BERNSTEIN'S INEQUALITY FOR MULTIVARIATE POLYNOMIALS ON THE STANDARD SIMPLEX

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Received 31 July 2003

The classical Bernstein pointwise estimate of the (first) derivative of a univariate algebraic polynomial on an interval has natural extensions to the multivariate setting. However, in several variables the domain of boundedness, even if convex, has a considerable geometric variety. In 1990, Y. Sarantopoulos satisfactorily settled the case of a centrally symmetric convex body by a method we may call "the method of inscribed ellipses." On the other hand, for the general case of nonsymmetric convex bodies we are only within a constant factor of an exact inequality. The best known results suggest relevance of the generalized Minkowski functional, and a natural conjecture for the exact Bernstein factor was formulated with this geometric quantity. This work deals with the most natural and simple nonsymmetric case, that of a standard simplex in $\mathbb{R}^d$, and computes the exact yield of the method of inscribed ellipses. Although the known general estimates of the Bernstein factor are improved for the simplex here, we find that not even the exact yield of the inscribed ellipse method reaches the conjecture. However, we also show that for an arbitrary convex body the subset of ridge polynomials satisfies the conjecture.

1. Introduction

If a univariate algebraic polynomial p is given with degree at most n, then by the classical Bernstein-Szegő inequality (see [1, 10, 11]), we have

$$|p'(x)| \le \frac{n\sqrt{\|p\|^2_{C[a,b]} - p^2(x)}}{\sqrt{(b-x)(x-a)}} \quad (a < x < b). \tag{1.1}$$

This inequality is sharp for every n and every point $x \in (a,b)$, as

$$\sup\left\{ \frac{|p'(x)|}{\sqrt{\|p\|^2_{C[a,b]} - p^2(x)}} : \deg p \le n,\ |p(x)| < \|p\|_{C[a,b]} \right\} = \frac{n}{\sqrt{(b-x)(x-a)}}. \tag{1.2}$$

We may say that the upper estimate (1.1) is exact, and the right-hand side is just the "true Bernstein factor" of the problem.

Journal of Inequalities and Applications 2005:2 (2005) 145–163
DOI: 10.1155/JIA.2005.145

In the multivariate setting a number of extensions were proved for this classical result. However, due to the geometric variety of possible convex sets replacing intervals of $\mathbb{R}$, our present knowledge is still not final. The exact Bernstein inequality is known only for symmetric convex bodies, and we are within a bound of some constant factor in the general, nonsymmetric case.

For more precise notation we may define formally for any topological vector space X, a subset $K \subset X$, and a point $\mathbf{x} \in K$ the nth "Bernstein factor" as

$$B_n(K,\mathbf{x}) := \frac{1}{n} \sup \left\{ \frac{\|Dp(\mathbf{x})\|}{\sqrt{\|p\|^2_{C(K)} - p^2(\mathbf{x})}} : \deg p \le n,\ |p(\mathbf{x})| < \|p\|_{C(K)} \right\}, \tag{1.3}$$

where $Dp(\mathbf{x})$ is the derivative of p at $\mathbf{x}$, and even for an arbitrary unit vector $\mathbf{y} \in X$

$$B_n(K,\mathbf{x},\mathbf{y}) := \frac{1}{n} \sup \left\{ \frac{\langle Dp(\mathbf{x}),\mathbf{y}\rangle}{\sqrt{\|p\|^2_{C(K)} - p^2(\mathbf{x})}} : \deg p \le n,\ |p(\mathbf{x})| < \|p\|_{C(K)} \right\}. \tag{1.4}$$

In the present paper, first we study the standard simplex of the d-dimensional Euclidean space $\mathbb{R}^d$. We find the exact yield of the possibly nicest available method—the method of inscribed ellipses, introduced into the subject by Sarantopoulos [9]—for arbitrary interior points of this nonsymmetric convex body. It will be seen that for this particular case this calculation improves upon the previously known general estimate. On the other hand, the perhaps most intriguing conjecture in the topic, which relates the "true Bernstein factor" to the "generalized Minkowski functional," remains still open even for the standard simplex.

On the other hand, all known lower estimates use ridge polynomials some way. So it is of interest to test, whether ridge polynomials can be used to disprove the conjecture. It turns out that ridge polynomials always satisfy the above-mentioned conjecture, even for general convex bodies in arbitrary normed spaces.

2. A review of Sarantopoulos' method of inscribed ellipses

Recall that a *convex body* in a topological vector space X (e.g., in $\mathbb{R}^d$) is a bounded, closed convex set that has a nonempty interior. Polynomials and continuous polynomials are defined over topological vector spaces, see, for example, [2]. The set of continuous polynomials over X will be denoted by $\mathcal{P} = \mathcal{P}(X)$ and polynomials in $\mathcal{P}$ with degree not exceeding n by $\mathcal{P}_n = \mathcal{P}_n(X)$. In this section, we review the inscribed ellipse method. Although for the reader's convenience we include short proofs, we emphasize that, unless otherwise stated, results in this section are due to Sarantopoulos [9].

Lemma 2.1 (inscribed ellipse lemma). *Let K be any subset in a vector space X. Suppose that $\mathbf{x} \in K$ and the ellipse*

$$\mathbf{r}(t) = \cos t\mathbf{a} + b\sin t\mathbf{y} + \mathbf{x} - \mathbf{a} \quad (t \in [-\pi,\pi)) \tag{2.1}$$

lies inside K. Then for any polynomial p_n of degree at most n the Bernstein-type inequality

$$|\langle Dp_n(\mathbf{x}),\mathbf{y}\rangle| \le \frac{n}{b}\sqrt{||p_n||^2_{C(K)} - p_n^2(\mathbf{x})} \tag{2.2}$$

holds true.

Proof. Consider the trigonometric polynomial $T(t) := p_n(\mathbf{r}(t))$ of degree at most n. Since $\mathbf{r}(t) \subset K$ we clearly have $\|T\| \le \|p_n\|_{C(K)}$. According to the Bernstein-Szegő inequality [10] (see also [11]) for trigonometric polynomials,

$$|T'(t)| \le n\sqrt{\|T\|^2 - T(t)^2} \le n\sqrt{||p_n||^2_{C(K)} - p_n(\mathbf{r}(t))^2} \quad (\forall t \in \mathbb{R}). \tag{2.3}$$

In particular, for $t = 0$, we get

$$|T'(0)| \le n\sqrt{||p_n||^2_{C(K)} - p_n^2(\mathbf{x})}. \tag{2.4}$$

By the chain rule

$$T'(0) = \langle Dp_n(\mathbf{r}(0)),\mathbf{r}'(0)\rangle = \langle Dp_n(\mathbf{x}),b\mathbf{y}\rangle, \tag{2.5}$$

which completes the proof. □

The Minkowski functional [5] of a convex body K is defined by

$$\varphi(\mathbf{x}) := \varphi(K,\mathbf{x}) := \inf\{\lambda > 0 : \mathbf{x} \in \lambda K\}. \tag{2.6}$$

Clearly, $\|\mathbf{x}\|_K := \varphi(\mathbf{x})$ is a norm on X if and only if K is centrally symmetric with respect to the origin. If K is not centrally symmetric, the same functional can be used; however, there is another extension, the "generalized Minkowski functional" $\alpha(K,x)$, which also goes back to Minkowski [5] and Radon [6], see also [3, 7]. For our present purposes we define $\alpha(K,\mathbf{x})$ for $\mathbf{x} \in \operatorname{int} K$ as follows. First, let

$$\gamma(K,\mathbf{x}) := \inf\left\{2\frac{\sqrt{\|\mathbf{x}-\mathbf{a}\|\,\|\mathbf{x}-\mathbf{b}\|}}{\|\mathbf{a}-\mathbf{b}\|} : \mathbf{a},\mathbf{b} \in \partial K, \text{ such that } \mathbf{x} \in [\mathbf{a},\mathbf{b}]\right\}. \tag{2.7}$$

Then we set

$$\alpha(K,\mathbf{x}) := \sqrt{1-\gamma^2(K,\mathbf{x})}. \tag{2.8}$$

For many other equivalent formulations, geometric properties, and applications in approximation theory, see [7] and the references therein.

