RIEMANN-STIELTJES OPERATORS FROM $F(p,q,s)$ SPACES TO α-BLOCH SPACES ON THE UNIT BALL

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Let $H(B)$ denote the space of all holomorphic functions on the unit ball $B \subset \mathbb{C}^n$. We investigate the following integral operators: $T_g(f)(z) = \int_0^1 f(tz)\Re g(tz)(dt/t)$, $L_g(f)(z) = \int_0^1 \Re f(tz)g(tz)(dt/t)$, $f \in H(B)$, $z \in B$, where $g \in H(B)$, and $\Re h(z) = \sum_{j=1}^n z_j(\partial h/\partial z_j)(z)$ is the radial derivative of h. The operator T_g can be considered as an extension of the Cesàro operator on the unit disk. The boundedness of two classes of Riemann-Stieltjes operators from general function space $F(p,q,s)$, which includes Hardy space, Bergman space, Q_p space, BMOA space, and Bloch space, to α-Bloch space $\mathcal{B}^\alpha$ in the unit ball is discussed in this paper.

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1. Introduction

Let $z = (z_1,\dots,z_n)$ and $w = (w_1,\dots,w_n)$ be points in the complex vector space $\mathbb{C}^n$ and

$$\langle z,w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n. \tag{1.1}$$

Let dv stand for the normalized Lebesgue measure on $\mathbb{C}^n$. For a holomorphic function f we denote

$$\nabla f = \left(\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_n}\right). \tag{1.2}$$

Let $H(B)$ denote the class of all holomorphic functions on the unit ball. Let $\Re f(z) = \sum_{j=1}^n z_j(\partial f/\partial z_j)(z)$ stand for the radial derivative of $f \in H(B)$ [21]. It is easy to see that, if $f \in H(B)$, $f(z) = \sum_\alpha a_\alpha z^\alpha$, where α is a multiindex, then

$$\Re f(z) = \sum_\alpha |\alpha| a_\alpha z^\alpha. \tag{1.3}$$

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The α-Bloch space $\mathcal{B}^\alpha(B)=\mathcal{B}^\alpha$, $\alpha>0$, is the space of all $f\in H(B)$ such that

$$b_\alpha(f)=\sup_{z\in B}\left(1-|z|^2\right)^\alpha\left|\Re f(z)\right|<\infty. \tag{1.4}$$

On $\mathcal{B}^\alpha$ the norm is introduced by

$$\|f\|_{\mathcal{B}^\alpha}=\left|f(0)\right|+b_\alpha(f). \tag{1.5}$$

With this norm $\mathcal{B}^\alpha$ is a Banach space. If $\alpha=1$, we denote $\mathcal{B}^\alpha$ simply by $\mathcal{B}$.

For $a,z\in B$, $a\neq 0$, let φ_a denote the Möbius transformation of B taking 0 to a defined by

$$\varphi_a(z)=\frac{a-P_a(z)-\sqrt{1-|z|^2}Q_a(z)}{1-\langle z,a\rangle}, \tag{1.6}$$

where $P_a(z)$ is the projection of z onto the one dimensional subspace of $\mathbb{C}^n$ spanned by a and $Q_a(z)=z-P_a(z)$ which satisfies (see [21])

$$\varphi_a\circ\varphi_a=\mathrm{id},\qquad \varphi_a(0)=a,\qquad \varphi_a(a)=0,\qquad 1-\left|\varphi_a(z)\right|^2=\frac{\left(1-|a|^2\right)\left(1-|z|^2\right)}{\left|1-\langle z,a\rangle\right|^2}. \tag{1.7}$$

Let $0<p,s<\infty$, $-n-1<q<\infty$. A function $f\in H(B)$ is said to belong to $F(p,q,s)=F(p,q,s)(B)$ (see [19, 20]) if

$$\|f\|^p_{F(p,q,s)}=\left|f(0)\right|^p+\sup_{a\in B}\int_B\left|\nabla f(z)\right|^p\left(1-|z|^2\right)^q g^s(z,a)\,dv(z)<\infty, \tag{1.8}$$

where $g(z,a)=\log|\varphi_a(z)|^{-1}$ is Green's function for B with logarithmic singularity at a.

