A UNIFYING APPROACH FOR CERTAIN CLASS OF MAXIMAL FUNCTIONS

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We establish L^p estimates for certain class of maximal functions with kernels in $L^q(\mathbf{S}^{n-1})$. As a consequence of such L^p estimates, we obtain the L^p boundedness of our maximal functions when their kernels are in $L(\log L)^{1/2}(\mathbf{S}^{n-1})$ or in the block space $B_q^{0,-1/2}(\mathbf{S}^{n-1})$, $q>1$. Several applications of our results are also presented.

1. Introduction and statement of results

Let $\mathbb{R}^n$, $n \geq 2$, be the n-dimensional Euclidean space and let $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma$. For nonzero $y \in \mathbb{R}^n$, we will let $y' = |y|^{-1}y$. Let Ω be an integrable function on $\mathbf{S}^{n-1}$ that is homogeneous of degree zero on $\mathbb{R}^n$ and satisfies the cancelation property

$$\int_{\mathbf{S}^{n-1}} \Omega(y')\,d\sigma(y') = 0. \tag{1.1}$$

Consider the maximal function $\mathcal{M}_\Omega$,

$$\mathcal{M}_\Omega(f)(x) = \sup_{h\in U}\left|\int_{\mathbb{R}^n} f(x-y)|y|^{-n}h(|y|)\Omega(y')\,dy\right|, \tag{1.2}$$

where U is the class of all $h \in L^2(\mathbb{R}_+, r^{-1}dr)$ with $\|h\|_{L^2(\mathbb{R}_+,r^{-1}dr)} \leq 1$.

The operator $\mathcal{M}_\Omega$ was introduced by Chen and Lin [7]. They showed that $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbb{R}^n)$ for all $p > 2n/(2n-1)$ provided that $\Omega \in \mathscr{C}(\mathbf{S}^{n-1})$. Recently, we have been able to show that the $L^p(\mathbb{R}^n)$ boundedness of $\mathcal{M}_\Omega$ still holds for all $p \geq 2$ if the condition $\Omega \in \mathscr{C}(\mathbf{S}^{n-1})$ is replaced by the more natural and weaker condition $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ [2]. Moreover, we showed that if the condition $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ is replaced by any condition in the form $\Omega \in L(\log L)^{r}(\mathbf{S}^{n-1})$ for some $r < 1/2$, then $\mathcal{M}_\Omega$ might fail to be bounded on L^2.

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On the other hand, when Ω lies in $B_s^{0,-1/2}(\mathbf{S}^{n-1})$, $s>1$, which is a special class of block spaces $B_q^{\kappa,\upsilon}(\mathbf{S}^{n-1})$ (see Section 5 for the definition), we were able to show that $\mathcal{M}_\Omega$ is bounded on L^p for all $p\geq 2$ [3]. Moreover, we showed that the condition $\Omega\in B_s^{0,-1/2}(\mathbf{S}^{n-1})$, $s>1$ is nearly optimal in the sense that the exponent $-1/2$ cannot be replaced by any smaller number for the L^2 boundedness of $\mathcal{M}_\Omega$ to hold. We remark here that block spaces have been introduced by Jiang and Lu to improve previously obtained L^p boundedness results for singular integrals [7]. It should be noted here that the relation between the spaces $B_s^{0,-1/2}(\mathbf{S}^{n-1})$ and $L(\log L)^{1/2}(\mathbf{S}^{n-1})$ is unknown.

