THE EXTENSION OF MONTGOMERY IDENTITY VIA FINK IDENTITY WITH APPLICATIONS

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The new extension of the weighted Montgomery identity is given by using Fink identity and is used to obtain some Ostrowski-type inequalities and estimations of the difference of two integral means.

1. Introduction

The following Ostrowski inequality is well known [10]:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{(x-(a+b)/2)^2}{(b-a)^2}\right](b-a)L, \quad x \in [a,b], \tag{1.1}$$

where $f : [a,b] \to \mathbb{R}$ is a differentiable function such that $|f'(x)| \le L$, for every $x \in [a,b]$.

The Ostrowski inequality has been generalized over the last years in a number of ways. Milovanović and Pečarić [8] and Fink [6] have considered generalizations of (1.1) in the form

$$\left| \frac{1}{n}\left(f(x) + \sum_{k=1}^{n-1} F_k(x)\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le K(n,p,x)\|f^{(n)}\|_p \tag{1.2}$$

which is obtained from the identity

$$\frac{1}{n}\left(f(x) + \sum_{k=1}^{n-1} F_k(x)\right) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{n!(b-a)}\int_a^b (x-t)^{n-1}k(t,x)f^{(n)}(t)\,dt, \tag{1.3}$$

where

$$\begin{aligned} F_k(x) &= \frac{n-k}{k!}\,\frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a}, \\ k(t,x) &= \begin{cases} t-a, & a \le t \le x \le b, \\ t-b, & a \le x < t \le b. \end{cases} \end{aligned} \tag{1.4}$$

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In fact, Milovanović and Pečarić have proved that

$$K(n,\infty,x) = \frac{(x-a)^{n+1}+(b-x)^{n+1}}{n(n+1)!(b-a)}, \tag{1.5}$$

while Fink gave the following generalizations of this result.

Theorem 1.1. *Let $f^{(n-1)}$ be absolutely continuous on $[a,b]$ and let $f^{(n)} \in L_p[a,b]$. Then inequality (1.2) holds with*

$$K(n,p,x) = \frac{\left[(x-a)^{nq+1}+(b-x)^{nq+1}\right]^{1/q}}{n!(b-a)} B\big((n-1)q+1,q+1\big)^{1/q}, \tag{1.6}$$

where $1 < p \le \infty$, $1/p+1/q = 1$, *B is the Beta function, and*

$$K(n,1,x) = \frac{(n-1)^{n-1}}{n^n n!(b-a)} \max\left[(x-a)^n,(b-x)^n\right]. \tag{1.7}$$

Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$ and $f' : [a,b] \to \mathbb{R}$ integrable on $[a,b]$. Then the Montgomery identity holds [9]:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + \int_a^b P(x,t)f'(t)dt, \tag{1.8}$$

where $P(x,t)$ is the Peano kernel defined by

$$P(x,t) = \begin{cases} \dfrac{t-a}{b-a}, & a \le t \le x, \\ \dfrac{t-b}{b-a}, & x < t \le b. \end{cases} \tag{1.9}$$

Now, we suppose $w : [a,b] \to [0,\infty\rangle$ is some probability density function, that is, an integrable function satisfying $\int_a^b w(t)dt = 1$, and $W(t) = \int_a^t w(x)dx$ for $t \in [a,b]$, $W(t) = 0$ for $t < a$, and $W(t) = 1$ for $t > b$. The following identity (given by Pečarić in [12]) is the weighted generalization of the Montgomery identity:

$$f(x) = \int_a^b w(t)f(t)dt + \int_a^b P_w(x,t)f'(t)dt, \tag{1.10}$$

where the weighted Peano kernel is

$$P_w(x,t) = \begin{cases} W(t), & a \le t \le x, \\ W(t)-1, & x < t \le b. \end{cases} \tag{1.11}$$

The aim of this paper is to give the extension of the weighted Montgomery identity (1.10) using identity (1.2), and further, obtain some new Ostrowski-type inequalities, as well as the generalizations of the estimations of the difference of two weighted integral means (generalizations of the results from [1, 3, 7, 11]).

