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A Weighted Inequality for Derivatives on the Half-line

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Some inequalities will be presented, which give weighted norm estimates for derivatives of functions defined on the half-line. These inequalities are related to Hardy's inequality, and they also generalize Hardy's inequality to higher derivatives. The results presented here are also analogous to some recently-derived inequalities for the derivatives of functions defined on the interval $[-1, 1]$, which have had important applications to the study of polynomial approximation on $[-1, 1]$.

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1 INTRODUCTION

Differentiation of a function $f(t)$ defined on the half-line $[0, \infty)$ corresponds in a natural way to the differentiation of $F(x)$ by t, where $F(x) = f(x^2)$, and $D_t F(x) = (2x)^{-1} F'(x)$. We will use this relationship and the generalized Hardy inequality to obtain some estimates for derivatives by t of even functions of x on $(-\infty, \infty)$ in terms of derivatives by x. Analogous results for the 2π-periodic even functions have been developed in Kilgore and Szabados [3] and in Kilgore [1], for differentiation by $x \leftrightarrow \cos\theta$. These results for periodic functions have been applied in Kilgore and Szabados [3] and in Kilgore [2] to prove some very basic properties of algebraic polynomial approximation on finite intervals, especially regarding the simultaneous approximation of derivatives. It is expected that the inequalities proved here will have similar applications in weighted spaces on the half-line.

For $1 \le p < \infty$, the function $F(x)$ (defined on the real line) is in L^p if

$$\|F\|_p := \left(\int_{-\infty}^{\infty} |F(x)|^p dx\right)^{\frac{1}{p}} < \infty.$$

We also write

$$\|F\|_\infty = \text{ess sup } |F|.$$

Weight functions will be introduced by placing them inside the norm. All weight functions used here will necessarily be even. To qualify as a weight, the function V will also be measurable, positive, and finite almost everywhere. The corresponding weighted norm of a function F will be $\|VF\|_p$, this includes the possibility that $p = \infty$.

We will also need to consider norms defined on various subintervals of $(-\infty, \infty)$. If (c, d) is a subinterval of $(-\infty, \infty)$, then we define

$$\|F\|_{p,(c,d)} = \left(\int_c^d |F(x)|^p dx\right)^{\frac{1}{p}}.$$

In particular, the norm of any even or any odd function F can be described on the interval $[0, \infty)$ or $(0, \infty)$ only; for $1 \le p \le \infty$ we have

$$\|F\|_p := 2^{\frac{1}{p}} \|F\|_{p,(0,\infty)}. \tag{1}$$

We are now ready to state the main results:

THEOREM 1 *Let $1 \le p \le \infty$, and let V be an even weight function such that with finite constant C the inequality*

$$\|(2x)^{-1} V(x) f(x)\|_{p,(0,\infty)} \le C \|V(x) f'(x)\|_{p,(0,\infty)} \tag{2}$$

of Hardy type holds for every absolutely continuous function f such that $f(0) = 0$. Let $F(x)$ be an even function such that $V(x)F^{(2r)}(x) \in L^p$ and $F^{(2r-1)}(x)$ is absolutely continuous. Then for $1 \le k \le r - 1$ the derivatives $D_t^k F(x)$ exist and are continuous. The derivative $D_t^r F(x)$ also exists and is continuous at all $x \ne 0$. Furthermore

$$\|V(x) D_t^r F(x)\|_p \le \sum_{j=1}^{r} A_{j,r} \|V(x) F^{(2j)}(x)\|_p, \tag{3}$$

in which $A_{j,r}$ are nonnegative constants independent of F.

Conversely, suppose that (3) holds for every r, for every even function $F(x)$ for which $F^{(2r-1)}(x)$ is absolutely continuous. Then the inequality (2) holds for the weight V.