However, it is known that $L^q(\mathbf{S}^{n-1})$ is properly contained in $L(\log L)^{1/2}(\mathbf{S}^{n-1})\cap B_s^{0,-1/2}(\mathbf{S}^{n-1})$ for all q, $s>1$. Moreover, it is not hard to see that every Ω in $L(\log L)^{1/2}(\mathbf{S}^{n-1})\cup B_s^{0,-1/2}(\mathbf{S}^{n-1})$ can be written as an infinite sum of functions in $L^q(\mathbf{S}^{n-1})$. This gives rise to the question whether the results pertaining the L^p boundedness of $\mathcal{M}_\Omega$ in [2, 3] can be obtained via certain corresponding L^p estimates with kernels in $L^q(\mathbf{S}^{n-1})$. It is one of our main goals in this paper to consider such problem. It should be pointed out here that a positive solution for this problem will not only make life easier when dealing with kernels in $L(\log L)^{1/2}(\mathbf{S}^{n-1})$ or $B_s^{0,-1/2}(\mathbf{S}^{n-1})$, but also will pave the way for extending several results that are known when kernels are in $L^q(\mathbf{S}^{n-1})$.

Our work in this paper will be mainly concerned with the following general class of maximal functions:

$$\mathcal{M}_{\Omega,P}(f)(x)=\sup_{h\in U}\left|\int_{\mathbb{R}^n}e^{iP(y)}f(x-y)|y|^{-n}h(|y|)\Omega(y')dy\right|,\tag{1.3}$$

where $P:\mathbb{R}^n\to\mathbb{R}$ is a real-valued polynomial.

Clearly, if $P(y)=0$, then $\mathcal{M}_{\Omega,P}=\mathcal{M}_\Omega$. For the significance of considering integral operators with oscillating kernels, we refer the readers to consult [1, 4, 11, 16, 19, 22–24], among others.

Our result concerning L^p estimates with kernels in $L^q(\mathbf{S}^{n-1})$ is the following theorem.

Theorem 1.1. *Let $\Omega\in L^q(\mathbf{S}^{n-1})$, $q>1$, be a homogeneous function of degree zero on $\mathbb{R}^n$ with $\|\Omega\|_1\leq 1$. Let $P:\mathbb{R}^n\to\mathbb{R}$ be a real-valued polynomial of degree d. Let $\mathcal{M}_{\Omega,P}$ be given by (1.3). Then*

$$\left\|\mathcal{M}_{\Omega,P}(f)\right\|_p\leq\left\{1+\log^{1/2}\left(e+\|\Omega\|_q\right)\right\}C_{p,q}\|f\|_p\tag{1.4}$$

for all $p\geq 2$, where $C_{p,q}=(2^{1/q'}/(2^{1/q'}-1))C_p$. Here $1/q'=1-1/q$ and C_p is a constant that may depend on the degree of the polynomial P but it is independent of the function Ω, the index q, and the coefficients of the polynomial P.

We remark here that the constant $C_{p,q}$ in Theorem 1.1 satisfies $C_{p,q}\to\infty$ as $q\to 1^+$. That is, the constant $C_{p,q}$ diverges when q tends to 1. This behavior of $C_{p,q}$ is natural since, by [2, Theorem B(b)], the special operator $\mathcal{M}_\Omega=\mathcal{M}_{\Omega,0}$ is not bounded on L^2 if the function Ω is assumed to satisfy only the sole condition that $\Omega\in L^1(\mathbf{S}^{n-1})$ (i.e., $q=1$).

By a suitable decomposition of the function Ω and an application of Theorem 1.1, we prove the following theorem which is a proper extension of the corresponding result in [2].

Theorem 1.2. *Suppose that $\Omega \in L(\log^+ L)^{1/2}(\mathbf{S}^{n-1})$ satisfying (1.1). Let $P : \mathbb{R}^n \to \mathbb{R}$ be a real-valued polynomial. Then $\mathcal{M}_{\Omega,P}$ is bounded on $L^p(\mathbb{R}^n)$ for all $p \geq 2$ with L^p bounds independent of the coefficients of the polynomial P.*

We should point out here that an alternative proof of Theorem 1.2 can be obtained by observing that $C_{p,q} \approx C_p/(q-1)$, where $C_{p,q}$ is the constant in Theorem 1.1, and then using a Yano-type extrapolation technique [27].

By another suitable application of Theorem 1.1, we will prove the following extension of [3, Theorem 1.2].