2. The extension of Montgomery identity via Fink identity

THEOREM 2.1. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is an absolutely continuous function on $[a,b]$ for some $n\geq 1$. If $w:[a,b]\to[0,\infty\rangle$ is some probability density function, then the following identity holds:*

$$\begin{aligned} f(x) &= \int_a^b w(t)f(t)dt-\sum_{k=1}^{n-1}F_k(x)+\sum_{k=1}^{n-1}\int_a^b w(t)F_k(t)dt \\ &\quad+\frac{1}{(n-1)!(b-a)}\int_a^b (x-y)^{n-1}k(y,x)f^{(n)}(y)dy \\ &\quad-\frac{1}{(n-1)!(b-a)}\int_a^b\left(\int_a^b w(t)(t-y)^{n-1}k(y,t)dt\right)f^{(n)}(y)dy. \end{aligned} \tag{2.1}$$

Proof. We apply identity (1.3) with $f'(t)$:

$$\begin{aligned} f'(t) &= -\sum_{k=1}^{n-1}\frac{n-k}{k!}\frac{f^{(k)}(a)(t-a)^k-f^{(k)}(b)(t-b)^k}{b-a}+n\frac{f(b)-f(a)}{b-a} \\ &\quad+\frac{1}{(n-1)!(b-a)}\int_a^b (t-y)^{n-1}k(y,t)f^{(n+1)}(y)dy \\ &= -\sum_{k=0}^{n-1}\frac{n-k}{k!}\frac{f^{(k)}(a)(t-a)^k-f^{(k)}(b)(t-b)^k}{b-a} \\ &\quad+\frac{1}{(n-1)!(b-a)}\int_a^b (t-y)^{n-1}k(y,t)f^{(n+1)}(y)dy. \end{aligned} \tag{2.2}$$

Now, by putting this formula in the weighted Montgomery identity (1.10), we obtain

$$\begin{aligned} f(x) &= \int_a^b w(t)f(t)dt \\ &\quad-\sum_{k=0}^{n-1}\frac{n-k}{k!}\int_a^b P_w(x,t)\frac{f^{(k)}(a)(t-a)^k-f^{(k)}(b)(t-b)^k}{b-a}dt \\ &\quad+\frac{1}{(n-1)!(b-a)}\int_a^b P_w(x,t)\left(\int_a^b (t-y)^{n-1}k(y,t)f^{(n+1)}(y)dy\right)dt. \end{aligned} \tag{2.3}$$

Further,

$$\begin{aligned}
&\int_a^b P_w(x,t)\frac{f^{(k)}(a)(t-a)^k - f^{(k)}(b)(t-b)^k}{b-a}dt\\
&\quad = \frac{f^{(k)}(a)(x-a)^{k+1} - f^{(k)}(b)(x-b)^{k+1}}{(b-a)(k+1)}\\
&\qquad - \int_a^b w(t)\frac{f^{(k)}(a)(t-a)^{k+1} - f^{(k)}(b)(t-b)^{k+1}}{(b-a)(k+1)}dt,\\
&\int_a^b P_w(x,t)(t-y)^{n-1}k(y,t)dt\\
&\quad = \frac{1}{n}(x-y)^n k(y,x) - \frac{1}{n}\int_a^b w(t)(t-y)^n k(y,t)dt.
\end{aligned} \tag{2.4}$$

Now, if we replace n with $n-1$, we will get (2.1). This identity is valid for $n-1 \geq 1$, that is, $n > 1$. □

Remark 2.2. We could also obtain identity (2.1) by applying identity (1.3) such that we multiply this identity by $w(x)$ and than integrate it to obtain

$$\begin{aligned}
\int_a^b w(x)f(x)dx = &-\sum_{k=1}^{n-1}\int_a^b w(x)F_k(x)dx + \left(\int_a^b w(x)dx\right)\frac{n}{b-a}\int_a^b f(t)dt\\
&+\frac{1}{(n-1)!(b-a)}\int_a^b\left(\int_a^b w(x)(x-t)^{n-1}k(t,x)dx\right)f^{(n)}(t)dt.
\end{aligned} \tag{2.5}$$

If we subtract this identity from (1.3) we will obtain (2.1).