We call $F(p,q,s)$ general function space because we can get many function spaces, such as BMOA space, Q_p space (see [9]), Bergman space, Hardy space, Bloch space, if we take special parameters of p,q,s in the unit disk setting, see [20]. If $q+s\leq -1$, then $F(p,q,s)$ is the space of constant functions.

For an analytic function $f(z)$ on the unit disk D with Taylor expansion $f(z)=\sum_{n=0}^\infty a_n z^n$, the Cesàro operator acting on f is

$$\mathcal{C}f(z)=\sum_{n=0}^\infty\left(\frac{1}{n+1}\sum_{k=0}^n a_k\right)z^n. \tag{1.9}$$

The integral form of $\mathcal{C}$ is

$$\mathcal{C}(f)(z)=\frac{1}{z}\int_0^z f(\zeta)\frac{1}{1-\zeta}d\zeta=\frac{1}{z}\int_0^z f(\zeta)\left(\ln\frac{1}{1-\zeta}\right)'d\zeta, \tag{1.10}$$

taking simply as a path of the segment joining 0 and z, we have

$$\mathscr{C}(f)(z)=\int_0^1 f(tz)\left(\ln\frac{1}{1-\zeta}\right)'\bigg|_{\zeta=tz}dt. \tag{1.11}$$

The following operator:

$$z\mathscr{C}(f)(z)=\int_0^z\frac{f(\zeta)}{1-\zeta}d\zeta, \tag{1.12}$$

is closely related to the previous operator and on many spaces the boundedness of these two operators is equivalent. It is well known that Cesàro operator acts as a bounded linear operator on various analytic function spaces (see [4, 8, 11–13, 16] and the references therein).

Suppose that $g\in H(D)$, the operator

$$J_gf(z)=\int_0^z f(\xi)dg(\xi)=\int_0^1 f(tz)zg'(tz)dt=\int_0^z f(\xi)g'(\xi)d\xi,\quad z\in D, \tag{1.13}$$

where $f\in H(D)$, was introduced in [10] where Pommerenke showed that J_g is a bounded operator on the Hardy space $H^2(D)$ if and only if $g\in \text{BMOA}$. The operator J_g acting on various function spaces have been studied recently in [1–3, 14, 17, 18].

Another operator was recently defined in [18], as follows:

$$I_gf(z)=\int_0^z f'(\xi)g(\xi)d\xi. \tag{1.14}$$

The above operators J_g, I_g can be naturally extended to the unit ball. Suppose that $g:B\to\mathbb{C}^1$ is a holomorphic map of the unit ball, for a holomorphic function f, define

$$T_gf(z)=\int_0^1 f(tz)\frac{dg(tz)}{dt}=\int_0^1 f(tz)\Re g(tz)\frac{dt}{t},\quad z\in B. \tag{1.15}$$

This operator is called Riemann-Stieltjes operator (or extended-Cesàro operator). It was introduced in [5], and studied in [5–7, 15, 17].

Here, we extend operator I_g for the case of holomorphic functions on the unit ball as follows:

$$L_gf(z)=\int_0^1 \Re f(tz)g(tz)\frac{dt}{t},\quad z\in B. \tag{1.16}$$

To the best of our knowledge operator L_g on the unit ball is introduced in the present paper for the first time.

The purpose of this paper is to study the boundedness of the two Riemann-Stieltjes operators T_g, L_g from $F(p,q,s)$ to α-Bloch space. The corollaries of our results generalized the former results and some results are new even in the unit disk setting.

In this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. $a \preceq b$ means that there is a positive constant C such that $a \le Cb$. Moreover, if both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \approx b$.

2. $T_g, L_g : F(p,q,s) \to \mathcal{B}^\alpha$

In order to prove our results, we need some auxiliary results which are incorporated in the following lemmas. The first one is an analogy of the following one-dimensional result:

$$\left(\int_0^z f(\zeta)g'(\zeta)d\zeta\right)' = f(z)g'(z), \qquad \left(\int_0^z f'(\zeta)g(\zeta)d\zeta\right)' = f'(z)g(z). \tag{2.1}$$

Lemma 2.1 [5]. *For every $f,g \in H(B)$, it holds that*

$$\Re[T_g(f)](z) = f(z)\Re g(z), \qquad \Re[L_g(f)](z) = \Re f(z)g(z). \tag{2.2}$$

Proof. Assume that the holomorphic function $f\Re g$ has the expansion $\sum_\alpha a_\alpha z^\alpha$. Then

$$\Re[T_g(f)](z) = \Re\int_0^1 \sum_\alpha a_\alpha (tz)^\alpha \frac{dt}{t} = \Re\left(\sum_\alpha \frac{a_\alpha}{|\alpha|} z^\alpha\right) = \sum_\alpha a_\alpha z^\alpha, \tag{2.3}$$

which is what we wanted to prove. The proof of the second formula is similar and will be omitted. □

The following lemma can be found in [19].