Theorem 1.3. *Suppose that $\Omega \in B_q^{0,-1/2}(\mathbf{S}^{n-1})$, $q > 1$, satisfying (1.1). Let $P : \mathbb{R}^n \to \mathbb{R}$ be a real-valued polynomial. Then $\mathcal{M}_{\Omega,P}$ is bounded on $L^p(\mathbb{R}^n)$ for all $p \geq 2$ with L^p bounds independent of the coefficients of the polynomial P.*

As an immediate consequence of Theorem 1.1 and the observation that

$$\left| T_{\Omega,P,h}(f)(x) \right| \leq \|h\|_{L^2(\mathbb{R}_+,r^{-1}dr)} \mathcal{M}_{\Omega,P}(f)(x), \tag{1.5}$$

we obtain the following result concerning oscillatory singular integrals.

Theorem 1.4. *Let $\Omega \in L^q(\mathbf{S}^{n-1})$, $q > 1$, be a homogeneous function of degree zero on $\mathbb{R}^n$ with $\|\Omega\|_1 \leq 1$. Let $P : \mathbb{R}^n \to \mathbb{R}$ be a real-valued polynomial of degree d and let $h \in L^2(\mathbb{R}_+, r^{-1}dr)$. Then the oscillatory singular integral operator $\mathcal{M}_{\Omega,P}$;*

$$T_{\Omega,P,h}(f)(x) = p \cdot v \int_{\mathbb{R}^n} e^{iP(y)} f(x-y)|y|^{-n} h(|y|)\Omega(y')dy \tag{1.6}$$

satisfies

$$\left|\left| T_{\Omega,P,h}(f) \right|\right|_p \leq \left\{ 1 + \log^{1/2}\left(e + \|\Omega\|_q\right) \right\} \|h\|_{L^2(\mathbb{R}_+,r^{-1}dr)} C_{p,q} \|f\|_p \tag{1.7}$$

for all $p \geq 2$, where $C_{p,q} = (2^{1/q'}/(2^{1/q'} - 1))C_p$. Here $1/q' = 1 - 1/q$ and C_p is a constant that may depend on the degree of the polynomial P but it is independent of the function Ω, the index q, and the coefficients of the polynomial P.

By Theorem 1.4, we obtain the following two results.

Corollary 1.5. *Let $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ be a homogeneous function of degree zero on $\mathbb{R}^n$ and satisfies (1.1). Let $P : \mathbb{R}^n \to \mathbb{R}$ be a real-valued polynomial of degree d and let $h \in L^2(\mathbb{R}_+, r^{-1}dr)$. Then the oscillatory singular integral operator $\mathcal{M}_{\Omega,P}$;*

$$T_{\Omega,P,h}(f)(x) = p \cdot v \int_{\mathbb{R}^n} e^{iP(y)} f(x-y)|y|^{-n} h(|y|)\Omega(y')dy \tag{1.8}$$

is bounded on L^p for all $p \geq 2$ with L^p bounds that may depend on the degree of the polynomial P but they are independent of the coefficients of the polynomial P.

Corollary 1.6. *Let $\Omega \in B_q^{0,-1/2}(\mathbf{S}^{n-1})$, $s > 1$, be a homogeneous function of degree zero on $\mathbb{R}^n$ and satisfies (1.1). Let $P : \mathbb{R}^n \to \mathbb{R}$ be a real-valued polynomial of degree d and let*

$h \in L^2(\mathbb{R}_+, r^{-1}dr)$. Then the oscillatory singular integral operator $\mathcal{M}_{\Omega,P}$;

$$T_{\Omega,P,h}(f)(x) = p \cdot v \int_{\mathbb{R}^n} e^{iP(y)} f(x-y)|y|^{-n} h(|y|)\Omega(y')dy \tag{1.9}$$

is bounded on L^p for all $p \geq 2$ with L^p bounds that may depend on the degree of the polynomial P but they are independent of the coefficients of the polynomial P.

Further applications of the results stated above will be presented in Section 6.

