximation and the $r$-th modulus of smoothness are based on two fundamental inequalities, the so called Jackson and Bernstein inequalities (cf. P.L. Butzer and R.J. Nessel [10, p. 99] or [12, p. 97, 202]).

Proposition 5.3 *For $r \in \mathbb{N}$ the following inequalities hold*

$$E_n[f] \le Mn^{-r}\|f^{(r)}\|_p, \qquad f \in W_{2\pi}^{p,r},\ n \in \mathbb{N}_0, \tag{5.1}$$
$$\|t_n^{(r)}\|_p \le Mn^r\|t_n\|_p, \qquad t_n \in \Pi_n,\ n \in \mathbb{N}_0. \tag{5.2}$$

On using these two inequalities and Theorem 5.2 we now can give a sufficient condition for the equivalence of best approximation and moduli of smoothness in terms of our growth relation.

Theorem 5.4 *If $t^r \prec \omega_r(f,t)$ for an $r \in \mathbb{N}$, then we have*

$$E_n[f] \sim \omega_r(f,1/n), \qquad n \to \infty.$$

*The constants involved by $\sim$ may depend on the given function $f$.*

*Proof* On using the Jackson inequality (5.1), we get for $n \in \mathbb{N}$ and arbitrary $g \in W_{2\pi}^{p,r}$,

$$E_n[f] \le E_n[f-g] + E_n[g] \le \|f-g\|_p + Mt^r\|g^{(r)}\|_p,$$

to obtain by Proposition 5.1

$$E_n[f] \le MK_r(f, n^{-r}) \le M\omega_r(f, 1/n). \tag{5.3}$$

Now, let $t_n^* \in \Pi_n$, $n \in \mathbb{N}$, be a polynomial of best approximation to $f$; then for $t, h \in (0,1]$ and $N \in \mathbb{N}$ such that $2^{-N} < h \le 2^{-N+1}$,

$$\omega_r(f,t) \le MK_r(f,t^r) \le M\{\|f - t_{2^N}^*\|_p + t^r\|(t_{2^N}^*)^{(r)}\|_p.$$

Expanding $(t_{2^N}^*)^{(r)}$ into a telescoping sum, we find

$$\|(t_{2^N}^*)^{(r)}\|_p \le \|(t_1^*)^{(r)}\|_p + \sum_{k=0}^{N-1} \|(t_{2^k}^*)^{(r)} - (t_{2^{k+1}}^*)^{(r)}\|_p.$$

Applying the Bernstein inequality (5.2) we obtain

$$\|(t_1^*)^{(r)}\|_p \le M\{\|t_1^* - f\|_p + \|f\|_p\} \le M\|f\|_p,$$

and

$$\begin{aligned}\|(t_{2^k}^*)^{(r)} - (t_{2^{k+1}}^*)^{(r)}\|_p &\le M2^{(k+1)r}\|t_{2^k}^* - t_{2^{k+1}}^*\|_p \\ &\le M2^{kr}\|f - t_{2^k}^*\|_p + M2^{(k+1)r}\|f - t_{2^{k+1}}^*\|_p.\end{aligned}$$

Inserting these estimates into the sum above, we deduce

$$\omega_r(f,t) \le M\left\{E_{2^N}[f] + t^r\|f\|_p + t^r\sum_{k=0}^{N} 2^{kr}E_{2^k}[f]\right\}. \tag{5.4}$$

By (5.3) and on applying Corollaries 4.5, 2.2 for $h^{-r}\omega_r(f,h) \succ 1$ we have

$$\sum_{k=0}^{N} 2^{kr}E_{2^k}[f] \le Mh^{-r}\omega_r(f,h).$$

We now define $e(h) := E_{[1/h]}[f]$, the index $[1/h] := \max\{n \in \mathbb{N}_0;\ n \le 1/h\}$ denoting the integer part of $1/h$; then recalling the monotonicity of the best approximation, these estimates imply

$$\omega_r(f,t) \le M\{e(h) + t^r h^{-r}\omega_r(f,h) + t^r\}$$

for some constant $M = M(f,r) > 0$. An application of Theorem 4.7 finally yields $e(t) \sim \omega_r(f,t)$, which proves the theorem. □

*Remark 5.1* The inequality

$$E_n[f] \leq M\omega_r(f, 1/n), \tag{5.5}$$

established above, is called the *direct* or *Jackson theorem*, while the estimate

$$\omega_r(f, t) \leq Mt^r \sum_{0 \leq k \leq 1/t} (k+1)^{r-1} E_k[f] \tag{5.6}$$

is called the *inverse* or *Bernstein theorem* on best trigonometric approximation. The latter estimate can be deduced by applying $f - t_0^*$ in (5.4) for $h = t$, and by using (4.8), taking into account that $\omega_r(f, t) = \omega_r(f - t_0^*, t)$ and $E_k[f] = E_k[f - t_0^*]$. Note that the inequalities (5.5) and (5.6) hold without the assumption of the preceding theorem, and the constants only depend on $r \in \mathbb{N}$.

The condition that the function $e(t) := E_{[1/t]}[f]$ belongs to $\Phi$ is necessary but not sufficient for the equivalence between the best approximation and modulus of smoothness. With the aid of Lemma 4.6 we obtain that $e(t)$ belongs to $\Phi$ if, for instance, the term $E_n[f]/E_{2n}[f]$ is bounded for $n \to \infty$. By Corollary 4.5, applied to the inverse theorem (5.6), we have

COROLLARY 5.5 *If* $e(t) = E_{[1/t]}[f] \in \Phi$ *such that* $t^r \prec e(t)$ *for an* $r \in \mathbb{N}$, *then*

$$E_n[f] \sim \omega_r(f, 1/n), \qquad n \to \infty.$$

Essentially Theorem 5.4 and Corollary 5.5 have already been proven for continuous functions in the fundamental work of S.B. Stečkin [24] (see also [13] for approximation in Banach spaces). In both papers there are comparison functions in use. This is now avoided by using the $\mathcal{O}$-RV functions combined with the relation $\prec$. The problem to find characterizations for $E_n[f] \sim \omega_r(f, 1/n)$ in terms of moduli of smoothness is referred to as the *Timan problem.* We finally prove the following theorem, due to R.K.S. Rathore [22], in the setting of $\mathcal{O}$-RV functions.

THEOREM 5.6 *For* $r \in \mathbb{N}$ *we have*

$$E_n[f] \sim \omega_r(f, 1/n), \qquad n \to \infty, \tag{5.7}$$

*if and only if*

$$\omega_r(f, t) \sim \omega_{r+1}(f, t), \qquad t \to 0^+. \tag{5.8}$$

*Proof* First, from the definition of the modulus of smoothness, we note that $\omega_{r+1}(f,t) \leq 2\omega_r(f,t)$. If (5.7) holds, then (5.8) follows on using the Jackson theorem (5.5) for $r+1$. Conversely, by Theorem 5.2 and Proposition 5.1 we obtain $t^{r+1} \prec t^r \preccurlyeq \omega_r(f,t)$. Hence we can apply Theorem 5.4 for $r+1$ to deduce $E_n[f] \sim \omega_{r+1}(f,1/n) \sim \omega_r(f,1/n)$, $n \to \infty$. □

The foregoing approach of using the growth relation can be applied to other problems of quantitative approximation provided one has at hand suitable inequalities of Jackson and Bernstein type and a corresponding $K$-functional, cf. [13], [18]. In particular, it is not required that the *orders* of the Jackson and Bernstein inequalities are power functions, which is in fact not necessarily the case; see e.g. the problem of best polynomial approximation in Freud-weighted spaces, cf. [13] or [19].

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