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On Weighted Hardy Inequalities on Semiaxis for Functions Vanishing at the Endpoints

MASHA NASYROVA[a] and VLADIMIR STEPANOV[b,*,†]

[a] *Khabarovsk State University, Department of Applied Mathematics, Tichookeanskaya, 136, Khabarovsk, 680035, Russia*
[b] *Computer Center of the Far-Eastern Branch of the Russian Academy of Sciences, Shelest 118–205, Khabarovsk, 680042, Russia*
E-mail: 1600@as.khabarovsk.su

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We study the weighted Hardy inequalities on the semiaxis of the form

$$\|Fu\|_2 \leq C \left\| F^{(k)} v \right\|_2 \tag{1}$$

for functions vanishing at the endpoints together with derivatives up to the order $k-1$. The case $k=2$ is completely characterized.

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1 INTRODUCTION

In the theory of the weighted Hardy inequalities for higher-order derivatives the problem, when functions vanish together with derivatives at the endpoints, is still undecided. The case $k = 1$ has been solved by P. Gurka [1] and activity in this area has increased in recent years mostly due to the efforts of A. Kufner,

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*Author for correspondence.

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who contributed to various other cases by himself and with coauthors as well [2–6].

We study the problem on the semiaxis, which is different in some aspects from the case of a finite interval [3,9] since the inequality (1), $k > 1$, becomes non-trivial with only one condition $F(\infty) = 0$. We give a complete solution for the case $k = 2$ and hope that our approach can be extended to the general situation on the semiaxis.

The inequality (1), $k > 1$, with the boundary conditions

$$F^{(i)}(0) = 0, \;\; F^{(j)}(\infty) = 0 \quad \text{for some } 0 \le i, j \le k-1 \tag{2}$$

becomes non-trivial, if the right-hand side of (1) for all functions, having absolutely continuous derivative $F^{(k-1)}(x)$ and satisfying (2), is a norm in a corresponding weighted space. We suppose throughout the paper, that $u(x)$ as well as $v(x)$ and $|v(x)|^{-1}$ be locally square integrable; the last is necessary and sufficient for the condition, that any function F satisfying $\left\|F^{(k)}v\right\|_2 < \infty$, can be represented by a Riemann-Liouville integral.

Let $\alpha=(\alpha_0, \alpha_1, \ldots, \alpha_{k-1})$, $\alpha_j=0, 1$; $j=0, 1, \ldots, k-1$; $|\alpha|= \sum_{0\le j\le k-1} \alpha_j$.

Put

$$F^{(\alpha)}(0) = 0 \Leftrightarrow F^{(j)}(0) = 0 \quad \text{for all } j, \text{ when } \alpha_j = 1,$$

$$F^{(\beta)}(\infty) = 0 \Leftrightarrow F^{(i)}(\infty) = 0 \quad \text{for all } i, \text{ when } \beta_i = 1$$

and

$$AC_2^k(\alpha,\beta) = \left\{F : \|F\|_{AC_2^k(\alpha,\beta)} \equiv \left\|F^{(k)}v\right\|_2 < \infty, \right.$$
$$\left. F^{(\alpha)}(0) = F^{(\beta)}(\infty) = 0\right\}.$$

For a finite interval and $k > 1$ the boundary value problem

$$F^{(k)}(x) = 0, \;\; F^{(\alpha)}(0) = 0, \;\; F^{(\beta)}(a) = 0 \tag{3}$$

has a non-trivial solution, if $1 \le |\alpha| + |\beta| \le k-1$. Hence, it might have only the trivial solution, only if $|\alpha| + |\beta| \ge k$. On the other hand, (3) has only the trivial solution, if $|\alpha| + |\beta| \ge 2(k-1)$ while if $|\alpha| + |\beta| \ge 2k$, then functions belonging to $AC_2^k(\alpha, \beta)$ can be approximated by C_0^∞ functions in the norm of $AC_2^k(\alpha, \beta)$. Thus, on a finite interval we have