Lemma 2.2. *Let K be a centrally symmetric convex body in a vector space X and let* $\mathbf{x} \in K$. *The ellipse* $\mathbf{r}(t) = \cos t\mathbf{x} + b\sin t\mathbf{y}$ $(t \in [-\pi,\pi))$ *lies in K whenever*

$$\|\mathbf{y}\|_K = 1, \qquad b = \sqrt{1-\|\mathbf{x}\|_K^2}. \tag{2.9}$$

Proof. The assertion is equivalent to $\|\mathbf{r}(t)\|_K \le 1$ for every t. By the triangle and Cauchy inequalities,

$$||\mathbf{r}(t)||_K \le |\cos t|\,\|\mathbf{x}\|_K + b|\sin t|\,\|\mathbf{y}\|_K \le \sqrt{\cos^2 t + \sin^2 t}\,\sqrt{\|\mathbf{x}\|_K^2 + b^2\|\mathbf{y}\|_K^2} = 1. \tag{2.10}$$

Lemma 2.2 is proved. □

The maximal chord of K in direction $\mathbf{v} \neq \mathbf{0}$ is

$$\tau(K,\mathbf{v}) := \sup\{\lambda > 0 : \exists \mathbf{y},\ \mathbf{z} \in K \text{ such that } \mathbf{z} = \mathbf{y} + \lambda\mathbf{v}\}, \tag{2.11}$$

see, for example, [12]. Note that in normed spaces the *width of* K is $w(K) := \inf\{\tau(K,\mathbf{v}) : \|\mathbf{v}\| = 1\}$, see, for example, [7]. Mutatis mutandis to the previous lemma we can deduce also the following variant.

LEMMA 2.3. *Let K be a centrally symmetric body in X, where $(X,\|\cdot\|)$ is a normed space. Let $\varphi_K = \|\cdot\|_K$ be the Minkowski functional (norm) generated by K. Then for every nonzero vector $\mathbf{y} \in X$ the ellipse $\mathbf{r}(t) = \cos t\mathbf{x} + b\sin t\mathbf{y}$ $(t \in [-\pi,\pi))$ lies in K with*

$$b := \frac{\sqrt{1-\varphi^2(K,\mathbf{x})}}{\varphi(K,\mathbf{y})}. \tag{2.12}$$

THEOREM 2.4. *Let p_n be any polynomial of degree at most n over the normed space X. Then for any unit vector $\mathbf{y} \in X$ the following Bernstein-type inequality holds:*

$$|\langle Dp_n(\mathbf{x}),\mathbf{y}\rangle| \le \frac{n\sqrt{||p_n||_{C(K)}^2 - p_n^2(\mathbf{x})}}{\sqrt{1-\|\mathbf{x}\|_K^2}}. \tag{2.13}$$

Proof. The proof follows from combining Lemmas 2.1 and 2.2. □

THEOREM 2.5. *Let K be a symmetric convex body and $\mathbf{y}$ a unit vector in the normed space X. Let p_n be any polynomial of degree at most n. Then,*

$$|\langle Dp_n(\mathbf{x}),\mathbf{y}\rangle| \le \frac{2n\sqrt{||p_n||_{C(K)}^2 - p_n^2(\mathbf{x})}}{\tau(K,\mathbf{y})\sqrt{1-\varphi^2(K,\mathbf{x})}}. \tag{2.14}$$

In particular,

$$||Dp_n(\mathbf{x})|| \le \frac{2n\sqrt{||p_n||_{C(K)}^2 - p_n^2(\mathbf{x})}}{w(K)\sqrt{1-\varphi^2(K,\mathbf{x})}}, \tag{2.15}$$

where $w(K)$ stands for the width of K.

Proof. Here we need to combine Lemmas 2.1 and 2.3 to obtain Theorem 2.5. □

It can be rather difficult to determine, or even to estimate the b-parameter of the "best ellipse," what can be inscribed into a convex body K through $\mathbf{x} \in K$ and tangential to direction of $\mathbf{y}$.

Definition 2.6. For arbitrary $K \subset X$ and $\mathbf{x} \in K$, $\mathbf{y} \in X$ the corresponding "best ellipse constants" are the extremal quantities

$$E(K,\mathbf{x},\mathbf{y}) := \sup\{b : \mathbf{r} \subset K \text{ with } \mathbf{r} \text{ as given in (2.1)}\}, \tag{2.16}$$

$$E(K,\mathbf{x}) := \inf\{E(K,\mathbf{x},\mathbf{y}) : \mathbf{y} \in X, \|\mathbf{y}\| = 1\}. \tag{2.17}$$

Clearly, the inscribed ellipse method yields Bernstein-type estimates whenever we can derive some estimate of the ellipse constants. In case of symmetric convex bodies, Sarantopoulos's Theorems 2.4 and 2.5 are sharp; for the nonsymmetric case we know only the following result.

Theorem 2.7 (Kroó-Révész [4]). *Let K be an arbitrary convex body, $\mathbf{x} \in \operatorname{int} K$ and $\|\mathbf{y}\| = 1$, where X can be an arbitrary normed space. Then,*

$$|\langle Dp_n(\mathbf{x}),\mathbf{y}\rangle| \leq \frac{2n\sqrt{\|p_n\|^2_{C(K)} - p_n^2(\mathbf{x})}}{\tau(K,\mathbf{y})\sqrt{1-\alpha(K,\mathbf{x})}}, \tag{2.18}$$

for any polynomial p_n of degree at most n. Moreover,

$$\|Dp_n(\mathbf{x})\| \leq \frac{2n\sqrt{\|p_n\|^2_{C(K)} - p_n^2(\mathbf{x})}}{w(K)\sqrt{1-\alpha(K,\mathbf{x})}} \leq \frac{2\sqrt{2}n\sqrt{\|p_n\|^2_{C(K)} - p_n^2(\mathbf{x})}}{w(K)\sqrt{1-\alpha^2(K,\mathbf{x})}}. \tag{2.19}$$

Note that in [4] the best ellipse is not found; the construction there gives only a good estimate, but not an exact value of (2.16) or (2.17). In fact, here we quoted [4] in a strengthened form: the original paper contains a somewhat weaker formulation only.

As mentioned above, one of the most intriguing questions of the topic is the following conjecture, formulated first in [7].