Throughout this paper, the letter C will stand for a constant that may vary at each occurrence, but it is independent of the essential variables.

2. Preliminary estimates

We start by recalling the following result in [10].

Lemma 2.1 (see [10]). *Let $\mathcal{P} = (P_1, \ldots, P_d)$ be a polynomial mapping from $\mathbb{R}^n$ into $\mathbb{R}^d$. Suppose that $\Omega \in L^1(\mathbf{S}^{n-1})$ and*

$$M_{\Omega,\mathcal{P}} f(x) = \sup_{j\in\mathbb{Z}} \int_{2^j \leq |y| < 2^{j+1}} |f(x-\mathcal{P}(y))|\,|y|^{-n}|\Omega(y')|\,dy. \tag{2.1}$$

Then for $1 < p \leq \infty$, there exist a constant $C_p > 0$ independent of Ω and the coefficients of $P_1, \ldots, P_d$ such that

$$\left|\left|M_{\Omega,\mathcal{P}} f\right|\right|_p \leq C_p \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \|f\|_p \tag{2.2}$$

for every $f \in L^p(\mathbb{R}^d)$.

Lemma 2.2 (van der Corput [26]). *Suppose ϕ is real valued and smooth in (a,b), and that $|\phi^{(k)}(t)| \geq 1$ for all $t \in (a,b)$. Then the inequality*

$$\left| \int_a^b e^{-i\lambda\phi(t)} \psi(t)dt \right| \leq C_k |\lambda|^{-1/k} \tag{2.3}$$

holds when

(i) *$k \geq 2$, or*

(ii) *$k = 1$ and ϕ' is monotonic.*

The bound C_k is independent of a, b, ϕ, and λ.

Lemma 2.3. *Let $\Omega \in L^q(\mathbf{S}^{n-1})$, $q > 1$, be a homogeneous function of degree zero on $\mathbb{R}^n$ with $\|\Omega\|_1 \leq 1$. Let $P(x) = \sum_{|\alpha|\leq d} a_\alpha x^\alpha$ be a real-valued polynomial of degree $d > 1$ such that $|x|^d$ is not one of its terms. For $k \in \mathbb{Z}$, let $E_{k,\Omega} : [1, \log(e + \|\Omega\|_q)] \times P(\mathbf{S}^{n-1}) \times \mathbb{R} \to \mathbb{C}$ and let $\mathbf{J}_{k,\Omega} : \mathbb{R}^n \to \mathbb{R}$ be given by*

$$\begin{gathered} E_{k,\Omega}(r, P(y'), s) = e^{-i[P(2^{-(k+1)\log(e+\|\Omega\|_q)} r y') + 2^{-(k+1)\log(e+\|\Omega\|_q)} s r]}, \\ \mathbf{J}_{k,\Omega}(\xi) = \int_1^{2^{2\log(e+\|\Omega\|_q)}} \left| \int_{\mathbf{S}^{n-1}} \Omega(y') E_{k,\Omega}(r, P(y'), \xi \cdot y') d\sigma(y') \right|^2 \frac{d}{r} r. \end{gathered} \tag{2.4}$$

Then, $\mathbf{J}_{k,\Omega}$ *satisfies*

$$\sup_{\xi\in\mathbb{R}^n} \mathbf{J}_{k,\Omega}(\xi) \le 2^{(k+1)/4q'} \log\left(e+\|\Omega\|_q\right)\left\{\sum_{|\alpha|=d} |a_\alpha|\right\}^{-\varepsilon/q'} C \tag{2.5}$$

for some $0<\varepsilon<1$, *where* C *is a constant that may depend on the degree of the polynomial* P *but it is independent of the function* Ω, *the index* q, *and the coefficients of the polynomial* P.