$$N_k = 2^{2k-1} + \frac{(2k)!}{2(k!)^2}$$

spaces $AC_2^k(\alpha,\beta)$, $k \le |\alpha| + |\beta| \le 2k$; however, for some of them $\|F\|_{AC_2^k(\alpha,\beta)}$ is not a norm. The last number is given by rather more complicated formulae. The situation on $(0,\infty)$ is slightly different and affected by the following observation. Let (1) holds and $\beta^{(1)}$ and $\beta^{(2)}$ be multiindices such, that $\min\{i : \beta_i^{(1)} = 1\} = \min\{i : \beta_i^{(2)} = 1\}$. Then $AC_2^k(\alpha,\beta^{(1)}) = AC_2^k(\alpha,\beta^{(2)})$. It implies an *heuristic principle*, that the characterization problems of (1) for functions with the boundary couples $(\alpha,\beta^{(1)})$ and $(\alpha,\beta^{(2)})$ are equivalent. Similar effects take place on $(-\infty,0)$ or $(-\infty,\infty)$.

For simplicity we restrict ourselves to case of L^2-norm, however the methods of the present paper in conjunction with the results [12–14] provide the L_p–L_q case as well.

Throughout the paper uncertainties of the form $0\cdot\infty$, $0/0$, ∞/∞ are taken equal to zero, the inequality $A \ll B$ means $A \le cB$ with an absolute constant c, perhaps, different in different places, however the relationship $A \approx B$ is interpreted as $A \ll B \ll A$ or $A = cB$. χ_E denotes the characteristic function of a set E.

2 REMARKS ON GENERAL CASE

If $k \ge 1$, then the following characterization are known.

(1) $k = 1$. $|\alpha| + |\beta| = 1$ [7], $|\alpha| + |\beta| = 2$ [1].
(2) $k > 1$. $|\alpha| = k$, $|\beta| = 0$ or $|\alpha| = 0$, $|\beta| = k$ [10].
(3) $k > 1$. $|\alpha| + |\beta| = k$, $\max\{j : \alpha_j = 1\} + 1 = \min\{i : \beta_i = 1\}$ [2].
(4) $k > 1$. Finite interval, $|\alpha| + |\beta| = k$ [9].
(5) $k > 1$. Finite interval, $|\alpha| + |\beta| = k + 1$, $\alpha_{k+1} = \beta_{k+1} = 1$ and the remaining parts of multiindices satisfy the Pólya condition [4–5].

Below (Theorem 1.2) we supplement this list by the following

(6) $k > 1$. Infinite interval (a,∞), $1 \le |\alpha| + |\beta| < k$, $|\beta| \ge 1$, $\alpha_j = 1$, $j = 0, 1, \dots, |\alpha|$, $|\alpha| + 1 = \min\{i : \beta_i = 1\}$.

Let $(a,b) \subseteq (-\infty,\infty)$, $a < b$ and $k \ge 1$. We need the following notations.

$$L_{2,v} = L_{2,v,(a,b)} = \left\{ f : \|fv\|_2^2 = \int_a^b |fv|^2 < \infty \right\},$$

$$I_k f(x) = I_{k,(a,b)} f(x) = \frac{1}{\Gamma(k)} \int_a^x (x-y)^{k-1} f(y)\,dy, \ x \in (a,b),$$

$$J_k g(x) = J_{k,(a,b)} g(x) = \frac{1}{\Gamma(k)} \int_x^b (y-x)^{k-1} g(y)\, dy, \ x \in (a,b),$$

$$A_{k,0}^2 = A_{k,0;(a,b),u,v}^2 = \sup_{a<t<b} \int_t^b (x-t)^{2(k-1)} |u(x)|^2\, dx \int_a^t |v|^{-2},$$

$$A_{k,1}^2 = A_{k,1;(a,b),u,v}^2 = \sup_{a<t<b} \int_t^b |u|^2 \int_a^t (t-x)^{2(k-1)} |v(x)|^{-2}\, dx,$$

$$B_{k,0}^2 = B_{k,0;(a,b),u,v}^2 = \sup_{a<t<b} \int_a^t (t-x)^{2(k-1)} |u(x)|^2\, dx \int_t^b |v|^{-2},$$

$$B_{k,1}^2 = B_{k,1;(a,b),u,v}^2 = \sup_{a<t<b} \int_a^t |u|^2 \int_t^b (x-t)^{2(k-1)} |v(x)|^{-2}\, dx,$$

$$A_k^2 = A_{k;(a,b),u,v}^2 = \max(A_{k,0}^2, A_{k,1}^2),$$

$$B_k^2 = B_{k;(a,b),u,v}^2 = \max(B_{k,0}^2, B_{k,1}^2).$$

Also we apply throughout the paper the following.