Remark 2.3. In the special case, if we take $w(t) = 1/(b-a)$, $t \in [a,b]$, we will have

$$\begin{aligned}
\frac{1}{b-a}\sum_{k=1}^{n-1}\int_a^b F_k(t)dt &= \frac{1}{b-a}\sum_{k=1}^{n-1}\frac{n-k}{k!}\int_a^b\frac{f^{(k-1)}(a)(t-a)^k - f^{(k-1)}(b)(t-b)^k}{b-a}dt\\
&= \sum_{k=1}^{n-1}\frac{n-k}{(k+1)!}\left[f^{(k-1)}(a)(b-a)^{k-1} + f^{(k-1)}(b)(a-b)^{k-1}\right],\\
\frac{1}{b-a}\int_a^b (t-y)^{n-1}k(y,t)dt &= k(y,b)\frac{(b-y)^n}{n(b-a)} - k(y,a)\frac{(a-y)^n}{n(b-a)}\\
&= \frac{(y-a)(b-y)^n}{n(b-a)} - \frac{(y-b)(a-y)^n}{n(b-a)}.
\end{aligned} \tag{2.6}$$

We denote

$$I_n = \frac{1}{n!(b-a)^2} \int_a^b [(y-a)(b-y)^n - (y-b)(a-y)^n] f^{(n)}(y)dy. \tag{2.7}$$

Then we have

$$I_n = \frac{1}{n!(b-a)^2} \int_a^b [(a-y)^n - (b-y)^n] f^{(n-1)}(y)dy + I_{n-1} = J_n + I_{n-1}, \tag{2.8}$$

where

$$I_0 = \frac{1}{(b-a)^2} \int_a^b (b-a) f(y)dy = \frac{1}{b-a} \int_a^b f(y)dy. \tag{2.9}$$

Further,

$$\begin{aligned} J_n &= \frac{1}{n!} [f^{(n-2)}(a)(b-a)^{n-2} + f^{(n-2)}(b)(a-b)^{n-2}] + J_{n-1}, \\ J_1 &= \frac{1}{(b-a)^2} \int_a^b (a-b) f(y)dy = -\frac{1}{b-a} \int_a^b f(y)dy. \end{aligned} \tag{2.10}$$

So,

$$J_n = \sum_{k=1}^{n-1} \frac{1}{(k+1)!} [f^{(k-1)}(a)(b-a)^{k-1} + f^{(k-1)}(b)(a-b)^{k-1}] + J_1, \tag{2.11}$$

and then

$$\begin{aligned} I_n &= \sum_{m=2}^{n} J_m + nJ_1 + I_0 \\ &= \sum_{k=1}^{n-1} \frac{n-k}{(k+1)!} [f^{(k-1)}(a)(b-a)^{k-1} + f^{(k-1)}(b)(a-b)^{k-1}] \\ &\quad - \frac{n-1}{b-a} \int_a^b f(y)dy. \end{aligned} \tag{2.12}$$

Consequently, identity (2.1) reduces to identity (1.3). So we may regard it as a weighted Fink identity.

Remark 2.4. Applying identity (2.1) with $x = a$ and $x = b$, we get

$$\begin{aligned} f(a) &= \int_a^b w(t)f(t)dt - \sum_{k=1}^{n-1} \frac{n-k}{k!} f^{(k-1)}(b)(a-b)^{k-1} + \sum_{k=1}^{n-1} \int_a^b w(t)F_k(t)dt \\ &\quad + \frac{1}{(n-1)!(b-a)} \int_a^b (a-y)^{n-1}(y-b)f^{(n)}(y)dy \\ &\quad - \frac{1}{(n-1)!(b-a)} \int_a^b \left(\int_a^b w(t)(t-y)^{n-1}k(y,t)dt \right) f^{(n)}(y)dy, \\ f(b) &= \int_a^b w(t)f(t)dt - \sum_{k=1}^{n-1} \frac{n-k}{k!} f^{(k-1)}(a)(b-a)^{k-1} + \sum_{k=1}^{n-1} \int_a^b w(t)F_k(t)dt \\ &\quad + \frac{1}{(n-1)!(b-a)} \int_a^b (b-y)^{n-1}(y-a)f^{(n)}(y)dy \\ &\quad - \frac{1}{(n-1)!(b-a)} \int_a^b \left(\int_a^b w(t)(t-y)^{n-1}k(y,t)dt \right) f^{(n)}(y)dy. \end{aligned} \tag{2.13}$$