Furthermore, for the constants $A_{j,r}$ in Theorem 1 the proof will make clear the following:

COROLLARY 1 *The constants $A_{j,r}$ in (3) are of the form $A_{j,r} = C_{2r-1}C_{2r-3} \cdots C_1 B_{j,2r}$. The constants $C_{2r-1}, \ldots, C_1$ depend only upon the weight V and are all bounded above by C, the constant of (2). The constants $B_{j,2r}$ depend only upon j and r and are independent of V and of p.*

More exact information about the constants $C_1, \ldots, C_{2r-1}$ may be found in Lemma 2.

Note that Theorem 1 does not state as a conclusion that $\|V(x)D_t^k F(x)\|_{p,(0,\infty)} < \infty$ for any particular value or values of k less than r. In fact, this conclusion does not follow without some additional hypothesis:

THEOREM 2 *Let $1 \le p \le \infty$, and let V be a weight function such that $\|V\|_p < \infty$ and such that with finite constant C' the inequality*

$$\|V(x)f(x)\|_{p,(0,\infty)} \le C'\|V(x)f'(x)\|_{p,(0,\infty)} \tag{4}$$

of Hardy type holds for every absolutely continuous function f such that $f(0) = 0$. Let $F(x)$ be an even function such that $V(x)F^{(2r)}(x) \in L^p$ and $F^{(2r-1)}(x)$ is absolutely continuous. Then for $0 \le k \le 2r$ the derivatives $F^{(k)}$ are in L^p with weight V, and also for $1 \le k \le r$ the derivatives $D_t^k F(x)$ are in L^p with weight V.

Now, inasmuch as Theorem 1 holds on the interval $[0, \infty)$, it also holds with essentially the same proof on any finite interval $[0, b)$. It will be seen from Lemma 1 that, on the finite interval $[0, b)$, the inequality (2) implies the inequality (4). These observations are summarized in

COROLLARY 2 *Theorem 1 and Theorem 2 still hold if the domain is given as a finite interval $(0, b)$ for some $b > 0$ instead of $(0, \infty)$. Moreover, in this case the inequality (2) implies the inequality (4).*

2 TOOLS

Muckenhoupt [4] has given a necessary and sufficient condition on two weight functions $V_1(x)$ and $V_0(x)$ for the generalized Hardy inequality

$$\|(2x)^{-1}V(x)f(x)\|_{p,(a,b)} \le C\|V(x)f'(x)\|_{p,(a,b)} \tag{5}$$

to hold for some finite C on an interval (a, b), with $-\infty \le a < b \le \infty$, for every absolutely continuous function F which is zero at a. The necessary and sufficient condition is that

$$B := \sup_{a<c<b} \|V_1(x)\|_{p,(c,b)} \|(V_0(x))^{-1}\|_{q,(a,c)} < \infty, \tag{6}$$

in which p and q satisfy $p^{-1} + q^{-1} = 1$ and $1 \le p, q \le \infty$. The values of B and C are related by

$$B \le C \le p^{\frac{1}{p}} q^{\frac{1}{q}} B, \tag{7}$$

with $B = C$ if $p = 1$ or $p = \infty$. Thus, (5) holds with finite C if and only if (6) gives a finite value of B.

Here we choose in particular $V_0 = V$, where V is the given, even weight, and we will almost invariably use $a = 0$. We further define not only V_1 but also a sequence of weights V_k for $k = 1, 2, \ldots$ by

$$V_k(x) := (2x)^{-1} V_{k-1}(x). \tag{8}$$

The following results are needed:

LEMMA 1 *Let V satisfy the inequality*

$$\|(2x)^{-1} V(x) f(x)\|_{p,(0,b)} \le C \|V(x) f'(x)\|_{p,(0,b)}$$

for every absolutely continuous function f such that $f(0) = 0$. Then $b < \infty$ implies

$$\|V(x) f(x)\|_{p,(0,b)} \le C' \|V(x) f'(x)\|_{p,(0,b)}$$

for every absolutely continuous function f such that $f(0) = 0$.