Proof of Lemma 2.3. First, we notice the following:

$$\mathbf{J}_{k,\Omega}(\xi) \le \log\left(e+\|\Omega\|_q\right), \tag{2.6}$$

$$\begin{aligned}\left(\mathbf{J}_{k,\Omega}(\xi)\right)^{q'} \le \|\Omega\|_q^{2q'} \iint_{\mathbf{S}^{n-1}} \Bigg| \int_1^{2^{2\log(e+\|\Omega\|_q)}} & E_{k,\Omega}(r,P(y'),\xi\cdot y') \\ & \times \overline{E_{k,\Omega}(r,P(z'),\xi\cdot z')}\frac{dr}{r}\Bigg|^{q'} d\sigma(y')d\sigma(z').\end{aligned} \tag{2.7}$$

Next, notice that

$$\begin{aligned}&P(2^{-\gamma_{k,\Omega}}ry') + 2^{-\gamma_{k,\Omega}}(\xi\cdot y')r - P(2^{-\gamma_{k,\Omega}}rz') + 2^{-\gamma_{k,\Omega}}(\xi\cdot z')r \\ &\quad = 2^{-\gamma_{k,\Omega}d}r^d\left\{\sum_{|\alpha|=d} a_\alpha y'^\alpha - \sum_{|\alpha|=d} a_\alpha z'^\alpha\right\} + 2^{-\gamma_{k,\Omega}}\xi\cdot(y'-z')r + H_k(r,y',z',\xi)\end{aligned} \tag{2.8}$$

with $(d^d/dr^d)H_k=0$, where $\gamma_{k,\Omega}=(k+1)\log(e+\|\Omega\|_q)$. Thus, by Lemma 2.2, we have

$$\left|\int_1^{2^{2\log(e+\|\Omega\|_q)}} E_{k,\Omega}(r,P(y'),\xi\cdot y')\overline{E_{k,\Omega}(r,P(z'),\xi\cdot z')}\frac{dr}{r}\right| \le \left|2^{-d\gamma_{k,\Omega}}\{P(y')-P(z')\}\right|^{-1/d}. \tag{2.9}$$

Now, by (2.9) and the inequality

$$\left|\int_1^{2^{2\log(e+\|\Omega\|_q)}} E_{k,\Omega}(r,P(y'),\xi\cdot y')\overline{E_{k,\Omega}(r,P(z'),\xi\cdot z')}\frac{dr}{r}\right| \le C\log\left(e+\|\Omega\|_q\right), \tag{2.10}$$

we obtain

$$\begin{aligned}&\left|\int_1^{2^{2\log(e+\|\Omega\|_q)}} E_{k,\Omega}(r,P(y'),\xi\cdot y')\overline{E_{k,\Omega}(r,P(z'),\xi\cdot z')}\frac{dr}{r}\right| \\ &\quad \le \left|2^{-d\gamma_{k,\Omega}}\{P(y')-P(z')\}\right|^{-1/4dq'} C\left\{\log\left(e+\|\Omega\|_q\right)\right\}^{1-1/4q'}.\end{aligned} \tag{2.11}$$

Therefore, by (2.7), (2.11), and [12, (3.11)], we obtain

$$\mathbf{J}_{k,\Omega}(\xi) \le 2^{\gamma_{k,\Omega/4q'}}\|\Omega\|_q^{2q'} C\left\{\log\left(e+\|\Omega\|_q\right)\right\}^{1-1/4q'}. \tag{2.12}$$

Hence, by (2.6) and (2.12), we get

$$\begin{aligned}\mathbf{J}_{k,\Omega}(\xi) &\leq 2^{\gamma_{k,\Omega}/4\log(e+\|\Omega\|_q)q'}\|\Omega\|_q^{2/\log(e+\|\Omega\|_q)}\log\left(e+\|\Omega\|_q\right)\\ &\leq 2^{(k+1)/4q'}\log\left(e+\|\Omega\|_q\right)C.\end{aligned}\tag{2.13}$$

This completes the proof. □

Now, we will need the following lemma.