Conjecture 2.8 (Révész and Sarantopoulos). *Let X be a topological vector space, and let K be a convex body in X. For every point $\mathbf{x} \in \operatorname{int} K$ and every (bounded) polynomial p of degree at most n over X, we have*

$$\|Dp(\mathbf{x})\| \leq \frac{2n\sqrt{\|p\|^2_{C(K)} - p^2(\mathbf{x})}}{w(K)\sqrt{1-\alpha^2(K,\mathbf{x})}}, \tag{2.20}$$

where $w(K)$ stands for the width of K.

3. The inscribed ellipse method for the standard simplex

We denote by $\mathcal{P}_n^d$ the space of polynomials of d variables and total degree $\leq n$. Let $\mathbb{S}^{d-1} := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}|_2 = 1\}$ be the unit sphere in $\mathbb{R}^d$, where $|\mathbf{x}|_2 := (\sum_{i=1}^d x_i^2)^{1/2}$ is the Euclidean norm of $\mathbf{x} = (x_1,\ldots,x_d) \in \mathbb{R}^d$. The derivative of $p_n \in \mathcal{P}_n^d$ in direction $\mathbf{y} \in \mathbb{S}^{d-1}$ will be denoted by $D_{\mathbf{y}} p_n$.

Let

$$\Delta := \Delta_d := \left\{(x_1,\ldots,x_d) : x_i \geq 0,\ i = 1,\ldots,d,\ \sum_{i=1}^d x_i \leq 1\right\} \tag{3.1}$$

be the standard simplex in $\mathbb{R}^d$. For fixed $\mathbf{x} = (x_1,\dots,x_d) \in \operatorname{int}\Delta$, and $\mathbf{y} = (y_1,\dots,y_d)$, $|\mathbf{y}|_2 = 1$, we consider the set of ellipses (2.1), where

$$\mathbf{a} = (a_1,\dots,a_d) \in \mathbb{R}^d, \quad b \in \mathbb{R}. \tag{3.2}$$

Clearly the best ellipse constant of Δ is the extremal quantity

$$E(\Delta,\mathbf{x},\mathbf{y}) = \max\{b : (a_1,\dots,a_d,b) \in \Omega\}, \tag{3.3}$$

where

$$\Omega := \{(a_1,\dots,a_d,b) : \mathbf{r}(t) \in \Delta \text{ for every } t \in [-\pi,\pi)\} \subset \mathbb{R}^{d+1}. \tag{3.4}$$

LEMMA 3.1. *The explicit formula*

$$E(\Delta,\mathbf{x},\mathbf{y}) = \left\{\frac{y_1^2}{x_1} + \cdots + \frac{y_d^2}{x_d} + \frac{(y_1+\cdots+y_d)^2}{1-x_1-\cdots-x_d}\right\}^{-1/2} \tag{3.5}$$

holds true.

Proof. In coordinate form

$$\begin{aligned}\Omega := \Big\{(a_1,\dots,a_d,b) : a_i\cos t + b y_i \sin t + x_i - a_i \geq 0 \ (i = 1,\dots,d),\\ \cos t\sum_{i=1}^d a_i + b\sin t\sum_{i=1}^d y_i + \sum_{i=1}^d x_i - \sum_{i=1}^d a_i \leq 1 \ \forall t \in [-\pi,\pi)\Big\}.\end{aligned} \tag{3.6}$$

We need the following simple lemma.

LEMMA 3.2. *The sets*

$$\begin{aligned}\mathcal{M} &= \{(a,b,c) : a\cos t + b\sin t \leq c \ \forall t \in [-\pi,\pi)\},\\ \mathcal{N} &= \{(a,b,c) : c \geq 0,\ a^2+b^2 \leq c^2\}\end{aligned} \tag{3.7}$$

coincide.

Proof. If $(a,b,c) \in \mathcal{M}$, then c has to be nonnegative. The case $a = b = 0$ is trivial, so we can assume that at least one of a and b is nonzero. Let ξ be defined to satisfy

$$\sin\xi = \frac{a}{\sqrt{a^2+b^2}}, \qquad \cos\xi = \frac{b}{\sqrt{a^2+b^2}}. \tag{3.8}$$

Since for all t $c \geq a\cos t + b\sin t = \sin(\xi+t)\sqrt{a^2+b^2}$, we obtain that $a^2+b^2 \leq c^2$.

Conversely, let $(a,b,c) \in \mathcal{N}$. The Cauchy-Schwartz inequality gives $(a\cos t + b\sin t)^2 \le a^2 + b^2 \le c^2$, that is, $(a,b,c) \in \mathcal{M}$, which concludes the proof. □

Using Lemma 3.2 and setting $Z := b^2$, we arrive at the problem

$$\max\{Z : \mathbf{z} := (a_1,\dots,a_d,Z) \in \Theta\}, \tag{3.9}$$

where

$$\begin{aligned}\Theta := \Bigg\{\mathbf{z} : Z \ge 0,\ Zy_i^2 \le x_i^2 - 2x_ia_i\ (i=1,\dots,d),\ x_i - a_i \ge 0\ (i=1,\dots,d),\\ Z\Big(\sum_{i=1}^d y_i\Big)^2 \le \Big(1-\sum_{i=1}^d x_i\Big)^2 + 2\Big(1-\sum_{i=1}^d x_i\Big)\sum_{i=1}^d a_i,\ 1-\sum_{i=1}^d x_i + \sum_{i=1}^d a_i \ge 0\Bigg\}.\end{aligned} \tag{3.10}$$

Note that $\Theta \subset \mathbb{R}^{d+1}$ is bounded, hence compact. Indeed, $\mathbf{x} \in \Delta$ and the inequalities $a_i \le x_i$, $i = 1,\dots,d$, $\sum_{i=1}^d a_i \ge \sum_{i=1}^d x_i - 1$ imply that $|a_i| \le 1$, $i = 1,\dots,d$. At least one of y_i, $i = 1,\dots,d$ is different from zero. If $y_j \ne 0$, then

$$Z \le \frac{x_j^2 - 2x_ja_j}{y_j^2} \le \frac{3}{y_j^2}. \tag{3.11}$$

It follows that the optimal value of the problem (3.9) is finite.

Next, we quote the Kuhn-Tucker Theorem, as given in [8, Corollary 28.3.1].

Lemma 3.3 (Kuhn-Tucker theorem). *Let the functions $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 0,\dots,m$, be differentiable, where f_i is convex for $i = 0,\dots,r$ and f_i is affine for $i = r+1,\dots,m$. Consider the extremal problem*

(P)

$$\min f_0(\mathbf{z}) \tag{3.12}$$

subject to constrains $f_1(\mathbf{z}) \le 0,\dots,f_r(\mathbf{z}) \le 0$, $f_{r+1}(\mathbf{z}) = 0,\dots,f_m(\mathbf{z}) = 0$. Suppose that the optimal value in (P) is not $-\infty$ and (P) has at least one feasible solution, which satisfies all the inequality constrains for $i = 1,\dots,r$ with strict inequality. Then $\mathbf{z}$ is a solution of (P) if and only if there exists numbers $\boldsymbol{\lambda} := (\lambda_1,\dots,\lambda_m)$ which together with $\mathbf{z}$ satisfy the following:

(a) $\lambda_i \ge 0$, $f_i(\mathbf{z}) \le 0$ *and* $\lambda_i f_i(\mathbf{z}) = 0$, $i = 1,\dots,r$,

(b) $f_i(\mathbf{z}) = 0$ *for* $i = r+1,\dots,m$,

(c) $\operatorname{grad} f_0(\mathbf{z}) + \sum_{i=1}^m \lambda_i \operatorname{grad} f_i(\mathbf{z}) = \mathbf{0}$.