THEOREM 1.1 [10–12] *Necessary and sufficient conditions for the inequality* (1) *to be valid on interval* (a, b) *are:*

(1) $A_k < \infty$, *when* $F(a) = F'(a) = \ldots = F^{(k)}(a) = 0$ *and then* $C \approx A_k$,

(2) $B_k < \infty$, *when* $F(b) = F'(b) = \ldots = F^{(k)}(b) = 0$ *and then* $C \approx B_k$.

Our first result is the following simple observation.

THEOREM 1.2 *Let* $-\infty < a < \infty$. *Necessary and sufficient conditions for the inequality (1) to be valid on* (a, ∞), *when* $F(\infty) = 0$, *is* $B_{k;(a,\infty),u,v} < \infty$ *and then* $C \approx B_{k;(a,\infty),u,v}$.

Proof Necessity follows from Theorem 1.1. For sufficiency suppose $\|F^{(k)}v\|_2 < \infty$, $F(\infty) = 0$ and $B_{k;(a,\infty),u,v} < \infty$. Denote $F^{(k)} = f$ and let

$$\tilde{F}(x) = J_k f(x) = \frac{1}{\Gamma(k)} \int_x^\infty (y-x)^{k-1} f(y)\, dy, \ x > a.$$

Since $B_{k;(a,\infty),u,v} < \infty$, we see, that

$$|\tilde{F}(x)| \leqslant \frac{1}{\Gamma(k)} \|fv\|_2 \left(\int_x^\infty (y-x)^{2(k-1)} |v(y)|^{-2}\, dy \right)^{1/2} \to 0, \ x \to \infty$$

and conclude, that $F = \tilde{F}$. Applying Theorem 1.1 again, we obtain the sufficiency.

Remark 1.1 Theorem 1.2 implies the limiting case of the assertion (6) above, when $|\alpha| = 0$, $|\beta| = 1$. The remaining part follows similarly by application of the result of [2]. The situation on the left semiaxis is similar, however, for the real line the only limiting cases of (6), that is $|\alpha| = 0$, $\beta = (1, 0, \dots)$ or $|\beta| = 0$, $\alpha = (1, 0, \dots)$, are valid.

3 THE CASE $k = 2$

In this section we consider the inequality

$$\|Fu\|_2 \le C \left\|F''v\right\|_2 \tag{3}$$

on $(0, \infty)$. The full list of non-trivial cases is the following.

$$\begin{array}{lll}
|\alpha| + |\beta| = 1. & (1.1) & F(\infty) = 0. \\
|\alpha| + |\beta| = 2. & (2.1) & F(0) = F(\infty) = 0, \\
 & (2.2) & F(0) = F'(0) = 0, \\
 & (2.3) & F(\infty) = F'(\infty) = 0, \\
 & (2.4) & F(0) = F'(\infty) = 0, \\
 & (2.5) & F'(0) = F(\infty) = 0. \\
|\alpha| + |\beta| = 3. & (3.1) & F(0) = F'(0) = F(\infty) = 0, \\
 & (3.2) & F(0) = F'(0) = F'(\infty) = 0, \\
 & (3.3) & F(0) = F(\infty) = F'(\infty) = 0, \\
 & (3.4) & F'(0) = F(\infty) = F'(\infty) = 0. \\
|\alpha| + |\beta| = 4. & (4.1) & F(0) = F'(0) = F(\infty) = F'(\infty) = 0.
\end{array}$$

The case (1.1) follows from Theorem 1.2, (2.2) and (2.3) – from Theorem 1.1, (2.4) – from [2]. Applying the heuristic principle we show that

$$(1.1) \Leftrightarrow (2.3),\ (2.1) \Leftrightarrow (3.3),\ (2.5) \Leftrightarrow (3.4),\ (3.1) \Leftrightarrow (4.1).$$

Thus, the only four cases of subsection $|\alpha| + |\beta| = 3$ need to be treated. Moreover, we show that the cases (3.2), (3.3) and (3.4) are similar to each other and begin with the first of them.