So, we get the generalized trapezoid identity

$$\begin{aligned} \frac{1}{2}[f(a)+f(b)] &= \int_a^b w(t)f(t)dt + \sum_{k=1}^{n-1} \int_a^b w(t)F_k(t)dt \\ &\quad - \frac{1}{2}\sum_{k=1}^{n-1} \frac{n-k}{k!} [f^{(k-1)}(a)(b-a)^{k-1} + f^{(k-1)}(b)(a-b)^{k-1}] \\ &\quad + \frac{1}{2(n-1)!(b-a)} \int_a^b [(a-y)^{n-1}(y-b) + (b-y)^{n-1}(y-a)] f^{(n)}(y)dy \\ &\quad - \frac{1}{(n-1)!(b-a)} \int_a^b \left(\int_a^b w(t)(t-y)^{n-1}k(y,t)dt \right) f^{(n)}(y)dy. \end{aligned} \tag{2.14}$$

Similarly, applying identity (2.1) with $x = (a+b)/2$, we get

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= \int_a^b w(t)f(t)dt + \sum_{k=1}^{n-1} \int_a^b w(t)F_k(t)dt \\ &\quad - \sum_{k=1}^{n-1} \frac{n-k}{2^k k!} [f^{(k-1)}(a)(b-a)^{k-1} + f^{(k-1)}(b)(a-b)^{k-1}] \\ &\quad + \frac{1}{(n-1)!(b-a)} \int_a^b \left(\frac{a+b}{2} - y\right)^{n-1} k\left(y, \frac{a+b}{2}\right) f^{(n)}(y)dy \\ &\quad - \frac{1}{(n-1)!(b-a)} \int_a^b \left(\int_a^b w(t)(t-y)^{n-1}k(y,t)dt \right) f^{(n)}(y)dy. \end{aligned} \tag{2.15}$$

We can regard this as the second Euler-Maclaurin formula (the generalized midpoint identity).

3. Ostrowski-type inequalities

We denote, for $n \geq 2$,

$$T_{w,n}(x) = \sum_{k=1}^{n-1} F_k(x) - \sum_{k=1}^{n-1} \int_a^b w(t) F_k(t)\,dt. \tag{3.1}$$

Theorem 3.1. *Assume (p,q) is a pair of conjugate exponents, that is, $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$. Let $|f^{(n)}|^p : [a,b] \to \mathbb{R}$ be an R-integrable function for some $n > 1$. Then, for $x \in [a,b]$, the following inequality holds:*

$$\begin{aligned} &\left| f(x) - \int_a^b w(t) f(t)\,dt + T_{w,n}(x) \right| \\ &\quad \leq \frac{1}{(n-2)!(b-a)} \left(\int_a^b \left| \int_a^b P_w(x,t)(t-y)^{n-2} k(y,t)\,dt \right|^q dy \right)^{1/q} \|f^{(n)}\|_p. \end{aligned} \tag{3.2}$$

The constant $(1/(n-2)!(b-a))(\int_a^b |\int_a^b P_w(x,t)(t-y)^{n-2}k(y,t)dt|^q dy)^{1/q}$ is sharp for $1 < p \leq \infty$ and is the best possible for $p = 1$.

Proof. From Theorem 2.1 we have

$$(x-y)^{n-1}k(y,x) - \int_a^b w(t)(t-y)^{n-1}k(y,t)\,dt = (n-1)\int_a^b P_w(x,t)(t-y)^{n-2}k(y,t)\,dt. \tag{3.3}$$

We denote $C_1(y) = (1/(n-2)!(b-a)) \int_a^b P_w(x,t)(t-y)^{n-2}k(y,t)dt$. We use identity (2.1) and apply the Hölder inequality to obtain

$$\begin{aligned} &\left| f(x) - \int_a^b w(t) f(t)\,dt + T_{w,n}(x) \right| \\ &\quad = \left| \int_a^b C_1(y) f^{(n)}(y)\,dy \right| \leq \left(\int_a^b |C_1(y)|^q dy \right)^{1/q} \|f^{(n)}\|_p. \end{aligned} \tag{3.4}$$

For the proof of the sharpness of the constant $(\int_a^b |C_1(y)|^q dy)^{1/q}$, we will find a function f for which the equality in (3.2) is obtained.