Clearly, if the system (a), (b), and (c) has a unique solution $(\mathbf{z},\boldsymbol{\lambda})$, then $\mathbf{z}$ is the unique solution of the problem (P).

Now we turn to the problem (3.9). In the notations of Lemma 3.3 we have $n = d+1$, $m = r = 2d+3$ (and thus, in particular, (b) becomes void), $\mathbf{z} = (\mathbf{a}, Z)$,

$$\begin{aligned}
f_0(\mathbf{a},Z) &= -Z, \\
f_1(\mathbf{a},Z) &= -Z, \\
f_{i+1}(\mathbf{a},Z) &= 2x_i a_i + y_i^2 Z - x_i^2 \quad (i = 1,\dots,d), \\
f_{d+2}(\mathbf{a},Z) &= -2\left(1-\sum_{i=1}^d x_i\right)\sum_{i=1}^d a_i + Z\left(\sum_{i=1}^d y_i\right)^2 - \left(1-\sum_{i=1}^d x_i\right)^2, \qquad (3.13)\\
f_{d+2+j}(\mathbf{a},Z) &= a_j - x_j \quad (j = 1,\dots,d), \\
f_{2d+3}(\mathbf{a},Z) &= \sum_{i=1}^d x_i - 1 - \sum_{i=1}^d a_i.
\end{aligned}$$

Differentiation with respect to $\mathbf{z}$ gives

$$\begin{aligned}
\operatorname{grad} f_0 = \operatorname{grad} f_1 &= (0,\dots,0,-1), \\
\operatorname{grad} f_2 &= (2x_1, 0,\dots,0, y_1^2), \\
&\vdots \\
\operatorname{grad} f_{d+1} &= (0,\dots,0,2x_d, y_d^2), \\
\operatorname{grad} f_{d+2} &= \left(-2\left(1-\sum_{i=1}^d x_i\right),\dots,-2\left(1-\sum_{i=1}^d x_i\right),\left(\sum_{i=1}^d y_i\right)^2\right), \qquad (3.14)\\
\operatorname{grad} f_{d+3} &= (1,0,\dots,0,0), \\
&\vdots \\
\operatorname{grad} f_{2d+2} &= (0,\dots,0,1,0), \\
\operatorname{grad} f_{2d+3} &= (-1,-1,\dots,-1,0).
\end{aligned}$$

Thus in problem (3.9) the system (a), (b), and (c) in $(\mathbf{z},\boldsymbol{\lambda}) = (a_1,\dots,a_d,Z,\lambda_1,\dots,\lambda_{2d+3})$ becomes

$$\lambda_i \geq 0, \quad i = 1,\dots,2d+3, \qquad (3.15)$$

$$Z \geq 0, \qquad (3.16)$$

$$2x_i a_i + y_i^2 Z - x_i^2 \leq 0, \quad i = 1,\dots,d, \qquad (3.17)$$

$$-2\left(1-\sum_{i=1}^d x_i\right)\sum_{i=1}^d a_i + Z\left(\sum_{i=1}^d y_i\right)^2 - \left(1-\sum_{i=1}^d x_i\right)^2 \leq 0, \qquad (3.18)$$

$$a_i - x_i \leq 0, \quad i = 1,\dots,d, \qquad (3.19)$$

$$\sum_{i=1}^d x_i - \sum_{i=1}^d a_i - 1 \leq 0, \qquad (3.20)$$

$$\lambda_1 Z = 0, \qquad (3.21)$$

$$\lambda_{i+1}(2x_ia_i + y_i^2 Z - x_i^2) = 0, \quad i = 1,\dots,d, \tag{3.22}$$

$$\lambda_{d+2}\left[-2\left(1-\sum_{i=1}^d x_i\right)\sum_{i=1}^d a_i + Z\left(\sum_{i=1}^d y_i\right)^2 - \left(1-\sum_{i=1}^d x_i\right)^2\right] = 0, \tag{3.23}$$

$$\lambda_{d+2+i}(a_i - x_i) = 0, \quad i = 1,\dots,d, \tag{3.24}$$

$$\lambda_{2d+3}\left(\sum_{i=1}^d x_i - \sum_{i=1}^d a_i - 1\right) = 0, \tag{3.25}$$

$$2\lambda_{i+1}x_i - 2\lambda_{d+2}\left(1-\sum_{i=1}^d x_i\right) + \lambda_{d+2+i} - \lambda_{2d+3} = 0, \quad i = 1,\dots,d, \tag{3.26}$$

$$-1 - \lambda_1 + \sum_{i=1}^d \lambda_{i+1}y_i^2 + \lambda_{d+2}\left(\sum_{i=1}^d y_i\right)^2 = 0. \tag{3.27}$$

If $x_j = a_j$ for some $j \in \{1,\dots,d\}$, then from (3.17) we get $x_j^2 + y_j^2 Z \le 0$, hence $x_j = 0$, a contradiction in view of $\mathbf{x} \in \operatorname{int}\Delta$.

Similarly, if $\sum_{i=1}^d a_i = \sum_{i=1}^d x_i - 1$ it follows from (3.18) that $1 - \sum_{i=1}^d x_i = 0$.

Therefore, from (3.24) and (3.25), $\lambda_{d+2+i} = 0$ for $i = 1,\dots,d+1$.

Now from these we will show that $\lambda_2 > 0,\dots,\lambda_{d+2} > 0$. Indeed, if λ_{d+2} were 0, then from (3.26) (due to $x_i > 0$, $i = 1,\dots,d$) it follows that $\lambda_{i+1} = 0$, $i = 1,\dots,d$ and then from (3.27) $\lambda_1 = -1$, which contradicts (3.15). So, $\lambda_{d+2} > 0$. Hence (3.26) leads to

$$\lambda_{i+1} = \frac{1-\sum_{i=1}^d x_i}{x_i}\lambda_{d+2} > 0, \quad \text{for } i = 1,\dots,d, \tag{3.28}$$

as stated. Then from (3.22)

$$a_i = \frac{x_i^2 - y_i^2 Z}{2x_i}, \quad \text{for } i = 1,\dots,d. \tag{3.29}$$

We substitute these expressions in (3.23) and derive

$$Z = \frac{1}{\sum_{i=1}^d (y_i^2/x_i) + \left(\sum_{i=1}^d y_i\right)^2/(1-\sum_{i=1}^d x_i)}. \tag{3.30}$$

Note that $Z > 0$ since $\mathbf{x} \in \operatorname{int}\Delta$ and $|\mathbf{y}| = 1$. Hence, from (3.21), $\lambda_1 = 0$. Substituting (3.28) in (3.27) yields

$$\lambda_{d+2} = \frac{1}{\left(\sum_{i=1}^d y_i\right)^2 + (1-\sum_{i=1}^d x_i)\sum_{i=1}^d (y_i^2/x_i)}. \tag{3.31}$$

This immediately implies the corresponding explicit formulas for λ_{i+1}, $i = 1,\dots,d$.

So, if the system (3.15)–(3.27) has a solution, it is unique. We have to check whether the so determined $a_1,\dots,a_d$, Z, $\lambda_1,\dots,\lambda_{2d+3}$ really solve (3.15)–(3.27).