Proof If $b < \infty$, then we have

$$\frac{1}{2b} \|V(x) f(x)\|_{p,(0,b)} \le \|(2x)^{-1} V(x) f(x)\|_{p,(0,b)} \le C \|V(x) f'(x)\|_{p,(0,b)}.$$

The lemma is proved with $C' \le 2bC$. □

LEMMA 2 *Let $V_0, V_1, \ldots$ be constructed according to (8). Let V_0 and V_1 satisfy (5) on the interval $[0, b)$, implying that (6) is satisfied with a finite constant B. Then for $k = 1, 2, \ldots$*

$$B_k := \sup \left[\|V_k\|_{p,(c,b)} \ \|V_{k-1}^{-1}\|_{q,(0,c)} \right] \le B < \infty. \tag{9}$$

Indeed, $B = B_1 \ge B_2 \ge \ldots$. for $k = 1, 2, \ldots$. Also

$$\|V_k F\|_{p,[0,b)} \le C_k \|V_{k-1} F'\|_{p,[0,b)} \tag{10}$$

for every function F which is absolutely continuous on $[0, b)$ and satisfies $F(0) = 0$. Furthermore, $C_k \le 2B_k \le 2B$.

Proof The lemma is obvious if $k = 1$. To complete the proof, let $0 < c < b$. Then for $k \geq 1$

$$\begin{aligned}\|V_k\|_{p,(c,b)}\|V_{k-1}^{-1}\|_{q,(0,c)} &= \|(2x)^{-1}V_{k-1}(x)\|_{p,(c,b)}\|2xV_{k-2}^{-1}(x)\|_{q,(0,c)}\\ &\leq (2c)^{-1}\|V_{k-1}\|_{p,(c,b)}\cdot(2c)\|V_{k-2}^{-1}\|_{q,(0,c)}\\ &\leq \|V_{k-1}\|_{p,(c,b)}\|V_{k-2}^{-1}\|_{q,(0,c)},\end{aligned}$$

and (9) follows immediately by induction for all $k > 1$. The other conclusions follow then, too, from (7). This concludes the proof of the lemma. □

Now, it is helpful in the proof of the Theorem to prove a weaker version first. We have

LEMMA 3 *Let $F(x) \in C^{2r}[-b, b]$ be an even function. Let $D_t F(x)$ signify its derivative with respect to $t = x^2$. Then for $r \geq 1$ the derivative $D_t^r F(x)$ exists. It is in $C[-b, b]$, and*

$$\|D_t^r F(x)\| \leq \sum_{j=1}^{r} \beta_{j,r}\|F^{(2j)}\|,$$

in which $F^{(j)}$ signifies the j^{th} derivative of F by x and in which $\beta_{j,r}$ are nonnegative constants independent of F.

Proof For $k \leq r$, the derivative $D_t^k F(x)$ is computed in terms of the first k derivatives of $F(x)$ by x. Therefore, since $F(x) \in C^{2r}[-b, b]$, the derivative $D_t^k F(x)$ clearly exists and is continuous for all $x \in [-b, b]$, such that $x \neq 0$.

To consider the situation at the special point 0, we can proceed inductively. Assuming for $k \leq r$ that $D_t^{k-1}F(x)$ exists and is continuous at $x = 0$, we note that by definition

$$D_t^k F(0) = \lim_{x\to 0}\frac{D_t^{k-1}F(x) - D_t^{k-1}F(0)}{x^2} \tag{11}$$

$$= \lim_{x\to 0}\frac{D_x D_t^{k-1}F(x)}{2x} = \lim_{x\to 0} D_t^k F(x). \tag{12}$$

Thus (11) shows that $D_t^k F(x)$ is continuous at zero provided that it exists there, and furthermore (11) provides a way to investigate its existence.