CASE (3.2). It is easy to see, that the characterization problem

$$\|Fu\|_2 \le C \left\|F''v\right\|_2, F(0) = F'(0) = F'(\infty) = 0$$

is equivalent to the following

$$\|(I_2 f)\, u\|_2 \leq C \|f v\|_2, \int_0^\infty f = 0. \tag{4}$$

Let $\tau \in (0, \infty)$ be defined by

$$\int_0^\tau |v|^{-2} = \int_\tau^\infty |v|^{-2} \tag{5}$$

and put

$$D_\tau^2 = \int_\tau^\infty |u|^2 \int_0^\tau (\tau - x)^2 \, |v(x)|^{-2}\, dx.$$

THEOREM 2.1 *The least possible constant C in (4) is sandwiched as follows*

$$C \approx A_{2;(0,\tau),u,v} + D_\tau + A_{1;(\tau,\infty),u,(x-\tau)^{-1}v(x)} + B_{1;(\tau,\infty),(x-\tau)u(x),v}. \tag{6}$$

Proof If $\int_0^\infty f = 0$, then for $x > \tau$ we have

$$I_2 f(x) = \int_0^x \left(\int_0^s f\right) ds = \int_0^\tau \left(\int_0^s f\right) ds + \int_\tau^x \left(\int_0^s f\right) ds =$$

$$\int_0^\tau (\tau - y) f(y) dy - \int_\tau^x \left(\int_s^\infty f\right) ds =$$

$$\int_0^\tau (\tau - y) f(y) dy - (x - \tau) \int_x^\infty f - \int_\tau^x (y - \tau) f(y) dy. \tag{7}$$

For the upper bound we use Theorem 1.1 and Cauchy's inequality and find

$$\|(I_2 f)\, u\|_2 \leq \left\|\chi_{[0,\tau]}\, (I_2 f)\, u\right\|_2 + \left\|\chi_{[\tau,\infty]} u\right\|_2 \left|\int_0^\tau (\tau - y) f(y) dy\right| +$$

$$\left\|\chi_{[\tau,\infty]}(x)(x-\tau)u(x) \int_x^\infty f\right\|_2 + \left\|\chi_{[\tau,\infty]}(x) u(x) \int_\tau^x (y - \tau) f(y) dy\right\|_2 \ll$$

$$\left(A_{2;(0,\tau),u,v} + D_\tau + A_{1;(\tau,\infty),u,(x-\tau)^{-1}v(x)} + B_{1;(\tau,\infty),(x-\tau)u(x),v}\right) \|f v\|_2 .$$

For the lower bound it is sufficient to arrange a one-to-one isometrical map ω_τ between the cones

$$\mathcal{L}_1 = \{f \in L_{2,v} : \ f \geq 0, \ \ \text{supp } f \subseteq [0, \tau]\}$$

and

$$\mathcal{L}_2 = \{f \in L_{2,v} : \ f \leq 0, \ \ \text{supp } f \subseteq [\tau, \infty]\}$$

such that

$$\int_0^\infty (f + \omega_\tau f) = 0, \ f \in \mathcal{L}_1 \tag{8}$$

We construct ω_τ below, using an idea from ([8], Section 1.8). Now, suppose (4) is true and the right-hand side of (5) is finite. Since for any function of the form $f + \omega_\tau f, \ f \in \mathcal{L}_1$ or $\omega_\tau^{-1} f + f, \ f \in \mathcal{L}_2$ the right-hand side of (7) is nonnegative, we obtain the following inequalities

$$\left\| \chi_{[0,\tau]} (I_2 f)\, u \right\|_2 \leq \|(I_2(f + \omega_\tau f))\, u\|_2 \leq C \, \|(f + \omega_\tau f) v\|_2 \leq$$
$$2C \left\| \chi_{[0,\tau]} f v \right\|_2, \ f \in \mathcal{L}_1$$

and

$$\left\| \chi_{[\tau,\infty]} (I_2 f)\, u \right\|_2 \leq 2C \left\| \chi_{[\tau,\infty]} f v \right\|_2, \ f \in \mathcal{L}_2.$$

Then the required lower bound follows from Theorem 1.1. For a possible infinite right hand side of (5) we proceed as before, replacing the weight $v(x)$ by $(1 + \varepsilon x) v(x)$ and then using limiting arguments with $\varepsilon \to 0$.