For $1 < p < \infty$, take f to be such that

$$f^{(n)}(y) = \operatorname{sgn} C_1(y) \cdot |C_1(y)|^{1/(p-1)}. \tag{3.5}$$

For $p = \infty$, take

$$f^{(n)}(y) = \operatorname{sgn} C_1(y). \tag{3.6}$$

For $p = 1$, we will prove that

$$\left| \int_a^b C_1(y) f^{(n)}(y) dt \right| \leq \max_{y \in [a,b]} |C_1(y)| \left(\int_a^b |f^{(n)}(y)| dy \right) \tag{3.7}$$

is the best possible inequality. Suppose that $|C_1(y)|$ attains its maximum at $y_0 \in [a,b]$. First we assume that $C_1(y_0) > 0$. For ε small enough, define $f_\varepsilon(y)$ by

$$f_\varepsilon(y) = \begin{cases} 0, & a \leq y \leq y_0, \\ \dfrac{1}{\varepsilon n!}(y - y_0)^n, & y_0 \leq y \leq y_0 + \varepsilon, \\ \dfrac{1}{n!}(y - y_0)^{n-1}, & y_0 + \varepsilon \leq y \leq b. \end{cases} \tag{3.8}$$

Then, for ε small enough,

$$\left| \int_a^b C_1(y) f^{(n)}(y) dy \right| = \left| \int_{y_0}^{y_0+\varepsilon} C_1(y) \frac{1}{\varepsilon} dy \right| = \frac{1}{\varepsilon} \int_{y_0}^{y_0+\varepsilon} C_1(y) dy. \tag{3.9}$$

Now, from inequality (3.7) we have

$$\frac{1}{\varepsilon} \int_{y_0}^{y_0+\varepsilon} C_1(y) dy \leq C_1(y_0) \int_{y_0}^{y_0+\varepsilon} \frac{1}{\varepsilon} dy = C_1(y_0). \tag{3.10}$$

Since

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{y_0}^{y_0+\varepsilon} C_1(y) dy = C_1(y_0), \tag{3.11}$$

the statement follows. In case $C_1(y_0) < 0$, we take

$$f_\varepsilon(y) = \begin{cases} \dfrac{1}{n!}(y - y_0 - \varepsilon)^{n-1}, & a \leq y \leq y_0, \\ -\dfrac{1}{\varepsilon n!}(y - y_0 - \varepsilon)^n, & y_0 \leq y \leq y_0 + \varepsilon, \\ 0, & y_0 + \varepsilon \leq y \leq b, \end{cases} \tag{3.12}$$

and the rest of the proof is the same as above. □

Remark 3.2. For $w(t) = 1/(b-a)$, $n = 2$, and $q = 1$ in Theorem 3.1, we get

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \left(x - \frac{a+b}{2}\right)(f(b)-f(a))\right| \\
&\quad \le \frac{1}{b-a}\left(\int_a^b \left|(x-y)k(y,x) - \frac{1}{b-a}\int_a^b (t-y)k(y,t)dt\right| dy\right) \|f''\|_\infty \\
&\quad = \frac{1}{b-a}\left(\int_a^b \left|(x-y)k(y,x) - \frac{(y-a)(b-y)}{2}\right| dy\right) \|f''\|_\infty \\
&\quad = \frac{1}{2(b-a)}\left(\int_a^x |(y-a)(2x-y-b)|\,dy + \int_x^b |(b-y)(-2x+y+a)|\,dy\right) \|f''\|_\infty \\
&\quad = \left(\frac{4}{3}\delta^3(x) - \frac{1}{2}\delta^2(x) + \frac{1}{24}\right)\|f''\|_\infty,
\end{aligned} \tag{3.13}$$

where $\delta(x) = |x-(a+b)/2|$.