Lemma 2.2. *For $0 < p,\ s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, if $f \in F(p,q,s)$, then $f \in \mathcal{B}^{(n+1+q)/p}$ and*

$$\|f\|_{\mathcal{B}^{(n+1+q)/p}} \le C\|f\|_{F(p,q,s)}. \tag{2.4}$$

The following lemma can be found in [15].

Lemma 2.3. *If $f \in \mathcal{B}^\alpha$, then*

$$|f(z)| \le C\begin{cases} |f(0)| + \|f\|_{\mathcal{B}^\alpha}, & 0<\alpha<1; \\ |f(0)| + \|f\|_{\mathcal{B}^\alpha}\log\dfrac{1}{1-|z|^2}, & \alpha = 1, \\ |f(0)| + \dfrac{\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha-1}}, & \alpha > 1, \end{cases} \tag{2.5}$$

for some C independent of f.

2.1. Case $p < n+1+q$.

In this section we consider the case $p < n+1+q$. Our first result is the following theorem.

Theorem 2.4. *Let g be a holomorphic function on B, $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $n+1+q \le p\alpha$, $p < n+1+q$. Then $T_g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded if and only if*

$$\sup_{z\in B}(1-|z|^2)^{\alpha+1-(n+1+q)/p}\,|\Re g(z)| < \infty. \tag{2.6}$$

Moreover, the following relationship:

$$||T_g||_{F(p,q,s)\to\mathcal{B}^\alpha} \approx \sup_{z\in B}(1-|z|^2)^{\alpha+1-(q+n+1)/p}\,|\Re g(z)| \tag{2.7}$$

holds.

Proof. For $f,g \in H(B)$, note that $T_g f(0) = 0$, by Lemmas 2.1, 2.2, and 2.3,

$$\begin{aligned} ||T_g f||_{\mathcal{B}^\alpha} &= \sup_{z\in B}(1-|z|^2)^{\alpha}\,|\Re(T_g f)(z)| \\ &= \sup_{z\in B}(1-|z|^2)^{\alpha}\,|f(z)|\,|\Re g(z)| \\ &\le C\|f\|_{\mathcal{B}^{(n+1+q)/p}}\sup_{z\in B}(1-|z|^2)^{\alpha+1-(n+1+q)/p}\,|\Re g(z)| \\ &\le C\|f\|_{F(p,q,s)}\sup_{z\in B}(1-|z|^2)^{\alpha+1-(n+1+q)/p}\,|\Re g(z)|. \end{aligned} \tag{2.8}$$

Therefore (2.6) implies that $T_g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded.

Conversely, suppose $T_g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded. For $w \in B$, let

$$f_w(z) = \frac{1-|w|^2}{(1-\langle z,w\rangle)^{(n+1+q)/p}}. \tag{2.9}$$

It is easy to see that

$$f_w(w) = \frac{1}{(1-|w|^2)^{(n+1+q)/p-1}}, \qquad |\Re f_w(w)| \approx \frac{|w|^2}{(1-|w|^2)^{(n+1+q)/p}}. \tag{2.10}$$

If $w = 0$ then $f_w \equiv 1$ obviously belongs to $F(p,q,s)$. From [19] we know that $f_w \in F(p,q,s)$, moreover there is a positive constant K such that $\sup_{w\in B}\|f_w\|_{F(p,q,s)} \le K$. Therefore

$$(1-|z|^2)^{\alpha}\,|f_w(z)\Re g(z)| = (1-|z|^2)^{\alpha}\,|\Re(T_g f_w)(z)| \le ||T_g f_w||_{\mathcal{B}^\alpha} \le K||T_g||_{F(p,q,s)\to\mathcal{B}^\alpha}, \tag{2.11}$$

for every $z,w \in B$.