To construct the map ω_τ we define for $f \in \mathcal{L}_1$

$$(\omega_\tau f)(x) = -\frac{f\left(\rho^{-1}(x)\right) \left|v\left(\rho^{-1}(x)\right)\right|^2}{|v(x)|^2}, \ x \geq \tau,$$

where $\rho : \ [0, \tau] \to [\tau, \infty]$ is given by

$$\int_0^s |v|^{-2} = \int_{\rho(s)}^\infty |v|^{-2}, \ s \in [0, \tau].$$

Then (8) is valid and

$$\left\| \chi_{[\tau,\infty]} f v \right\|_2 = \left\| \chi_{[\tau,\infty]} (\omega_\tau f) v \right\|_2, \ f \in \mathcal{L}_2.$$

Theorem 2.1 is proved.

Remark 2.1 Theorem 2.1 has a complete analog for a finite interval, $(-\infty, 0)$ or $(-\infty, \infty)$ and for an L_p–L_q setting. For a finite interval the L_p–L_q version of Theorem 2.1 follows from the characterization (6) above, which has independently been proved by a different method in [5].

CASE (3.4). In this case the characterization of

$$\|Fu\|_2 \leq C \left\|F''v\right\|_2, \; F'(0) = F(\infty) = F'(\infty) = 0$$

is equivalent to the problem

$$\|(J_2 g)\, u\|_2 \leq C \,\|gv\|_2\,, \; \int_0^\infty g = 0, \tag{9}$$

which is dual to (4). Put

$$D_{1,\tau}^2 = \int_0^\tau |u|^2 \int_\tau^\infty (x-\tau)^2\, |v(x)|^{-2}\, dx.$$

THEOREM 2.2 *The least possible constant C in (9) is estimated by*

$$C \approx B_{2;(\tau,\infty),u,v} + D_{1,\tau} + A_{1;(0,\tau),(\tau-x)u(x),v} + B_{1;(0,\tau),u,(\tau-x)^{-1}v(x)}. \tag{10}$$

Proof This time we use a decomposition for $0 < x < \tau$ of the form

$$J_2 g(x) = \int_\tau^\infty (y-\tau) g(y)\, dy - (\tau - x) \int_0^x g - \int_x^\tau (\tau - y) g(y)\, dy$$

and proceed as in the proof of Theorem 2.1.

CASE (3.3). Now we have

$$\|Fu\|_2 \leq C \left\|F''v\right\|_2, \; F(0) = F(\infty) = F'(\infty) = 0,$$

which equivalent to the problem

$$\|(J_2 g)\, u\|_2 \leq C \,\|gv\|_2\,, \; \int_0^\infty y\, g(y)\, dy = 0. \tag{11}$$

Making the substitution $y\, g(y) \to g(y)$, we make (11) equivalent to

$$\left\|\left(\overline{J_2} g\right) u\right\|_2 \leq C \,\|gv_1\|_2\,, \; \int_0^\infty g = 0,$$

where $v_1(x) = \frac{1}{x}v(x)$ and

$$\overline{J_2}g(x) = \int_x^\infty \left(1 - \frac{x}{y}\right) g(y)\, dy.$$

By a known criterion [14]

$$\left\|\overline{J_2}\right\|_{L_{2,v_1,(\tau,\infty)} \to L_{2,u,(\tau,\infty)}} = \left\|J_2\right\|_{L_{2,v,(\tau,\infty)} \to L_{2,u,(\tau,\infty)}}.$$

Thus, using the decomposition

$$\begin{aligned}\overline{J_2}g(x) &= \frac{x}{\tau}\int_\tau^\infty (y-\tau)g(y)\frac{dy}{y}\\ &\quad - \frac{\tau - x}{\tau}\int_0^x g - \frac{x}{\tau}\int_x^\tau (\tau - y)g(y)\frac{dy}{y},\ 0 < x < \tau\end{aligned}$$

and arguing as in the proof of Theorems 2.1 and 2.2 we obtain

THEOREM 2.3 *The least constant C in (11) is estimated by*

$$\begin{aligned}C \approx B_{2;(\tau,\infty),u,v} + \tau^{-1}\big(&D_{2,\tau} + A_{1;(0,\tau),(\tau-x)u(x),x^{-1}v(x)}\\ &+B_{1;(0,\tau),xu(x),(\tau-x)^{-1}v(x)}\big),\end{aligned} \tag{12}$$

where

$$D_{2,\tau}^2 = \int_0^\tau x^2|u(x)|^2\, dx \int_\tau^\infty (x-\tau)^2|v(x)|^{-2}\, dx.$$

CASE (3.1). Now we have equivalence of

$$\|Fu\|_2 \le C\left\|F''v\right\|_2,\ F(0) = F'(0) = F(\infty) = 0$$

and

$$\|(I_2 f)\, u\|_2 \le C\, \|fv\|_2,\ \int_0^\infty \left(\int_0^y f\right) dy = 0, \tag{13}$$

where the integral is interpreted in the Riemann sense.