From the considerations until now it is clear that (3.15)–(3.18) and (3.21)–(3.27) are satisfied. It remains to consider (3.19) and (3.20).

By (3.29), inequalities (3.19) are equivalent to

$$x_i - a_i = \frac{x_i^2 + y_i^2 Z}{2x_i} \geq 0, \quad i = 1,\dots,d, \tag{3.32}$$

which hold true, because $Z > 0$.

To prove that (3.20) holds, we substitute the values (3.29) for a_i, $i = 1,\dots,d$, which gives

$$Z\sum_{i=1}^{d}\frac{y_i^2}{x_i} \leq 2 - \sum_{i=1}^{d} x_i. \tag{3.33}$$

Substituting formula (3.30) for Z, the left-hand side is seen to remain below 1, while the right-hand side exceeds 1, provided $\mathbf{x} \in \operatorname{int}\Delta$. Thus also (3.20) holds true.

So, the optimal solution of the problem (3.9) is Z as given in (3.30). Clearly, $E(\Delta,\mathbf{x},\mathbf{y}) = \sqrt{Z}$, and Lemma 3.1 follows. □

Theorem 3.4. *Let $p_n \in \mathcal{P}_n^d$. Then for every $\mathbf{x} \in \operatorname{int}\Delta$ and $\mathbf{y} \in \mathbb{S}^{d-1}$,*

$$\left| D_{\mathbf{y}} p_n(\mathbf{x}) \right| \leq \frac{n\sqrt{\|p_n\|_{C(\Delta)}^2 - p_n^2(\mathbf{x})}}{E(\Delta,\mathbf{x},\mathbf{y})}, \tag{3.34}$$

where $E(\Delta,\mathbf{x},\mathbf{y})$ is as given in (3.5).

Proof. According to Lemma 3.1, the ellipse (2.1) with the parameter $b = E(\Delta,\mathbf{x},\mathbf{y})$ from (3.5) belongs to Δ. Hence the method of inscribed ellipses (cf. Lemma 2.1) gives (3.34). Theorem 3.4 is proved. □

4. Comparisons and improvements for the standard triangle

In this section, we restrict ourselves to the case $d = 2$. First, we see that estimate (3.34) is better than (2.18) when $K = \Delta$.

We denote the vertices of Δ by $O = (0,0)$, $A = (1,0)$, $B = (0,1)$, and the centroid (i.e., mass point) of Δ by $M = (1/3,1/3)$.

A calculation shows that $1 - \alpha(\Delta,\mathbf{x}) = 2r(\mathbf{x})$, with

$$r := r(\mathbf{x}) := \min\{x_1, x_2, 1 - x_1 - x_2\} = \begin{cases} x_1, & \mathbf{x} \in \triangle OMB, \\ x_2, & \mathbf{x} \in \triangle OMA, \\ 1 - x_1 - x_2, & \mathbf{x} \in \triangle AMB, \end{cases} \tag{4.1}$$

and if $\mathbf{y} = (\cos\varphi, \sin\varphi)$ $(0 \le \varphi \le \pi)$, then

$$\tau(\Delta,\mathbf{y}) = \begin{cases} \dfrac{1}{y_1+y_2}, & \varphi \in \left[0,\dfrac{\pi}{2}\right], \\ \dfrac{1}{y_2}, & \varphi \in \left(\dfrac{\pi}{2},\dfrac{3\pi}{4}\right], \\ -\dfrac{1}{y_1}, & \varphi \in \left(\dfrac{3\pi}{4},\pi\right]. \end{cases} \tag{4.2}$$

Theorem 4.1. *The inequality*

$$\frac{1}{E(\Delta,\mathbf{x},\mathbf{y})} \le \frac{2}{\tau(\Delta,\mathbf{y})\sqrt{1-\alpha(\Delta,\mathbf{x})}} \tag{4.3}$$

holds true for every $\mathbf{x} \in \operatorname{int}\Delta$ *and* $\mathbf{y} \in \mathbb{S}^1$. *The equality occurs if and only if*

(a) $\mathbf{x} \in (OM)$, $\mathbf{y} = (-\sqrt{2}/2, \sqrt{2}/2)$;
(b) $\mathbf{x} \in (AM)$, $\mathbf{y} = (0,1)$;
(c) $\mathbf{x} \in (BM)$, $\mathbf{y} = (1,0)$;
(d) $\mathbf{x} = M$, $\mathbf{y} \in \{(0,1),(1,0),(-\sqrt{2}/2,\sqrt{2}/2)\}$.

Proof. Combining (3.5) and (4.1), inequality (4.3) can be written in the equivalent form:

$$\sum_{i=1}^{3} p_i(\mathbf{x})q_i(\mathbf{y}) \le \frac{2}{\tau^2(\Delta,\mathbf{y})}, \tag{4.4}$$

where

$$\begin{gathered} \big(p_1(\mathbf{x}),p_2(\mathbf{x}),p_3(\mathbf{x})\big) := r(\mathbf{x})\left(\frac{1}{x_1},\frac{1}{x_2},\frac{1}{1-x_1-x_2}\right), \\ \big(q_1(\mathbf{y}),q_2(\mathbf{y}),q_3(\mathbf{y})\big) := \left(y_1^2,y_2^2,(y_1+y_2)^2\right). \end{gathered} \tag{4.5}$$

Note that $p_i(\mathbf{x}) \le 1$ for all $\mathbf{x} \in \operatorname{int}\Delta$ and $i = 1,2,3$. Hence,

$$\sum_{i=1}^{3} p_i(\mathbf{x})q_i(\mathbf{y}) \le \sum_{i=1}^{3} q_i(\mathbf{y}). \tag{4.6}$$

Now the validity of (4.4) will follow from

$$\sum_{i=1}^{3} q_i(\mathbf{y}) \le \frac{2}{\tau^2(\Delta,\mathbf{y})}. \tag{4.7}$$

Depending on φ, (4.7) is equivalent to $y_1y_2 \ge 0$ if $\varphi \in [0,\pi/2]$ or $y_1(y_1+y_2) \le 0$ if $\varphi \in (\pi/2,3\pi/4]$ or $y_2(y_1+y_2) \le 0$ if $\varphi \in (3\pi/4,\pi]$. It is easily seen that these inequalities hold true. Moreover, (4.7) is an equality if and only if $\mathbf{y} \in \{(0,1),(1,0),(-\sqrt{2}/2,\sqrt{2}/2)\}$.

It remains only to characterize the cases when (4.4) is an equality. This means that both (4.6) and (4.7) are equalities.

If $\mathbf{x}$ is an interior point for some of the triangles OMA, OMB, and AMB, then $p_i(\mathbf{x}) < 1$ for all i, hence (4.6) is a strict inequality.

If $\mathbf{x} \in (OM)$, then $p_1(\mathbf{x}) = p_2(\mathbf{x}) = 1$ and $p_3(\mathbf{x}) < 1$. So, (4.6) can be equality only if $q_3(\mathbf{y}) = 0$, that is, $\mathbf{y} = (-\sqrt{2}/2, \sqrt{2}/2)$. In the latter case also (4.7) is an equality. Similarly, one can prove statements (b) and (c). Finally, if $\mathbf{x} = M$, then $p_1 = p_2 = p_3 = 1$ and (4.4) is an equality if and only if $\mathbf{y} \in \{(0,1),(1,0),(-\sqrt{2}/2,\sqrt{2}/2)\}$. Theorem 4.1 is proved. □

In the next theorem we give a new estimation for $Dp_n(\mathbf{x})$.