From this and (2.10), we get

$$(1-|w|^2)^{\alpha+1-(n+1+q)/p}\,|\Re g(w)| = (1-|w|^2)^{\alpha}\,|f_w(w)\Re g(w)| \le K||T_g||_{F(p,q,s)\to\mathcal{B}^\alpha}, \tag{2.12}$$

from which (2.6) follows. From the above proof, we see that (2.7) holds. □

Theorem 2.5. *Let g be a holomorphic function on B, $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $n+1+q \le p\alpha$, $p < n+1+q$. Then $L_g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded if and only if*

$$\sup_{z\in B}\left(1-|z|^2\right)^{\alpha-(n+1+q)/p}|g(z)| < \infty. \tag{2.13}$$

Moreover, the following relationship:

$$\|L_g\|_{F(p,q,s)\to\mathcal{B}^\alpha} \approx \sup_{z\in B}\left(1-|z|^2\right)^{\alpha-(n+1+q)/p}|g(z)| \tag{2.14}$$

holds.

Proof. Assume that (2.13) holds. Let $f(z) \in F(p,q,s) \subset \mathcal{B}^{(n+1+q)/p}$, then

$$\sup_{z\in B}\left(1-|z|^2\right)^{(n+1+q)/p}|\Re f(z)| < \infty. \tag{2.15}$$

Therefore by Lemmas 2.1 and 2.2 we have

$$\begin{aligned}
\|L_g f\|_{\mathcal{B}^\alpha} &= \sup_{z\in B}\left(1-|z|^2\right)^{\alpha}|\Re(L_g f)(z)| \\
&= \sup_{z\in B}\left(1-|z|^2\right)^{\alpha}|\Re f(z)|\,|g(z)| \\
&\le \sup_{z\in B}\left(1-|z|^2\right)^{(n+1+q)/p}|\Re f(z)| \sup_{z\in B}\left(1-|z|^2\right)^{\alpha-(n+1+q)/p}|g(z)| \\
&\le C\|f\|_{\mathcal{B}^{(n+1+q)/p}} \sup_{z\in B}\left(1-|z|^2\right)^{\alpha-(n+1+q)/p}|g(z)| \\
&\le C\|f\|_{F(p,q,s)} \sup_{z\in B}\left(1-|z|^2\right)^{\alpha-(n+1+q)/p}|g(z)|.
\end{aligned} \tag{2.16}$$

Here we used the fact $L_g f(0) = 0$. It follows that L_g is bounded.

Conversely, if $L_g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded. Let $\beta(z,w)$ denote the Bergman metric between two points z and w in B. It is well known that

$$\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}. \tag{2.17}$$

For $a \in B$ and $r > 0$ the set

$$D(a,r) = \{z \in B : \beta(a,z) < r\}, \quad a \in B, \tag{2.18}$$

is a Bergman metric ball at a with radius r. It is well known that (see [21])

$$\frac{\left(1-|a|^2\right)^{n+1}}{|1-\langle a,z\rangle|^{2(n+1)}} \approx \frac{1}{\left(1-|z|^2\right)^{n+1}} \approx \frac{1}{\left(1-|a|^2\right)^{n+1}} \approx \frac{1}{|D(a,r)|} \tag{2.19}$$

when $z \in D(a,r)$. For $w \in B$, let $f_w(z)$ be defined by (2.9), then by (2.10) and (2.19) we have

$$
\begin{aligned}
&(1-|w|^2)^{-2(n+1+q)/p}|g(w)|^2|w|^4 \\
&\quad\approx |\Re f_w(w)g(w)|^2 \le \frac{C}{(1-|w|^2)^{n+1}}\int_{D(w,r)}|\Re f_w(z)|^2|g(z)|^2 dv(z) \\
&\quad\le \frac{C}{(1-|w|^2)^{n+1}}\int_{D(w,r)}|\Re f_w(z)|^2|g(z)|^2(1-|z|^2)^{2\alpha}\frac{1}{(1-|z|^2)^{2\alpha}}dv(z) \qquad (2.20)\\
&\quad\le C\int_{D(w,r)}\frac{dv(z)}{(1-|z|^2)^{2\alpha+n+1}}\sup_{z\in D(w,r)}(1-|z|^2)^{2\alpha}|\Re f_w(z)|^2|g(z)|^2 \\
&\quad\le \frac{C}{(1-|w|^2)^{2\alpha}}\|L_g f_w\|^2_{\mathcal{B}^\alpha},
\end{aligned}
$$

that is,

$$
(1-|w|^2)^{2\alpha-2(n+1+q)/p}|g(w)|^2|w|^4 \le C\|L_g f_w\|^2_{\mathcal{B}^\alpha} \le CK^2\|L_g\|^2_{F(p,q,s)\to\mathcal{B}^\alpha}. \tag{2.21}
$$