We need the following simple note.

LEMMA *A locally integrable function f satisfies*

$$\int_0^\infty \left(\int_0^y f \right) dy = 0 \tag{14}$$

if and only if there exist $\lambda \in (0, \infty)$ such, that

$$\int_0^\lambda f = 0 \quad \text{and} \quad \int_0^\lambda \left(\int_y^\lambda f \right) dy = \int_\lambda^\infty \left(\int_\lambda^y f \right) dy. \tag{15}$$

Proof Suppose (14) be valid, then the first assertion in (15) is trivial. Then we have

$$0 = \int_0^\infty \left(\int_0^y f \right) dy = \int_0^\lambda \left(\int_0^y f \right) dy + \int_\lambda^\infty \left(\int_0^y f \right) dy =$$

$$- \int_0^\lambda \left(\int_y^\lambda f \right) dy + \int_\lambda^\infty \left(\int_\lambda^y f \right) dy.$$

Conversing the last line, we prove the "if" part.

Suppose $\lambda \in (0, \infty)$ and put

$$\tau_\lambda : \int_0^{\tau_\lambda} |v|^{-2} = \int_{\tau_\lambda}^\lambda |v|^{-2}.$$

THEOREM 2.4 *The least possible constant C in (13) is given by*

$$C \approx A_{2;(0,\infty),u,v} + \sup_{\lambda>0} \left(A_{1;(\tau_\lambda,\lambda),u,(x-\tau_\lambda)^{-1}v(x)} + B_{1;(\tau_\lambda,\lambda),(x-\tau_\lambda)u(x),v} \right). \tag{16}$$

Proof We begin with the upper bound. Let $f \in L_{2,v}$ and (14) be valid. Hence, (15) is true for some $\lambda \in (0, \infty)$. Then for $x > \lambda$ we have

$$I_2 f(x) = \int_0^\lambda (\lambda - y) f(y)\, dy + \int_\lambda^x (x - y) f(y)\, dy.$$

Using this and applying Theorems 1.1 and 2.1, we obtain

$$\|(I_2 f)\, u\|_2 \leq \left\| \chi_{[0,\lambda]}\, (I_2 f)\, u \right\|_2 + \left\| \chi_{[\lambda,\infty]}\, (I_2 f)\, u \right\|_2 \ll$$

$$\sup_{\lambda>0}\Bigg(A_{2;(0,\tau_\lambda),u,v}+\int_{\tau_\lambda}^{\lambda}|u|^2\int_0^{\tau_\lambda}(\tau_\lambda-x)^2|v(x)|^{-2}\,dx+$$
$$A_{1;(\tau_\lambda,\lambda),u,(x-\tau_\lambda)^{-1}v(x)}+B_{1;(\tau_\lambda,\lambda),(x-\tau_\lambda)u(x),v}+$$
$$\int_\lambda^\infty|u|^2\int_0^\lambda(\lambda-x)^2|v(x)|^{-2}\,dx+A_{2;(\lambda,\infty),u,v}\Bigg)\|fv\|_2$$
$$\ll\Bigg(A_{2;(0,\infty),u,v}+\sup_{\lambda>0}\big(A_{1;(\tau_\lambda,\lambda),u,(x-\tau_\lambda)^{-1}v(x)}+B_{1;(\tau_\lambda,\lambda),(x-\tau_\lambda)u(x),v}\big)\Bigg)$$
$$\|fv\|_2.$$

For the lower bound we suppose $\lambda\in(0,\infty)$ and define the function $\rho_1:[0,\tau_\lambda]\to[\tau_\lambda,\lambda]$ by

$$\int_0^s|v|^{-2}=\int_{\rho_1(s)}^{\lambda}|v|^{-2},\quad s\in[0,\tau_\lambda].$$