In view of (11), the lemma follows immediately if $r = 1$. To handle the more general case, we need first a formula which more explicitly gives the derivative $D_t^k F(x)$ in terms of derivatives by x. By induction, one may obtain

$$D_t^k F(x) = \frac{\sum_{j=1}^{k} a_{j,k}(x)F^{(j)}(x)}{(2x)^{2k-1}}, \tag{13}$$

valid for all $x \neq 0$ and $k \leq r$. Furthermore, in (13) the coefficients $a_{j,k}(x)$ are polynomials which are independent of F, depend only upon j, k, and can be recursively computed. Specifically, if it is agreed that $a_{j,k}$ is zero if $j < 1$ or if $j > k$, we can write

$$a_{j,k+1}(x) = 2x(a'_{j,k}(x) + a_{j-1,k}(x)) - (2k-1)a_{j,k}(x), \tag{14}$$

starting with $a_{1,1} = 1$. In general it is seen that the degree of $a_{j,k}$ is bounded by $k-1$, and $a_{j,k}$ is even if j is odd and odd if j is even. From these two observations it also follows that the degree of $a_{j,k}$ is bounded by $k-2$ in case that k is even and j is odd and in case that k is odd and j is even, that is, if k and j disagree in parity.

Let us show the lemma first under the stronger assumption that $D_t^r F(0)$ exists. By (11) and (13) we have for $k \leq r$

$$D_t^k F(0) = \lim_{x \to 0} \frac{\sum_{j=1}^{k} a_{j,k}(x) F^{(j)}(x)}{(2x)^{2k-1}},$$

implying that the numerator of the fraction within the limit is zero when $x = 0$. Thus, using the mean value theorem of Cauchy we can rewrite (13) as

$$D_t^k F(x) = \frac{\sum_{j=1}^{k+1} [a'_{j,k}(x_1) + a_{j-1,k}(x_1)] F^{(j)}(x_1)}{(2k-1)(2)(2x_1)^{2k-2}} \tag{15}$$

for some x_1 between x and 0. If $k = 1$ the denominator on the right in (15) is 2, so that

$$D_t F(x) = \frac{1}{2} \sum_{j=1}^{2} [a'_{j,1}(x_1) + a_{j-1,1}(x_1)] F^{(j)}(x_1).$$

In view of (14) we note that this gives

$$|D_t F(x)| = \frac{1}{2} |F^{(2)}(x_1)| \leq \frac{1}{2} \|F^{(2)}\|_{[-b,b]}.$$

We proceed under the assumption that $k > 1$. Since $D_t^k F(0)$ is the limit of the fraction on the right in (15) as $x_1 \to 0$, the numerator must in turn be zero when $x_1 = 0$. If j is odd, we know that $F^{(j)}(0) = 0$. Therefore, independently of F

$$\sum_{j=1}^{[\frac{k+1}{2}]} [a'_{2j,k}(0) + a_{2j-1,k}(0)] F^{(2j)}(0) = 0.$$

Now in particular the functions x^{2m}, where $m = 1, 2, \ldots$ satisfy

$$D_x^{2j}\ x^{2m}|_{x=0} = 0 \ \text{ for } \ j < m,$$

while

$$D_x^{2m}\ x^{2m}|_{x=0} = (2m)! \neq 0.$$

Successively choosing $F(x) = x^{2m}$ for $m = [\frac{k+1}{2}], \ldots, 1$ therefore shows that

$$a'_{2j,k}(0) + a_{2j-1,k}(0) = 0 \ \text{ for } \ j = 1, \ldots, [\frac{k+1}{2}].$$

Recalling that $a_{j,k}$ is even when j is odd and odd when j is even, we see from this argument that, when j is even, the expression $a'_{j,k}(x) + a_{j-1,k}(x)$ (which is an even polynomial) is zero at $x = 0$ and thus contains a factor of $2x^2$. And when j is odd, then $a'_{j,k}(x) + a_{j-1,k}(x)$ is odd and because of its oddness must contain a factor of $2x$. Therefore we may write for $j = 1, \ldots, k+1$

$$a'_{j,k}(x) + a_{j-1,k}(x) = 2x\ b_{j,k+1}(x). \tag{16}$$

We further note that the degree of $b_{j,k+1}$ is not more than $k-2$, and if j and $k+1$ disagree in parity, then the degree of $b_{j,k+1}$ is not more than $k-3$. Thus we may also write

$$D_t^k F(x) = \frac{\sum_{j=1}^{k+1} b_{j,k+1}(x_1) F^{(j)}(x_1)}{(2k-1)(2)(2x_1)^{2k-3}}, \tag{17}$$

in which $b_{j,k+1}$ is odd if j is even and even if j is odd, and is of degree not more than $k-3$ if j and $k+1$ disagree in parity.