THEOREM 4.2. *Let $p_n \in \mathcal{P}_n^2$. Then for every $\mathbf{x} \in \operatorname{int}\Delta$,*

$$|Dp_n(\mathbf{x})|_2 \le nE(\mathbf{x})\sqrt{\|p_n\|_{C(\Delta)}^2 - p_n^2(\mathbf{x})}, \tag{4.8}$$

where

$$E(\mathbf{x}) = \sqrt{\frac{x_1(1-x_1)+x_2(1-x_2)+D(\mathbf{x})}{2x_1x_2(1-x_1-x_2)}} \tag{4.9}$$

with

$$D(\mathbf{x}) = \sqrt{\left[x_1(1-x_1)+x_2(1-x_2)\right]^2 - 4x_1x_2(1-x_1-x_2)}. \tag{4.10}$$

Remark 4.3. The inequality

$$\left[x_1(1-x_1)+x_2(1-x_2)\right]^2 - 4x_1x_2(1-x_1-x_2) > \left[x_1(1-x_1)-x_2(1-x_2)\right]^2 \tag{4.11}$$

holds true for $\mathbf{x} \in \operatorname{int}\Delta$, hence $D(\mathbf{x}) > 0$.

Proof of Theorem 4.2. Clearly, the inscribed ellipse method from Section 2 implies

$$|Dp_n(\mathbf{x})|_2 \le nE(\mathbf{x})\sqrt{\|p_n\|_{C(\Delta)}^2 - p_n^2(\mathbf{x})}, \tag{4.12}$$

where

$$E(\mathbf{x}) := \frac{1}{E(\Delta,\mathbf{x})}. \tag{4.13}$$

It remains to prove the explicit representation (4.9) for $E(\mathbf{x})$. From (3.5),

$$\frac{1}{E^2(\Delta,\mathbf{x},\mathbf{y})} = \frac{y_1^2}{x_1} + \frac{y_2^2}{x_2} + \frac{(y_1+y_2)^2}{1-x_1-x_2} = \frac{\mathbf{y}^T P\mathbf{y}}{x_1x_2(1-x_1-x_2)}, \tag{4.14}$$

where

$$P = \begin{bmatrix} x_2(1-x_2) & x_1x_2 \\ x_1x_2 & x_1(1-x_1) \end{bmatrix}. \tag{4.15}$$

Since P is a positive definite symmetric matrix, it follows that

$$\max\{\mathbf{y}^T P\mathbf{y} : |\mathbf{y}| = 1\} = \lambda_1, \tag{4.16}$$

where λ_1 is the largest eigenvalue of P.

Now, a computation gives

$$\lambda_1 = \frac{1}{2}[x_1(1-x_1) + x_2(1-x_2) + D(\mathbf{x})], \tag{4.17}$$

which implies (4.9). Theorem 4.2 is proved. □

In the next theorem we improve Theorem 2.7 for $K = \Delta$.

THEOREM 4.4. *Let* $p_n \in \mathcal{P}_n^2$ *and* $\|p_n\|_{C(\Delta)} = 1$. *Then for every* $\mathbf{x} \in \operatorname{int}\Delta$,

$$|Dp_n(\mathbf{x})|_2 \le \frac{\sqrt{3}n\sqrt{\|p_n\|_{C(\Delta)}^2 - p_n^2(\mathbf{x})}}{w(\Delta)\sqrt{1-\alpha(\Delta,\mathbf{x})}}. \tag{4.18}$$

Proof. According to Theorem 4.2, it suffices to prove

$$E(\mathbf{x}) \le \frac{\sqrt{3}}{w(\Delta)\sqrt{1-\alpha(\Delta,\mathbf{x})}}, \quad \mathbf{x} \in \operatorname{int}\Delta. \tag{4.19}$$

Since $w(\Delta) = \sqrt{2}/2$, this is equivalent to

$$E^2(\mathbf{x})(1-\alpha(\Delta,\mathbf{x})) \le 6, \quad \text{for } \mathbf{x} \in \operatorname{int}\Delta. \tag{4.20}$$

We set besides (4.1) also

$$u = x_1(1-x_1), \qquad v = x_2(1-x_2), \qquad w = \frac{x_1x_2(1-x_1-x_2)}{r}. \tag{4.21}$$

Then (4.20) can be rewritten as

$$u + v + \sqrt{(u+v)^2 - 4rw} \le 6w. \tag{4.22}$$

Since $6w - (u+v) \ge 0$, we arrive at the inequality

$$w[9w - 3(u+v) + r] \ge 0. \tag{4.23}$$

To prove (4.23), we define the numbers p and q by

$$\{p,q,w\} := \{x_1(1-x_1-x_2), x_2(1-x_1-x_2), x_1x_2\}. \tag{4.24}$$

Note that $w \ge p$ and $w \ge q$. It is seen that $u + v \le 2w + p + q$ and $rw = pq/r \ge 3pq$. Hence, $w[9w - 3(u+v) + r] \ge 3(w-p)(w-q) \ge 0$. Theorem 4.4 is proved. □

Remark 4.5. The constant 6 in inequality (4.20) cannot be replaced by smaller one, since $E^2(M)(1-\alpha(\Delta,M)) = 6$.

In the next theorem we present the estimation corresponding to $1 - \alpha^2(\Delta,\mathbf{x})$.

THEOREM 4.6. *Let* $p_n \in \mathcal{P}_n^2$ *and* $\|p_n\|_{C(\Delta)} = 1$. *Then for every* $\mathbf{x} \in \operatorname{int}\Delta$,

$$|Dp_n(\mathbf{x})|_2 \le \frac{\sqrt{3+\sqrt{5}}n\sqrt{\|p_n\|_{C(\Delta)}^2 - p_n^2(\mathbf{x})}}{w(\Delta)\sqrt{1-\alpha^2(\Delta,\mathbf{x})}}. \tag{4.25}$$

Remark 4.7. This improves the constant in Theorem 2.7 but falls short of Conjecture 2.8, since $2\sqrt{2} = 2.8284\cdots > \sqrt{3+\sqrt{5}} = 2.2882\cdots > 2$.

Proof. Arguing as in Theorem 4.4, it suffices to prove that

$$E^2(\mathbf{x})\left(1-\alpha^2(\Delta,\mathbf{x})\right) \le 2(3+\sqrt{5}), \quad \text{for } \mathbf{x} \in \operatorname{int}\Delta. \tag{4.26}$$

Note first that

$$1-\alpha^2(\Delta,\mathbf{x}) = \begin{cases} 4x_1(1-x_1), & \mathbf{x} \in \triangle OMB, \\ 4x_2(1-x_2), & \mathbf{x} \in \triangle OMA, \\ 4(x_1+x_2)(1-x_1-x_2), & \mathbf{x} \in \triangle AMB. \end{cases} \tag{4.27}$$

Let u, v, w, and r be as in Theorem 4.4. Set $c := 3+\sqrt{5}$. Then (4.26) is equivalent to

$$\left(u+v+\sqrt{(u+v)^2-4rw}\right)(1-r) \le cw \Longleftrightarrow f(\mathbf{x}) := c^2w - 2c(u+v)(1-r) + 4r(1-r)^2 \ge 0. \tag{4.28}$$

For $\beta \in [0,1/3]$ we consider the line segments

$$\begin{aligned} [O_\beta, B_\beta] &= \{(\beta,t) : \beta \le t \le 1-2\beta\}, \\ [O_\beta, A_\beta] &= \{(t,\beta) : \beta \le t \le 1-2\beta\}, \\ [A_\beta, B_\beta] &= \{(t, 1-\beta-t) : \beta \le t \le 1-2\beta\}. \end{aligned} \tag{4.29}$$

We will prove that for every $\beta \in [0,1/3]$, $f(\mathbf{x}) \ge 0$ on these segments.