Given $f\in L_{2,v}$, supp $f\subseteq[0,\tau_\lambda]$, $f(x)\geq 0$ we define $f_1(x)$ to be zero outside $[\tau_\lambda,\lambda]$ and

$$f_1(x)=-\frac{f\left(\rho_1^{-1}(x)\right)\left|v\left(\rho_1^{-1}(x)\right)\right|^2}{|v(x)|^2},\quad x\in(\tau_\lambda,\lambda).$$

Then, by the argument from the proof of Theorem 2.1 we have

$$\left\|\chi_{[0,\tau_\lambda]}fv\right\|_2=\left\|\chi_{[\tau_\lambda,\lambda]}f_1v\right\|_2,$$

$$\int_0^{\tau_\lambda}f+\int_{\tau_\lambda}^{\lambda}f_1=0. \tag{17}$$

Now, given $\lambda\in(0,\infty)$ we find $\mu=\mu(\lambda)\in(0,\infty)$ from equation

$$\int_0^\lambda x^2|v(x)|^{-2}\,dx=\int_\lambda^\mu(\mu-x)^2|v(x)|^{-2}\,dx$$

and define the function $\rho_2:[0,\lambda]\to[\lambda,\mu]$ by

$$\int_0^s x^2|v(x)|^{-2}\,dx=\int_{\rho_2(s)}^{\mu}(\mu-x)^2|v(x)|^{-2}\,dx. \tag{18}$$

Put

$$f_0 = f + f_1$$

and define $f_2(x) = 0$, $x \notin [\lambda, \mu]$ and

$$f_2(x) = \frac{(\mu - x) f_0\left(\rho_2^{-1}(x)\right) \left|v\left(\rho_2^{-1}(x)\right)\right|^2}{\rho_2^{-1}(x)\, |v(x)|^2}, \; x \in (\lambda, \mu). \tag{19}$$

Then

$$\|f_2 v\|_2^2 = \|f_0 v\|_2^2 = \|f v\|_2^2 + \|f_1 v\|_2^2.$$

Now, put

$$\tilde{f} = f_0 + f_2.$$

Then (17) implies

$$\int_0^\lambda \tilde{f} = 0.$$

Let us show that

$$\int_0^\lambda \left(\int_y^\lambda \tilde{f}\right) dy = \int_\lambda^\infty \left(\int_\lambda^y \tilde{f}\right) dy.$$

Indeed, applying (18) and (19) we find

$$\int_\lambda^\infty \left(\int_\lambda^y \tilde{f}\right) dy = \int_\lambda^\mu (\mu - y) f_2(y)\, dy = \int_0^\lambda (\mu - \rho_2(s)) f_2(\rho_2(s))\, d\rho_2(s)$$
$$= \int_0^\lambda s f_0(s)\, ds = \int_0^\lambda \left(\int_y^\lambda \tilde{f}\right) dy.$$

The lemma implies that $\tilde{f}$ satisfies (14) and, consequently, admissible for (13). Since functions f_0 form a cone used for the lower bound in the proof of Theorem 2.1, we obtain

$$C \gg \sup_{\lambda>0} \left(A_{2;(0,\tau_\lambda),u,v} + D_{\tau_\lambda} + A_{1;(\tau_\lambda,\lambda),u,(x-\tau_\lambda)^{-1}v(x)} + B_{1;(\tau_\lambda,\lambda),(x-\tau_\lambda)u(x),v}\right)$$
$$\gg \sup_{\lambda>0} \left(A_{1;(\tau_\lambda,\lambda),u,(x-\tau_\lambda)^{-1}v(x)} + B_{1;(\tau_\lambda,\lambda),(x-\tau_\lambda)u(x),v}\right).$$

For the remaining part of lower bound we suppose, that $0 < \lambda < \sigma < \infty$ and choose $\mu \in (\lambda, \sigma)$ such that

$$\int_\lambda^\mu (\sigma - x)^2 |v(x)|^{-2}\, dx = \int_\mu^\sigma (\sigma - x)^2 |v(x)|^{-2}\, dx.$$