Using these observations, we may again apply the mean value theorem of Cauchy to (17), obtaining a fraction on the right which is evaluated at a point x_2 lying between 0 and x_1. If $k = 2$ the denominator on the right side of (17) is reduced to a constant times $2x_2$, and the coefficient polynomials $b_{j,3}$ are zero if j is even, constants if j is odd. One application of Cauchy's theorem, followed by a norm estimate, completes the argument. If on the other hand $k > 2$, we may remove a common factor of $2x_2$ from numerator and denominator. In general, it is possible to complete $k-1$ like steps, the ℓth of which consists of two actions. The first action in the ℓth step is to apply Cauchy's mean value theorem, giving the evaluation at a point x_ℓ lying between 0 and $x_{\ell-1}$ of the right side of the equation obtained from the completion of the previous step. The second action in the ℓth step is to cancel the factor of $2x_\ell$ In each of these $k-1$ like steps, a factor of $2x^2$ is removed from the numerator and denominator. After the ℓth step, it is possible to write

$$D_t^k F(x) = \frac{\sum_{j=1}^{k+\ell} b_{j,k+\ell}(x_\ell) F^{(j)}(x_\ell)}{(2k-1)(2k-3)\dots(2k-2\ell+1)(2^\ell)(2x_\ell)^{2k-2\ell-1}}, \quad (18)$$

in which the coefficient functions $b_{j,k+\ell}$ are defined for $j = 1, \dots, k+\ell$ by

$$b'_{j,k+\ell-1}(x) + b_{j-1,k+\ell-1}(x) = 2x\, b_{j,k+\ell}(x). \quad (19)$$

Note that after this reduction it necessarily follows that the polynomial $b_{j,k+\ell}$ is of degree at most $k-\ell-1$, and furthermore that if j disagrees in parity with $k+\ell$ the degree is not more than $k-\ell-2$. Finally, after these $k-1$ steps, a single power of $2x$ remains in the denominator, and we have

$$D_t^k F(x) = \frac{\sum_{j=1}^{2k-1} b_{j,2k-1}(x_k) F^{(j)}(x_{k-1})}{(2k-1)(2k-3)\dots(3)(2^{k-1})(2x)}, \quad (20)$$

in which the coefficient polynomials $b_{j,2k-1}$ are in fact all constants, and furthermore they are zero if j is even.

This permits one more application of the mean value theorem of Cauchy, and using the notation

$$b'_{j,2k-1}(x) + b_{j-1,2k-1}(x) = b_{j,2k}, \quad (21)$$

we can write the result as

$$D_t^k F(x) = \frac{\sum_{j=1}^{k} b_{2j,2k} F^{(j)}(x_k)}{(2k-1)(2k-3)\dots(3)(2^k)}, \quad (22)$$

in which x_k lies between x_{k-1} and 0. This enables us to write

$$|D_t^k F(x)| \le \sum_{j=1}^{k} \beta_{j,k} |F^{(j)}(x_k)| \le \sum_{j=1}^{k} \beta_{j,k} \|F^{(j)}\|_{[-b,b]},$$

in which

$$\beta_{j,k} = \frac{|b_{2j,2k}|}{(2k-1)(2k-3)\dots(3)(2^k)}.$$

The lemma is now established for all functions $F \in C^{2r}$ for which $D_t^r F(x)$ is known to exist at 0. But then, since the behavior of the coefficient polynomials $a_{j,k}$ is completely independent of the function $F(x)$, and since $D_t^k F(0) = \lim_{x\to 0} D_t^k F(x)$, it follows that $D_t^k F(0)$ exists if the limit exists. If $F \in C^{2r}$, then (22) demonstrates that the appropriate limits do exist. Therefore $D_t^k F(x)$ exists and is continuous at all values of x, for the indices $k = 0, \dots, r$, and the proof of the lemma is completed. □