If instead of $q = 1$ ($p = \infty$) we put $p = 1$, then, similarly we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \left(x - \frac{a+b}{2}\right)(f(b)-f(a))\right| \\
&\quad \le \frac{1}{2(b-a)} \max\left\{ \max_{y\in[a,x]} |(y-a)(2x-y-b)|, \max_{y\in[x,b]} |(b-y)(-2x+y+a)| \right\} \|f''\|_1 \\
&\quad = \frac{1}{4}\left[\frac{1}{4} + \left|\frac{1}{4} - 2\left(x - \frac{a+b}{2}\right)^2\right|\right]\|f''\|_1.
\end{aligned} \tag{3.14}$$

These two inequalities are proved in [5].

Corollary 3.3. *Suppose that all the assumptions of Theorem 3.1 hold. Then the following inequality holds:*

$$\begin{aligned}
&\left| f(x) - \int_a^b w(t)f(t)dt + T_{w,n}(x)\right| \\
&\quad \le \frac{1}{(n-1)!(b-a)}\left(\int_a^b [(b-y)(y-a)^{n-1} + (y-a)(b-y)^{n-1}]^q dy\right)^{1/q} \|f^{(n)}\|_p.
\end{aligned} \tag{3.15}$$

Proof. Since $0 \le W(t) \le 1$, $t \in [a,b]$, so $|P_w(x,t)| \le 1$. Then, for every $y \in [a,b]$, we have

$$\begin{aligned}
&\left|\int_a^b P_w(x,t)(t-y)^{n-2}k(y,t)dt\right| \\
&\quad\le \int_a^b |P_w(x,t)|\,|(t-y)^{n-2}k(y,t)|\,dt \\
&\quad\le \int_a^b |(t-y)^{n-2}k(y,t)|\,dt \\
&\quad= \left[\int_a^y (y-t)^{n-2}(b-y)dt + \int_y^b (t-y)^{n-2}(y-a)dt\right] \\
&\quad= \frac{1}{n-1}\left[(b-y)(y-a)^{n-1} + (y-a)(b-y)^{n-1}\right].
\end{aligned} \tag{3.16}$$

So,

$$\begin{aligned}
&\left(\int_a^b \left|\int_a^b P_w(x,t)(t-y)^{n-1}k(y,t)dt\right|^q dy\right)^{1/q} \\
&\quad\le \frac{1}{n-1}\left(\int_a^b \left[(b-y)(y-a)^{n-1} + (y-a)(b-y)^{n-1}\right]^q dt\right)^{1/q}
\end{aligned} \tag{3.17}$$

and, by applying (3.2), the inequality is proved. □

Remark 3.4. Inequality (3.15) reduces to the following: for $n = 2$,

$$\begin{aligned}
&\left|f(x) - \int_a^b w(t)f(t)dt + T_{w,2}(x)\right| \\
&\quad\le \frac{2}{b-a}\left(\int_a^b (b-y)^q(y-a)^q dy\right)^{1/q} \|f''\|_p \\
&\quad= 2(b-a)^{(q+1)/q}\left(\int_0^1 (1-s)^q s^q ds\right)^{1/q} \|f''\|_p \\
&\quad= 2(b-a)^{(q+1)/q} B(q+1,q+1)^{1/q} \|f''\|_p.
\end{aligned} \tag{3.18}$$

For $n = 3$,

$$\begin{aligned}
&\left|f(x) - \int_a^b w(t)f(t)dt + T_{w,3}(x)\right| \\
&\quad\le \frac{1}{2(b-a)}\left(\int_a^b (b-y)^q(y-a)^q(b-a)^q dy\right)^{1/q} \|f''\|_p \\
&\quad= \frac{1}{2}(b-a)^{(2q+1)/q} B(q+1,q+1)^{1/q} \|f'''\|_p.
\end{aligned} \tag{3.19}$$

Remark 3.5. If we use the identities (2.14) and (2.15) for $n = 2$ and $w(t) = 1/(b-a)$, $t \in [a,b]$, and then apply the Hölder inequality with $p = \infty$, $q = 1$, we will obtain

$$\begin{aligned} \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt \right| &\le \frac{(b-a)^2}{12}\|f''\|_\infty, \\ \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| &\le \frac{(b-a)^2}{24}\|f''\|_\infty. \end{aligned} \tag{3.20}$$