Taking supremum in the last inequality over the set $1/2 \le |w| < 1$ and noticing that by the maximum modulus principle there is a positive constant C independent of $g \in H(B)$ such that

$$
\sup_{|w|\le 1/2}(1-|w|^2)^{\alpha-(q+n+1)/p}|g(w)| \le C\sup_{1/2\le|w|<1}|w|^4(1-|w|^2)^{\alpha-(q+n+1)/p}|g(w)|. \tag{2.22}
$$

Therefore

$$
\sup_{z\in B}(1-|w|^2)^{\alpha-(q+n+1)/p}|g(w)| < C\|L_g\|_{F(p,q,s)\to\mathcal{B}^\alpha}, \tag{2.23}
$$

the result follows. □

Remark 2.6. Note that if $\alpha < (q+n+1)/p$ in Theorem 2.5, then the condition (2.13) is equivalent to $g \equiv 0$.

Corollary 2.7. *Let g be a holomorphic function on B, $\alpha > 0$. Then the operator $T_g : A^2 \to \mathcal{B}^\alpha$ is bounded if and only if*

$$
\sup_{z\in B}(1-|z|^2)^{\alpha-(n+1)/2}|\Re g(z)| < \infty. \tag{2.24}
$$

$L_g : A^2 \to \mathcal{B}^\alpha$ is bounded if and only if

$$
\sup_{z\in B}(1-|z|^2)^{\alpha-(n+1)/2-1}|g(z)| < \infty. \tag{2.25}
$$

$T_g : H^2 \to \mathcal{B}^\alpha$ is bounded if and only if

$$\sup_{z\in B}\left(1-|z|^2\right)^{\alpha-n/2}\left|\Re g(z)\right| < \infty. \tag{2.26}$$

$L_g : H^2 \to \mathcal{B}^\alpha$ is bounded if and only if

$$\sup_{z\in B}\left(1-|z|^2\right)^{\alpha-n/2-1}\left|g(z)\right| < \infty. \tag{2.27}$$

2.2. Case $p > n+1+q$

Theorem 2.8. *Let g be a holomorphic function on B, $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $\alpha \geq 0$, $n+1+q \leq p\alpha$, $p > n+1+q$. Then $T_g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded if and only if $g \in \mathcal{B}^\alpha$. Moreover, the following relationship:*

$$\|T_g\|_{F(p,q,s)\to\mathcal{B}^\alpha} \approx \sup_{z\in B}\left(1-|z|^2\right)^{\alpha}\left|\Re g(z)\right| \tag{2.28}$$

holds.

Proof. Since $f \in F(p,q,s) \subset \mathcal{B}^{(n+1+q)/p}$, by Lemmas 2.1, 2.2, and 2.3,

$$\begin{aligned}\|T_g f\|_{\mathcal{B}^\alpha} &= \sup_{z\in B}\left(1-|z|^2\right)^{\alpha}\left|f(z)\right|\left|\Re g(z)\right| \\ &\leq C\|f\|_{F(p,q,s)}\sup_{z\in B}\left(1-|z|^2\right)^{\alpha}\left|\Re g(z)\right|.\end{aligned} \tag{2.29}$$

Therefore $g \in \mathcal{B}^\alpha$ implies that $T_g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded.

Conversely, suppose $T_g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded. For $w \in B$, let

$$f_w(z) = \frac{\left(1-|w|^2\right)^{(p+n+1+q)/p}}{\left(1-\langle z,w\rangle\right)^{2(n+1+q)/p}} - \frac{1-|w|^2}{\left(1-\langle z,w\rangle\right)^{(n+1+q)/p}} + 1. \tag{2.30}$$

Then it is easy to see that

$$f_w(w) = 1, \quad \left|\Re f_w(z)\right| \leq \frac{C\left(1-|w|^2\right)}{\left|1-\langle z,w\rangle\right|^{(n+1+q+p)/p}}, \quad \left|\Re f_w(w)\right| \approx \frac{|w|^2}{\left(1-|w|^2\right)^{(n+1+q)/p}}. \tag{2.31}$$