Proof of Theorem 1 Let F be any even function such that $F^{(2r-1)}$ is absolutely continuous. While proving Lemma 3, we used the representation (13) for the derivative $D_t^k F(x)$. It follows by inspection from (13) that $D_t^k F$ is defined and continuous for $x \neq 0$ for the indices $k = 1, \ldots, r$. Lemma 3 shows immediately that $D_t^{(k)} F(x)$ is also continuous at 0 for $k = 0, \ldots, r-1$.

We will now show that (2) implies (3). Note that because of (1) we can work entirely on the interval $(0, \infty)$. Therefore, in this proof we will not distinguish notationally whether the norm is taken on the half-line or whole line; the expression $\|F(x)\|_p$ can henceforth mean either, or both of these, provided that the use is consistent within a formula, equation, or inequality.

We show first that (3) is true when $r = 1$. We have $D_t F(x) = \frac{F'(x)}{2x}$, and F' is odd and absolutely continuous, whence

$$F'(x) = \int_0^x F''(y)\, dy.$$

Now, with $V_0 = V$ and with $V_1(x) = (2x)^{-1} V(x)$ we immediately have from (2) that

$$\|V(x) D_t F(x)\|_p = \|V_1(x) F'(x)\|_p \leq C \|V_0(x) F''(x)\|_p. \tag{23}$$

Our theorem follows for the case that $r = 1$.

Turning now to the case that $r > 1$, we again employ (13). Using this representation for the derivative, we can carry out a sequence of $k - 1$ like steps which are very similar in construction to those in Lemma 3 which lead from (13) to (20). The only difference is that in the ℓth step we use the norm estimate (10) instead of using Cauchy's mean value theorem, and then, as the parity conditions on the coefficient polynomials and as the construction of the coefficient polynomials obtained after using (10) is identical to that obtained from Cauchy's theorem, we can follow up with a cancellation of a common factor of $2x$ from numerator and denominator. Specifically, using the same notations for the coefficient polynomials as in Lemma 3:

The numerator on the right in (13) is a continuous odd function and thus zero at 0. Thus, using (10) with weight functions V_{2k-1} and V_{2k-2}, we can obtain

$$\|V(x) D_t^k F(x)\|_p \leq C_{2k-1} \|V(x) \frac{\sum_{j=1}^{k+1} [a'_{j,k}(x) + a_{j-1,k}(x)] F^{(j)}(x)}{(2)(2x)^{2k-2}} \|_p, \tag{24}$$

in which $a_{0,k}$ and $a_{k+1,k}$ are both set equal to zero.

If $k = 1$ the denominator on the right is now 2, and we can go directly to (27). If $1 < k \le r$, we must continue the argument. Exactly as in Lemma 3, we can cancel from the fraction on the right in (24) a common factor of $2x$, obtaining

$$\|V(x)D_t^k F(x)\|_p \le C_{2k-1}\|V(x)\frac{\sum_{j=1}^{k+1} b_{j,k+1}(x)F^{(j)}(x)}{(2x)^{2k-3}}\|_p, \tag{25}$$

in which the coefficient polynomials $b_{j,k+1}(x)$ are given in (16). This completes the first of the $k-1$ like steps. In general, in the ℓth step of the mentioned $k-1$ like steps,we begin with the output of the previous step, apply (10) with weights $V_{2k-2\ell+1}$ and $V_{2k-2\ell}$, cancel the common factor of $2x$ from numerator and denominator, and end with

$$\|V(x)D_t^k F(x)\|_p \le C_{2k-1}C_{2k-3}\cdots C_{2k-2\ell+1}\|V(x)\frac{\sum_{j=1}^{k+\ell} b_{j,k+\ell}(x)F^{(j)}(x)}{(2x)^{2k-2\ell-1}}\|_p \tag{26}$$

in which the coefficients $b_{j,k+\ell}(x)$ are given in (19).