By doing the same for $n = 3$, we have

$$\begin{aligned} \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt - \frac{b-a}{12}[f'(b)-f'(a)] \right| &\le \frac{(b-a)^3}{192}\|f'''\|_\infty, \\ \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt + \frac{b-a}{24}[f'(b)-f'(a)] \right| &\le \frac{(b-a)^3}{192}\|f'''\|_\infty. \end{aligned} \tag{3.21}$$

The first two inequalities were obtained in [4] and the last two in [2].

4. Estimations of the difference of two weighted integral means

In this section, we will denote, for $n > 1$,

$$T_{w,n}^{[a,b]}(x) = \sum_{k=1}^{n-1} F_k^{[a,b]}(x) - \sum_{k=1}^{n-1}\int_a^b w(t)F_k^{[a,b]}(t)dt, \tag{4.1}$$

for a function $f : [a,b] \to \mathbb{R}$ such that $f^{(n-1)}$ is an absolutely continuous function on $[a,b]$.

The following results are generalizations of the results from [3] in two cases. The first case is when $[c,d] \subseteq [a,b]$ and the second is when $[a,b] \cap [c,d] = [c,b]$. Other two possible cases, when $[a,b] \cap [c,d] \ne \emptyset$ ($[a,b] \subset [c,d]$ and $[a,b] \cap [c,d] = [a,d]$) are simply got by change $a \leftrightarrow c$, $b \leftrightarrow d$.

Theorem 4.1. *Let $f : [a,b] \cup [c,d] \to \mathbb{R}$ be such that $f^{(n-1)}$ is an absolutely continuous function on $[a,b]$ for some $n > 1$, and let $w : [a,b] \to [0,\infty\rangle$ and $u : [c,d] \to [0,\infty\rangle$ be some probability density functions. Then, if $[a,b] \cap [c,d] \ne \emptyset$ and $x \in [a,b] \cap [c,d]$,*

$$\int_a^b w(t)f(t)dt - \int_c^d u(t)f(t)dt - T_{w,n}^{[a,b]}(x) + T_{u,n}^{[c,d]}(x) = \int_{\min\{a,c\}}^{\max\{b,d\}} K_n(x,y)f^{(n)}(y)dy, \tag{4.2}$$

where, in case $[c,d] \subseteq [a,b]$,

$$K_n(x,y) = \begin{cases} \dfrac{-1}{(n-2)!(b-a)}\left[\displaystyle\int_a^b P_w(x,t)(t-y)^{n-2}k^{[a,b]}(y,t)dt\right], & y \in [a,c], \\ \dfrac{-1}{(n-2)!(b-a)}\left[\displaystyle\int_a^b P_w(x,t)(t-y)^{n-2}k^{[a,b]}(y,t)dt\right] & \\ +\dfrac{1}{(n-2)!(d-c)}\left[\displaystyle\int_c^d P_u(x,t)(t-y)^{n-2}k^{[c,d]}(y,t)dt\right], & y \in \langle c,d], \\ \dfrac{-1}{(n-2)!(b-a)}\left[\displaystyle\int_a^b P_w(x,t)(t-y)^{n-2}k^{[a,b]}(y,t)dt\right], & y \in \langle d,b], \end{cases} \tag{4.3}$$

and in case $[a,b] \cap [c,d] = [c,b]$,

$$K_n(x,y) = \begin{cases} \dfrac{-1}{(n-2)!(b-a)}\left[\displaystyle\int_a^b P_w(x,t)(t-y)^{n-2}k^{[a,b]}(y,t)dt\right], & y \in [a,c], \\ \dfrac{-1}{(n-2)!(b-a)}\left[\displaystyle\int_a^b P_w(t)(t-y)^{n-2}k^{[a,b]}(y,t)dt\right] & \\ +\dfrac{1}{(n-2)!(d-c)}\left[\displaystyle\int_c^d P_u(t)(t-y)^{n-2}k^{[c,d]}(y,t)dt\right], & y \in \langle c,b], \\ \dfrac{1}{(n-2)!(d-c)}\left[\displaystyle\int_c^d P_u(t)(t-y)^{n-2}k^{[c,d]}(y,t)dt\right], & y \in \langle b,d]. \end{cases} \tag{4.4}$$