By [19], we know that $f_w \in F(p,q,s)$, moreover there exists a constant L such that $\sup_{z\in B}\|f_w\|_{F(p,q,s)} \le L$. Hence

$$\begin{aligned}|\Re g(w)|^2 = |f_w(w)\Re g(w)|^2 &\le \frac{C}{(1-|w|^2)^{n+1}}\int_{D(w,r)} |f_w(z)|^2 |\Re g(z)|^2 dv(z)\\ &\le \frac{C}{(1-|w|^2)^{n+1}}\int_{D(w,r)} |f_w(z)|^2 |\Re g(z)|^2 (1-|z|^2)^{2\alpha}\frac{1}{(1-|z|^2)^{2\alpha}}dv(z)\\ &\le C\int_{D(w,r)}\frac{dv(z)}{(1-|z|^2)^{2\alpha+n+1}}\sup_{z\in D(w,r)}(1-|z|^2)^{2\alpha}|f_w(z)|^2|\Re g(z)|^2\\ &\le \frac{C}{(1-|w|^2)^{2\alpha}}\|T_g f_w\|^2_{\mathcal{B}^\alpha},\end{aligned} \tag{2.32}$$

that is,

$$(1-|w|^2)^{\alpha}|\Re g(w)| \le C\|T_g f_w\|_{\mathcal{B}^\alpha} \le CL\|T_g\|_{F(p,q,s)\to\mathcal{B}^\alpha} \tag{2.33}$$

for every $w \in B$. The result follows. □

Theorem 2.9. *Let g be a holomorphic function on B, $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $\alpha \ge 0$, $n+1+q \le p\alpha$, $p > n+1+q$. Then $L_g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded if and only if*

$$\sup_{z\in B}(1-|z|^2)^{\alpha-(n+1+q)/p}|g(z)| < \infty. \tag{2.34}$$

Moreover, the following relationship:

$$\|L_g\|_{F(p,q,s)\to\mathcal{B}^\alpha} \approx \sup_{z\in B}(1-|z|^2)^{\alpha-(n+1+q)/p}|g(z)| \tag{2.35}$$

holds.

Proof. Suppose (2.34) holds. Let $f(z) \in F(p,q,s) \subset \mathcal{B}^{(n+1+q)/p}$, then

$$\sup_{z\in B}(1-|z|^2)^{(n+1+q)/p}|\Re f(z)| < \infty. \tag{2.36}$$

Hence

$$\begin{aligned}\|L_g f\|_{\mathcal{B}^\alpha} &= \sup_{z\in B}(1-|z|^2)^{\alpha}|\Re f(z)||g(z)|\\ &\le \sup_{z\in B}(1-|z|^2)^{(n+1+q)/p}|\Re f(z)|\sup_{z\in B}(1-|z|^2)^{\alpha-(n+1+q)/p}|g(z)|\\ &\le C\|f\|_{\mathcal{B}^{(n+1+q)/p}}\sup_{z\in B}(1-|z|^2)^{\alpha-(n+1+q)/p}|g(z)|\\ &\le C\|f\|_{F(p,q,s)}\sup_{z\in B}(1-|z|^2)^{\alpha-(n+1+q)/p}|g(z)|.\end{aligned} \tag{2.37}$$

It follows that L_g is bounded.

Conversely, if $L_g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded, for $w \in B$, let $f_w(z)$ be defined by (2.30). Then by (2.31),

$$\begin{aligned}
&(1-|w|^2)^{-2(n+1+q)/p}|g(w)|^2|w|^4 \\
&\quad\approx |\Re f_w(w)g(w)|^2 \le \frac{C}{(1-|w|^2)^{n+1}}\int_{D(w,r)}|\Re f_w(z)|^2|g(z)|^2 dv(z) \\
&\quad\le \frac{C}{(1-|w|^2)^{n+1}}\int_{D(w,r)}|\Re f_w(z)|^2|g(z)|^2(1-|z|^2)^{2\alpha}\frac{1}{(1-|z|^2)^{2\alpha}}dv(z) \\
&\quad\le C\int_{D(w,r)}\frac{dv(z)}{(1-|z|^2)^{2\alpha+n+1}}\sup_{z\in D(w,r)}(1-|z|^2)^{2\alpha}|\Re f_w(z)|^2|g(z)|^2 \\
&\quad\le \frac{C}{(1-|w|^2)^{2\alpha}}||L_g f_w||^2_{\mathcal{B}^\alpha},
\end{aligned} \tag{2.38}$$