Lemma 2.4. *Let* $\Omega \in L^q(\mathbf{S}^{n-1})$, $q > 1$, *be a homogeneous function of degree zero on* $\mathbb{R}^n$ *with* $\|\Omega\|_1 \leq 1$. *Then*

$$\left|\left|\mathcal{M}_\Omega(f)\right|\right|_p \leq \log^{1/2}\left(e+\|\Omega\|_q\right)\left\{\frac{2^{1/q'}}{2^{1/q'}-1}\right\}C_p\|f\|_p \tag{2.14}$$

for all $p \geq 2$ *with constants* C_p *independent of the function* Ω *and the index* q.

We remark here that since $L^q(\mathbf{S}^{n-1}) \subset L\log^{1/2}L$, it follows from [2, Theorem B(a)] that $\|\mathcal{M}_\Omega\|_p \leq \|\Omega\|_q C_p$ for all $p \geq 2$. But, clearly the constant $\{1+\log^{1/2}(e+\|\Omega\|_q)\}$ in (2.14) is sharper than the constant $\|\Omega\|_q$ that can be deduced from [2, Theorem B(a)]. However, the former constant can be obtained by following a similar argument as in the proof of Theorem B(a) in [2] and keeping track of certain constants. For completeness, we, below, present the main ideas of the proof.

Proof of Lemma 2.4. Choose a collection of $\mathscr{C}^\infty$ functions $\{\omega_k\}_{k\in\mathbb{Z}}$ on $(0,\infty)$ with the properties $\sup(\omega_k) \subseteq [2^{-\log(e+\|\Omega\|_q)(k+1)}, 2^{-\log(e+\|\Omega\|_q)(k-1)}]$, $0 \leq \omega_k \leq 1$, $\sum_{k\in\mathbb{Z}}\omega_k(u) = 1$, $|(d^s\omega_k/du^s)(u)| \leq C_s u^{-s}$, where C_s is a constant independent of $\log(e+\|\Omega\|_q)$. For $k \in \mathbb{Z}$, let G_k be the operator defined by $(G_k(f))\hat{}(\xi) = \omega_k(|\xi|)\hat{f}(\xi)$. Let

$$E_j(f)(x) = \left(\sum_{k\in\mathbb{Z}}\int_1^{2^{2\log(e+\|\Omega\|_q)}}\left|\int_{\mathbf{S}^{n-1}}\Omega(y')G_{k+j}(f)\left(x-2^{k\log(e+\|\Omega\|_q)}ry'\right)d\sigma(y')\right|^2 r^{-1}dr\right)^{1/2}. \tag{2.15}$$

Then

$$\mathcal{M}_\Omega(f)(x) \leq \sum_{j\in\mathbb{Z}}E_j(f)(x). \tag{2.16}$$

By exactly the same argument in [2], we obtain

$$\left|\left|E_j(f)\right|\right|_2 \leq C2^{-\beta|j|/q'}\log^{1/2}\left(e+\|\Omega\|_q\right)\|f\|_2. \tag{2.17}$$

On the other hand, by a duality argument; see (3.24)-(3.25) for similar argument, we get

$$\left|\left|E_j(f)\right|\right|_p \leq C\log^{1/2}\left(e+\|\Omega\|_q\right)\|f\|_p \tag{2.18}$$

for all $2 < p < \infty$. Thus, by interpolation between (2.17) and (2.18), we have

$$\left|\left|E_j(f)\right|\right|_p \leq C2^{-\varepsilon(|j|/q')}\log^{1/2}\left(e+\|\Omega\|_q\right)\|f\|_p \tag{2.19}$$

for some $\varepsilon > 0$ and for all $2 \leq p < \infty$, and $j \in \mathbb{Z}$ with constant C independent of Ω, k, and j. Hence, (2.14) follows by (2.16) and (2.19). This completes the proof. □