Proof. We subtract identity (2.1) for intervals $[a,b]$ and $[c,d]$ to get formula (4.2). □

Theorem 4.2. *Assume* (p,q) *is a pair of conjugate exponents, that is,* $1 \le p,\ q \le \infty$, $1/p + 1/q = 1$. *Let* $|f^{(n)}|^p : [a,b] \to \mathbb{R}$ *be an R-integrable function for some* $n > 1$. *Then*

$$\begin{aligned} &\left| \int_a^b w(t)f(t)dt - \int_c^d u(t)f(t)dt - T_{w,n}^{[a,b]}(x) + T_{u,n}^{[c,d]}(x) \right| \\ &\qquad \le \left(\int_{\min\{a,c\}}^{\max\{b,d\}} |K_n(x,y)|^q dy \right)^{1/q} \|f^{(n)}\|_p \end{aligned} \tag{4.5}$$

for every $x \in [a,b] \cap [c,d]$. *The constant* $(\int_{\min\{a,c\}}^{\max\{b,d\}} |K_n(x,y)|^q dy)^{1/q}$ *in inequality (4.5) is sharp for* $1 < p \le \infty$ *and is the best possible for* $p = 1$.

Proof. Use identity (4.2) and apply the Hölder inequality to obtain

$$\begin{aligned} &\left| \int_a^b w(t)f(t)dt - \int_c^d u(t)f(t)dt - T_{w,n}^{[a,b]}(x) + T_{u,n}^{[c,d]}(x) \right| \\ &\qquad \le \int_{\min\{a,c\}}^{\max\{b,d\}} |K_n(x,y)|\,|f^{(n)}(y)|\,dy \le \left(\int_{\min\{a,c\}}^{\max\{b,d\}} |K_n(x,y)|^q dy \right)^{1/q} \|f^{(n)}\|_p, \end{aligned} \tag{4.6}$$

which proves inequality (4.5). The proofs for sharpness and best possibility are as in Theorem 3.1. □

Corollary 4.3. *Suppose that all the assumptions of Theorem 4.2 hold. Then, for* $x \in [a,b] \cap [c,d]$,

$$\left| \int_a^b w(t)f(t)dt - \int_c^d u(t)f(t)dt - T_{w,n}^{[a,b]}(x) + T_{u,n}^{[c,d]}(x) \right| \le \frac{2}{(n-1)!}\left(\int_a^{\max\{b,d\}} \left| (y-a)^{n-1} + \left(\max\{b,d\} - y\right)^{n-1} \right|^q dy \right)^{1/q} \|f^{(n)}\|_p. \tag{4.7}$$

Proof. We have

$$K_n(x,y) = \frac{-1}{(n-2)!} \int_{\min\{a,c\}}^{\max\{b,d\}} \left[P_w(x,t)\frac{k^{[a,b]}(y,t)}{b-a} - P_u(x,t)\frac{k^{[c,d]}(y,t)}{d-c} \right](t-y)^{n-2}dt \tag{4.8}$$

because $P_w(x,t) = 0$, for $x \notin [a,b]$ and $P_u(x,t) = 0$, for $x \notin [c,d]$. Since

$$-1 \le P_w(x,t), P_u(x,t), \frac{k^{[a,b]}(y,t)}{b-a}, \frac{k^{[c,d]}(y,t)}{d-c} \le 1, \tag{4.9}$$

we get

$$\left| P_w(x,t)\frac{k^{[a,b]}(y,t)}{b-a} - P_u(x,t)\frac{k^{[c,d]}(y,t)}{d-c} \right| \le 2, \tag{4.10}$$

and then we have

$$\left| K_n(x,y) \right| \le \frac{2}{(n-2)!} \int_a^{\max\{b,d\}} |t-y|^{n-2}dt = \frac{2\left((y-a)^{n-1} + \left(\max\{b,d\} - y\right)^{n-1} \right)}{(n-1)!}. \tag{4.11}$$

□

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