that is,

$$(1-|w|^2)^{2\alpha-2(n+1+q)/p}|g(w)|^2|w|^4 \le C||L_g f_w||^2_{\mathcal{B}^\alpha} \le CL||L_g||^2_{F(p,q,s)\to\mathcal{B}^\alpha}. \tag{2.39}$$

Similarly to the proof of Theorem 2.5, we get the desired result. □

2.3. Case $p = n+1+q$

Theorem 2.10. *Let g be a holomorphic function on B, $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $s > n$, $\alpha \ge 1$, $p = n+1+q$. Then $T_g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded if and only if*

$$\sup_{z\in B}(1-|z|^2)^\alpha \log\frac{1}{1-|z|^2}|\Re g(z)| < \infty. \tag{2.40}$$

Moreover the following relationship:

$$||T_g||_{F(p,q,s)\to\mathcal{B}^\alpha} \approx \sup_{z\in B}(1-|z|^2)^\alpha \log\frac{1}{1-|z|^2}|\Re g(z)| \tag{2.41}$$

holds.

Proof. Since $f \in F(p,q,s) \subset \mathcal{B}$, by Lemmas 2.1, 2.2, and 2.3,

$$\begin{aligned}
||T_g f||_{\mathcal{B}^\alpha} &= \sup_{z\in B}(1-|z|^2)^\alpha|\Re(T_g f)(z)| \\
&= \sup_{z\in B}(1-|z|^2)^\alpha|f(z)||\Re g(z)| \\
&\le C\|f\|_{F(p,q,s)}\sup_{z\in B}(1-|z|^2)^\alpha\log\frac{1}{1-|z|^2}|\Re g(z)|.
\end{aligned} \tag{2.42}$$

Therefore (2.40) implies that T_g is a bounded operator from $F(p,q,s)$ to $\mathcal{B}^\alpha$.

Conversely, suppose T_g is a bounded operator from $F(p,q,s)$ to $\mathcal{B}^\alpha$. For $w \in B$, let

$$f_w(z) = \log \frac{1}{1 - \langle z, w \rangle}. \tag{2.43}$$

Then by [19] we see that $f_w \in F(p,q,s)$ and

$$f_w(w) = \log \frac{1}{1 - |w|^2}, \quad |\Re f_w(w)| \approx \frac{|w|^2}{(1 - |w|^2)}. \tag{2.44}$$

Moreover there is a positive constant M such that $\sup_{w\in B} \|f\|_{F(p,q,s)} \le M$. Hence

$$\begin{aligned}
&\left(\log \frac{1}{1 - |w|^2}\right)^2 |\Re g(w)|^2 \\
&\quad = |f_w(w)\Re g(w)|^2 \le \frac{C}{(1 - |w|^2)^{n+1}} \int_{D(w,r)} |f_w(z)|^2 |\Re g(z)|^2 dv(z) \\
&\quad \le \frac{C}{(1 - |w|^2)^{n+1}} \int_{D(w,r)} |f_w(z)|^2 |\Re g(z)|^2 (1 - |z|^2)^{2\alpha} \frac{1}{(1 - |z|^2)^{2\alpha}} dv(z) \\
&\quad \le C \int_{D(w,r)} \frac{dv(z)}{(1 - |z|^2)^{2\alpha+n+1}} \sup_{z\in D(w,r)} (1 - |z|^2)^{2\alpha} |f_w(z)|^2 |\Re g(z)|^2 \\
&\quad \le \frac{C}{(1 - |w|^2)^{2\alpha}} ||T_g f_w||^2_{\mathcal{B}^\alpha},
\end{aligned} \tag{2.45}$$

that is,

$$(1 - |w|^2)^\alpha \left(\log \frac{1}{1 - |w|^2}\right) |\Re g(w)| \le C||T_g f_w||_{\mathcal{B}^\alpha} \le CM||T_g||_{F(p,q,s)\to\mathcal{B}^\alpha}. \tag{2.46}$$

The result follows. □

Theorem 2.11. *Let g be a holomorphic function on B,* $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $s > n$, $\alpha \ge 1$, $p = n+1+q$. *Then* $L_g : F(p,q,s) \to \mathcal{B}^\alpha$ *is bounded if and only if*

$$\sup_{z\in B} (1 - |z|^2)^{\alpha-1} |g(z)| < \infty. \tag{2.47}$$

Moreover the following relationship:

$$||L_g||_{F(p,q,s)\to\mathcal{B}^\alpha} \approx \sup_{z\in B} (1 - |z|^2)^{\alpha-1} |g(z)| \tag{2.48}$$

holds.