(1) Let $\mathbf{x} = (\beta,t) \in [O_\beta, B_\beta]$. We have

$$u = \beta(1-\beta), \qquad v = t(1-t), \qquad r = \beta, \qquad w = t(1-\beta-t). \tag{4.30}$$

Hence

$$f(\mathbf{x}) = g(t) := \left[2c(1-\beta) - c^2\right]t^2 + c(c-2)(1-\beta)t + \beta(1-\beta)^2(4-2c). \tag{4.31}$$

The leading coefficient of g is negative, so it is sufficient to show that $g(\beta) \ge 0$ and $g(1-2\beta) \ge 0$. We have $g(\beta) = \beta[c-2(1-\beta)][(1-\beta)(c-2) - c\beta]$. Since

$$\frac{c-2}{c} \ge \frac{\beta}{1-\beta} \quad \text{for every } \beta \in \left[0, \frac{1}{3}\right], \tag{4.32}$$

it follows that $g(\beta) \ge 0$. Next we compute

$$g(1-2\beta) = (4-10c)\beta^3 - 2(c^2-8c+4)\beta^2 + (c^2-6c+4)\beta. \tag{4.33}$$

Using $c^2 = 6c-4$ we get

$$g(1-2\beta) = 2\beta^2\left[(2-5c)\beta + 2c\right] \ge 0 \quad \text{for every } \beta \in \left[0, \frac{1}{3}\right]. \tag{4.34}$$

(2) Let $\mathbf{x} = (t,\beta) \in [O_\beta, A_\beta]$. Then

$$u = t(1-t), \qquad v = \beta(1-\beta), \qquad r = \beta, \qquad w = t(1-t-\beta) \tag{4.35}$$

and $f(\mathbf{x}) = g(t)$ is the same as in case (1).

(3) Let $\mathbf{x} = (t, 1-\beta-t) \in [A_\beta, B_\beta]$. Then

$$\begin{gathered} u = t(1-t), \quad v = (t+\beta)(1-t-\beta), \quad r = \beta, \quad w = t(1-\beta-t), \\ f(\mathbf{x}) = h(t) := [4c(1-\beta) - c^2]t^2 + c(1-\beta)[c - 4(1-\beta)]t + 2\beta(1-\beta)^2(2-c). \end{gathered} \tag{4.36}$$

Since the leading coefficient is negative and, by continuity from cases (1) and (2), $h(\beta) = f(B_\beta) \geq 0$ and $h(1-2\beta) = f(A_\beta) \geq 0$, we conclude that $f(\mathbf{x}) \geq 0$ on $[A_\beta, B_\beta]$. Theorem 4.6 is proved. □

Remarks 4.8. (1) The constant in (4.26) cannot be replaced by a smaller one. To prove this, we consider

$$(BM) = \left\{(t, 1-2t) : 0 < t < \frac{1}{3}\right\}. \tag{4.37}$$

If $\mathbf{x} \in (BM)$, then

$$\begin{aligned} E^2(\mathbf{x}) &= \frac{3 - 5t + \sqrt{25t^2 - 22t + 5}}{2t(1-2t)}, \\ 1 - \alpha^2(\Delta, \mathbf{x}) &= 4t(1-t). \end{aligned} \tag{4.38}$$

Hence

$$\begin{aligned} E^2(\mathbf{x})(1 - \alpha^2(\Delta,\mathbf{x})) &= \frac{2(1-t)(3-5t+\sqrt{25t^2-22t+5})}{1-2t}, \\ \lim_{\mathbf{x}\to B, \mathbf{x}\in(BM)} E^2(\mathbf{x})(1-\alpha^2(\Delta,\mathbf{x})) &= 2(3+\sqrt{5}). \end{aligned} \tag{4.39}$$

(2) $\lim_{\mathbf{x}\to B} E^2(\mathbf{x})(1-\alpha^2(\Delta,\mathbf{x}))$ does not exist. This can be seen calculating limits on the intervals (B, E_k), where $E_k = (1/k, 0) \in (OA)$ $(k > 1)$. We skip the details.

(3) Observe that Theorems 4.2, 4.4, and 4.6 show that not even the exact yield of the inscribed ellipse method can reach Conjecture 2.8.

5. Lower estimations by ridge functions

We consider the following question. All known lower estimates for the Bernstein factors used some kind of ridge polynomials, that is, polynomials composed from a linear form and some (in fact, a Chebyshev) polynomial. Can one sharpen these lower estimates to the extent that Conjecture 2.8 will be disproved?

Recall that in a normed space X the support function of $K \subset X$ is $h(K, \mathbf{v}^*) := \sup_K \mathbf{v}^*$ $(\mathbf{v}^* \in X^*)$, and the width of K in direction $\mathbf{v}^* \in S^* := \{\mathbf{v}^* \in X^* : \|\mathbf{v}^*\| = 1\}$ is

$w(K,\mathbf{v}^*) := h(K,\mathbf{v}^*) + h(K,-\mathbf{v}^*)$. Ridge polynomials $\mathcal{R}$ are defined as

$$\mathcal{R}_n := \{p \in \mathcal{P} : p(\mathbf{x}) = P(L(\mathbf{x})),\ L \in X^*,\ P \in \mathcal{P}_n^1\}, \quad \mathcal{R} := \bigcup_{n=1}^{\infty} \mathcal{R}_n. \tag{5.1}$$

We denote for any $\mathbf{v}^* \in S^*$ the linear expression

$$t(\mathbf{x}) := t(K,\mathbf{v}^*,\mathbf{x}) := \frac{2\langle \mathbf{v}^*,\mathbf{x}\rangle - h(K,\mathbf{v}^*) + h(K,-\mathbf{v}^*)}{w(K,\mathbf{v}^*)}. \tag{5.2}$$

In the following we assume, as we may, that ridge polynomials are expressed by using some $L(x) = t(K,\mathbf{v}^*,\mathbf{x})$.