Obviously, $\mu \to \infty$, whereas $\sigma \to \infty$. Let $f \in L_{2,v}$, $\operatorname{supp} f \subseteq [\lambda, \mu]$, $f(x) \geq 0$. Define the function $\rho : [\lambda, \mu] \to [\mu, \sigma]$ such that

$$\int_\lambda^s (\sigma - x)^2 |v(x)|^{-2}\, dx = \int_{\rho(s)}^\sigma (\sigma - x)^2 |v(x)|^{-2}\, dx$$

and define $f_1(x) = 0$, $x \notin [\mu, \sigma]$ and

$$f_1(x) = -\frac{(\sigma - x) f\left(\rho^{-1}(x)\right) \left|v\left(\rho^{-1}(x)\right)\right|^2}{\left(\sigma - \rho^{-1}(x)\right) |v(x)|^2}, \quad x \in (\mu, \sigma).$$

Put

$$\tilde{f} = f + f_1.$$

Then it is routine to verify that

$$\|\tilde{f} v\|_2^2 = 2\|f v\|_2^2$$

and

$$\int_0^\infty \left(\int_0^y \tilde{f} \right) dy = 0.$$

Applying Theorem 1.1 we find

$$C \gg A_{2;(\lambda,\mu),u,v}, \quad 0 < \lambda < \mu < \infty.$$

Letting $\sigma \to \infty$ and then $\lambda \to 0$, we obtain the remaining part of the lower bound.

Theorem 2.4 is proved.

Case (2.5). We have

$$\|Fu\|_2 \leq C \left\|F'' v\right\|_2, \quad F'(0) = F(\infty) = 0. \tag{20}$$

THEOREM 2.5 *The estimate for the least constant C in (20) is established by* (10).

Proof Since

$$\{F : F'(0) = F(\infty) = 0\} \supset \{F : F'(0) = F(\infty) = F'(\infty) = 0\},$$

we find by Theorem 2.2, that

$$C \gg B_{2;(\tau,\infty),u,v} + D_{1,\tau} + A_{1;(0,\tau),(\tau-x)u(x),v} + B_{1;(0,\tau),u,(\tau-x)^{-1}v(x)}.$$

Now, suppose

$$B_{2;(\tau,\infty),u,v} + D_{1,\tau} + A_{1;(0,\tau),(\tau-x)u(x),v} + B_{1;(0,\tau),u,(\tau-x)^{-1}v(x)} < \infty$$

and, consequently,

$$\int_s^\infty (x-s)^2 |v(x)|^{-2}\, dx < \infty, \ s > \tau.$$

Hence, the function

$$\tilde{F}(s) = \int_s^\infty (x-s) F''(x)\, dx$$

is defined for all $s > 0$, $\tilde{F}(\infty) = 0$ and $\tilde{F}'' = F''$. It implies that $\tilde{F} = F$ and by Theorem 2.2 we obtain the upper bound.

CASE (2.1). Now we have

$$\|Fu\|_2 \le C \left\|F''v\right\|_2, \ F(0) = F(\infty) = 0. \tag{21}$$

Analogously to the case (2.5) we prove the following.

THEOREM 2.6 *The estimate for the least constant C in* (21) *is established by* (12).

CASE (4.1). Now the problem is to characterize

$$\|Fu\|_2 \le C \left\|F''v\right\|_2, \ F(0) = F'(0) = F(\infty) = F'(\infty) = 0. \tag{22}$$

THEOREM 2.7 *The estimate for the least constant C in* (22) *is established by* (16).

Proof The upper bound immediately follows from Theorem 2.4. For the lower bound suppose (22) is valid. Then it is valid for the weight $v_\varepsilon(x) = (1+\varepsilon x)|v(x)|$ instead of $v(x)$. It is easy to see that

$$\{F : \|F''v_\varepsilon\|_2 < \infty,\ F(0) = F'(0) = F(\infty) = F'(\infty) = 0\}$$

$$= \{F : \|F''v_\varepsilon\|_2 < \infty,\ F(0) = F'(0) = F(\infty) = 0\}.$$

Consequently, by Theorem 2.4 we have

$$C \gg A_{2;(a,\infty),u,v_\varepsilon} + \sup_{\lambda>0}\left(A_{1;(\tau_\lambda,\lambda),u,(x-\tau_\lambda)^{-1}v_\varepsilon(x)} + B_{1;(\tau_\lambda,\lambda),(x-\tau_\lambda)u(x),v_\varepsilon}\right).$$

The Fatou theorem brings the required lower bound.

Remark 2.2 The characterization of cases (2.1), (2.5) and (4.1) on a finite interval is different (for (2.1) and (2.5) see [6]) and (4.1) is so far unknown.

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