Proof. Suppose (2.47) holds. Let $f(z) \in F(p,q,s) \subset \mathcal{B}$, then $\sup_{z\in B}(1-|z|^2)|\Re f(z)| < \infty$. By Lemmas 2.1 and 2.2 we have

$$\begin{aligned} \|L_g f\|_{\mathcal{B}^\alpha} &= \sup_{z\in B}\left(1-|z|^2\right)^\alpha |\Re f(z)|\,|g(z)| \\ &\le C\|f\|_{\mathcal{B}} \sup_{z\in B}\left(1-|z|^2\right)|g(z)| \\ &\le C\|f\|_{F(p,q,s)} \sup_{z\in B}\left(1-|z|^2\right)^{\alpha-1}|g(z)|. \end{aligned} \tag{2.49}$$

It follows that L_g is bounded.

Conversely, if $L_g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded. For $w \in B$, let $f_w(z)$ be defined by (2.43), then by (2.44) we have

$$\begin{aligned} &\left(1-|w|^2\right)^{-2}|g(w)|^2|w|^4 \\ &\quad\approx |\Re f_w(w)g(w)|^2 \le \frac{C}{\left(1-|w|^2\right)^{n+1}}\int_{D(w,r)}|\Re f_w(z)|^2|g(z)|^2 dv(z) \\ &\quad\le \frac{C}{\left(1-|w|^2\right)^{n+1}}\int_{D(w,r)}|\Re f_w(z)|^2|g(z)|^2\left(1-|z|^2\right)^{2\alpha}\frac{1}{\left(1-|z|^2\right)^{2\alpha}}dv(z) \\ &\quad\le C\int_{D(w,r)}\frac{dv(z)}{\left(1-|z|^2\right)^{2\alpha+n+1}}\sup_{z\in D(w,r)}\left(1-|z|^2\right)^{2\alpha}|\Re f_w(z)|^2|g(z)|^2 \\ &\quad\le \frac{C}{\left(1-|w|^2\right)^{2\alpha}}\|L_g f_w\|^2_{\mathcal{B}^\alpha}, \end{aligned} \tag{2.50}$$

that is,

$$\left(1-|w|^2\right)^{2\alpha-2}|g(w)|^2|w|^4 \le C\|L_g f_w\|^2_{\mathcal{B}^\alpha}. \tag{2.51}$$

Similarly to the proof of Theorem 2.5, we get the desired result. □

Similarly to the proof of Theorems 2.10 and 2.11, we can obtain the following results. We omit the details.

Corollary 2.12. *Let g be a holomorphic function on B, $0 < p < \infty$, and $\alpha \ge 1$. Then $T_g : Q_p \to \mathcal{B}^\alpha$ is bounded if and only if*

$$\sup_{z\in B}\left(1-|z|^2\right)^\alpha \log\frac{1}{1-|z|^2}|\Re g(z)| < \infty. \tag{2.52}$$

$L_g : Q_p \to \mathcal{B}^\alpha$ is bounded if and only if

$$\sup_{z\in B}\left(1-|z|^2\right)^{\alpha-1}|g(z)| < \infty. \tag{2.53}$$

Corollary 2.13. *Let g be a holomorphic function on B. Then $L_g : \mathcal{B} \to \mathcal{B}$ is bounded if and only if $g \in H^\infty$.*

Especially, we have the following known result (see [6, 15, 17]).

Corollary 2.14. *Let g be a holomorphic function on B. Then $T_g : \mathcal{B} \to \mathcal{B}$ is bounded if and only if*

$$\sup_{z\in B} (1-|z|^2)\log\frac{1}{1-|z|^2}\,|\Re g(z)| < \infty. \tag{2.54}$$

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