Definition 5.1. For any $n \in \mathbb{N}$ the corresponding "ridge Bernstein constant" is

$$C_n(K,\mathbf{x},\mathbf{y}) := \frac{1}{n} \sup_{R\in\mathcal{R}_n,\, |R(\mathbf{x})|<\|R\|_{C(K)}} \frac{|\langle DR(\mathbf{x}),\mathbf{y}\rangle|}{\sqrt{\|R\|_{C(K)}^2 - R^2(\mathbf{x})}}. \tag{5.3}$$

Theorem 5.2. *For every convex body K and $\mathbf{x} \in \operatorname{int} K$, $\mathbf{y} \in S^*$, it holds that*

$$C_n(K,\mathbf{x},\mathbf{y}) \le \frac{2}{\tau(K,\mathbf{y})} \frac{1}{\sqrt{1-\alpha^2(K,\mathbf{x})}}. \tag{5.4}$$

Proof. By the chain rule we have for any $R \in \mathcal{R}_n$, $R = P(t(\mathbf{x}))$, the formula

$$|\langle DR(\mathbf{x}),\mathbf{y}\rangle| = \left| P'(t(\mathbf{x})) \frac{2}{w(K,\mathbf{v}^*)} \langle \mathbf{v}^*,\mathbf{y}\rangle \right| \le \frac{2}{\tau(K,\mathbf{y})} |P'(t(\mathbf{x}))|. \tag{5.5}$$

Applying the Bernstein-Szegő inequality for $s \in (-1,1)$, we get

$$\frac{|P'(s)|}{\sqrt{\|P\|_{C[-1,1]}^2 - P^2(s)}} \le \frac{n}{\sqrt{1-s^2}}. \tag{5.6}$$

Note that for T_n, the classical Chebyshev polynomial of degree n, (and only for that) this last inequality is sharp. Putting $s := t(\mathbf{x})$ and combining the previous two inequalities, we are led to

$$\frac{1}{n} \frac{|\langle DR(\mathbf{x}),\mathbf{y}\rangle|}{\sqrt{\|R\|_{C(K)}^2 - R(\mathbf{x})}} \le \frac{2}{\tau(K,\mathbf{y})} \frac{1}{\sqrt{1-t^2(K,\mathbf{v}^*,\mathbf{x})}}. \tag{5.7}$$

Taking supremum with respect to $\mathbf{v}^* \in S^*$ on the right-hand side, we obtain a bound independent of $\mathbf{v}^*$. In fact, according to [7, Proposition 4.1], the supremum is a maximum and is equal to $(2/\tau(K,\mathbf{y}))(1/\sqrt{1-\alpha^2(K,\mathbf{x})})$. Thus taking supremum also on the left-hand side, Theorem 5.2 follows. □

It follows from the definitions and Lemma 2.1 that

$$C_n(K,\mathbf{x},\mathbf{y}) \le B_n(K,\mathbf{x},\mathbf{y}) \le \frac{1}{E(K,\mathbf{x},\mathbf{y})}. \tag{5.8}$$

For the case of the standard simplex we will prove a converse inequality.

Theorem 5.3. *For every* $\mathbf{x} \in \operatorname{int}\Delta$ *and* $\mathbf{y} \in \mathbb{S}^{d-1}$, *the following inequality holds:*

$$\frac{1}{E(\Delta,\mathbf{x},\mathbf{y})} \le \sqrt{d}C_n(\Delta,\mathbf{x},\mathbf{y}). \tag{5.9}$$

Proof. Let

$$\mathbf{v}_i^* := (0,\dots,0,1,0,\dots,0), \quad i = 1,\dots,d, \tag{5.10}$$

be the ith unit vector in $\mathbb{R}^d$ and

$$\mathbf{v}_{d+1}^* := \frac{1}{\sqrt{d}}(1,1,\dots,1). \tag{5.11}$$

We denote

$$T_n(K,\mathbf{v}^*,\mathbf{x}) := T_n(t(\mathbf{x})) \tag{5.12}$$

and define

$$C_n^{(i)}(\Delta,\mathbf{x},\mathbf{y}) := \frac{1}{n}\frac{|D_{\mathbf{y}}T_n(K,\mathbf{v}_i^*,\mathbf{x})|}{\sqrt{1-T_n^2(K,\mathbf{v}_i^*,\mathbf{x})}}, \quad i = 1,\dots,d+1. \tag{5.13}$$

A computation gives

$$\begin{aligned} C_n^{(i)}(\Delta,\mathbf{x},\mathbf{y}) &= \frac{|y_i|}{\sqrt{x_i(1-x_i)}}, \quad i = 1,\dots,d, \\ C_n^{(d+1)}(\Delta,\mathbf{x},\mathbf{y}) &= \frac{|\sum_{i=1}^d y_i|}{\sqrt{\sum_{i=1}^d x_i(1-\sum_{i=1}^d x_i)}}. \end{aligned} \tag{5.14}$$

Hence, from (3.5),

$$\begin{aligned} \frac{1}{E(\Delta,\mathbf{x},\mathbf{y})} &= \sqrt{\sum_{i=1}^d (1-x_i)\left[C_n^{(i)}(\Delta,\mathbf{x},\mathbf{y})\right]^2 + \sum_{i=1}^d x_i\left[C_n^{(d+1)}(\Delta,\mathbf{x},\mathbf{y})\right]^2} \\ &\le \sqrt{d}\max_{i=1,\dots,d+1} C_n^{(i)}(\Delta,\mathbf{x},\mathbf{y}) \le \sqrt{d}C_n(\Delta,\mathbf{x},\mathbf{y}). \end{aligned} \tag{5.15}$$

Theorem 5.3 is proved. □

Corollary 5.4. *For every* $\mathbf{x} \in \operatorname{int}\Delta$ *and* $\mathbf{y} \in \mathbb{S}^{d-1}$, *it holds that*

$$1 \le \frac{B_n(\Delta,\mathbf{x},\mathbf{y})}{C_n(\Delta,\mathbf{x},\mathbf{y})} \le \sqrt{d}, \qquad 1 \le \frac{B_n(\Delta,\mathbf{x})}{C_n(\Delta,\mathbf{x})} \le \sqrt{d}. \tag{5.16}$$

Note that in [4] it is proved that for every $\mathbf{x}_0 \in \operatorname{int}K$ there is a direction $\mathbf{y}_0$ and a ridge polynomial $T_n(K,\mathbf{v}_0^*,\mathbf{x})$ such that

$$\frac{1}{n}\frac{D_{\mathbf{y}_0}T_n(K,\mathbf{v}_0^*,\mathbf{x})}{\sqrt{1-T_n(K,\mathbf{v}_0^*,\mathbf{x})^2}} = \frac{2}{\tau(K,\mathbf{y}_0)\sqrt{1-\alpha^2(K,\mathbf{x}_0)}}. \tag{5.17}$$

Consequently,

$$C_n(K, \mathbf{x}_0, \mathbf{y}_0) \geq \frac{2}{\tau(K, \mathbf{y}_0)\sqrt{1 - \alpha^2(K, \mathbf{x}_0)}}. \tag{5.18}$$

Hence, for every $\mathbf{x}_0 \in \operatorname{int} K$ there is a $\mathbf{y}_0$ such that

$$\frac{B_n(K, \mathbf{x}_0, \mathbf{y}_0)}{C_n(K, \mathbf{x}_0, \mathbf{y}_0)} \leq \sqrt{2}. \tag{5.19}$$

Comparing this with Corollary 5.4, we see that (for the case of the simplex) the latter ratio remains uniformly bounded for all $\mathbf{x}$ and $\mathbf{y}$.

Acknowledgments

This joint work was done during the postdoctoral fellowship of the first author at the Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, supported by the "Mathematics in Information Society" project in the framework of the European Community's "Confirming the International Role of Community Research" program. The first author was supported in part by the Bulgarian Ministry of Science under Project MM802/98. The work of the second author was supported by the French-Hungarian Scientific and Technological Governmental Cooperation, Project no. F-10/04 and by the Hungarian-Spanish Scientific and Technological Governmental Cooperation, Project no. E-38/04.

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