After $k-1$ steps, each consisting of an application of (10) followed by a cancellation of a factor of $2x$, we reach

$$\|V(x)D_t^k F(x)\|_p \le C_{2k-1}C_{2k-3}\cdots C_3\left\|V(x)\frac{\sum_{j=1}^{2k-1} b_{j,2k-1}(x)F^{(j)}(x)}{2x}\right\|_p,$$

in which $b_{1,2k-1},\ldots,b_{2k-1,2k-1}$ are polynomials of degree zero, and each of the polynomials $b_{2j,2k-1}$ is odd and therefore equal to zero.

After one more application of the argument leading to (23) we therefore reach

$$\begin{aligned}\|V(x)D_t^k F(x)\|_p \le& C_{2k-1}C_{2k-3}\cdots C_1\left\|V(x)\sum_{j=1}^{k} b_{2j,2k}F^{(2j)}(x)\right\|_p\\ \le& C_{2k-1}C_{2k-3}\cdots C_1\sum_{j=1}^{k} B_{j,2k}\|V(x)F^{(2j)}(x)\|_p,\end{aligned} \tag{27}$$

in which the coefficients $b_{2j,2k}$ are as defined in (21), and

$$B_{j,2k} = |b_{2j,2k}|.$$

Finally, with the help of (27) we may define the constants $A_{j,r}$ in (27) by

$$A_{j,r} = C_{2r-1}C_{2r-3}\cdots C_1|B_{2j,2r}|.$$

To prove the converse statement in the theorem, it suffices to notice that, in case $r = 1$, the equation (3) is simply a restatement of the requirements that must be satisfied by V_0 and V_1 in order for (2) to hold for the absolutely continuous odd function $F'(x)$ on the interval $[0, \infty)$.

The proof of the theorem is now completed. □

Proof of Theorem 2 First we note that $F = F^{(0)}, \dots, F^{(2r-1)}$ are all absolutely continuous and $\|VF^{(2r)}\| < \infty$ by hypothesis. From (4) we obtain, as the odd derivatives of F are odd functions and therefore zero at $x = 0$

$$\|V(x)F^{(2k-1)}(x)\|_{p,(0,\infty)} \le C'\|V(x)F^{(2k)}(x)\|_{p,(0,\infty)}$$

for $k = 1, \dots, r$. Because of (1), the inequality

$$\|V(x)F^{(2k-1)}(x)\|_p \le C'\|V(x)F^{(2k)}(x)\|_p$$

also follows. Also, for every even derivative $F^{(2k)}$, for $k = 0, \dots, r-1$ there is a constant c_{2k} such that $F^{(2k)}(0) = c_{2k}$. Therefore, according to (4) and (1) we have

$$\|V(x)(F^{(2k)}(x) - c_{2k})\|_p \le C'\|V(x)F^{(2k+1)}(x)\|_p. \tag{28}$$

Since $\|V\|_p < \infty$, it now follows that

$$\begin{aligned}\|V(x)F^{(2k)}(x)\|_p &\le \|V(x)(F^{(2k)}(x) - c_{2k})\|_p + \|c_{2k}V(x)\|_p \\ &\le C'\|V(x)F^{(2k+1)}(x)\|_p + |c_{2k}|\|V\|_p \end{aligned} \tag{29}$$

for $k = 0, \dots, r-1$. Now, in view of the fact that $\|VF^{(2r)}\| < \infty$, the combination of (29) and (28) demonstrates $\|VF^{(k)}\| < \infty$ for $k = 0, \dots, 2r$, and since (2) implies (3) it also follows that $\|V(x)D_t^kF(x)\| < \infty$ for $k = 0, \dots, r$. □

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