3. Proof of Theorem 1.1

Proof of Theorem 1.1. We will argue by induction on the degree of the polynomial P. If $d = \deg(P) = 0$, then (1.4) follows easily from Lemma 2.4. In fact, if $d = 0$, then by duality it can be easily seen that

$$\mathcal{M}_{\Omega,P}(f)(x) \leq C\mathcal{M}_{\Omega}(f)(x). \tag{3.1}$$

Thus, by Lemma 2.4, we have

$$\begin{aligned} ||\mathcal{M}_{\Omega,P}(f)||_p &\leq \left\{\frac{2^{1/q'}}{2^{1/q'}-1}\right\} \log^{1/2}(e+\|\Omega\|_q)C_p\|f\|_p \\ &\leq \left\{\frac{2^{1/q'}}{2^{1/q'}-1}\right\}\{1+\log^{1/2}(e+\|\Omega\|_q)\}C_p\|f\|_p \end{aligned} \tag{3.2}$$

for all $p \geq 2$.

Now, if $d = 1$, that is, $P(y) = \vec{a} \cdot y$ for some $\vec{a} \in \mathbb{R}^n$, then by (3.2), we have

$$\begin{aligned} ||\mathcal{M}_{\Omega,P}(f)||_p &\leq \left\{\frac{2^{1/q'}}{2^{1/q'}-1}\right\}\{\log^{1/2}\|\Omega\|_q\}C_p\|g\|_p \\ &= \left\{\frac{2^{1/q'}}{2^{1/q'}-1}\right\}\{1+\log^{1/2}(e+\|\Omega\|_q)\}C_p\|f\|_p, \end{aligned} \tag{3.3}$$

where $g(y) = e^{-iP(y)}f(y)$.

Next, assume that (1.4) holds for all polynomials Q of degree less than or equal to $d > 1$. Let

$$P(x) = \sum_{|\alpha|\leq d+1} a_\alpha x^\alpha \tag{3.4}$$

be a polynomial of degree $d+1$. Then by duality, we have

$$\mathcal{M}_{\Omega,P}(f)(x) = \left(\int_0^\infty \left|\int_{\mathbf{S}^{n-1}} e^{iP(ry')}\Omega(y')f(x-ry')d\sigma(y')\right|^2 r^{-1}dr\right)^{1/2}. \tag{3.5}$$

We may assume that P does not contain $|x|^{d+1}$ as one of its terms. By dilation invariance, we may also assume that

$$\sum_{|\alpha|=d+1} |a_\alpha| = 1. \tag{3.6}$$

We now choose a collection $\{\omega_k\}_{k\in\mathbb{Z}}$ of $\mathscr{C}^\infty$ functions defined on $(0,\infty)$ that satisfy the following properties:

$$\operatorname{supp}(\psi_k) \subseteq \left[2^{-\log(e+\|\Omega\|_q)(k+1)}, 2^{-\log(e+\|\Omega\|_q)(k-1)}\right], \quad 0 \le \psi_k \le 1, \qquad \sum_{k\in\mathbb{Z}} \psi_k(u) = 1. \tag{3.7}$$

Set

$$\eta_\infty(u) = \sum_{k=-\infty}^{0} \psi_k(u), \qquad \eta_0(u) = \sum_{k=1}^{\infty} \psi_k(u). \tag{3.8}$$

Then,

$$\eta_\infty(u) + \eta_0(u) = 1, \qquad \operatorname{supp}(\eta_\infty(u)) \subset \left[2^{-\log(e+\|\Omega\|_q)}, \infty\right), \qquad \operatorname{supp}(\eta_0(u)) \subset (0,1]. \tag{3.9}$$

Define the operators $\mathcal{S}_{\Omega,P,\infty}$ and $\mathcal{S}_{\Omega,P,0}$ by

$$\mathcal{S}_{\Omega,P,\infty}(f)(x) = \left(\int_{2^{-\log(e+\|\Omega\|_q)}}^{\infty} \left|\eta_\infty(r)\int_{\mathbf{S}^{n-1}} e^{iP(ry')}\Omega(y')f(x-ry')d\sigma(y')\right|^2 r^{-1}dr\right)^{1/2},$$
$$\mathcal{S}_{\Omega,P,0}(f)(x) = \left(\int_{0}^{1} \left|\eta_0(r)\int_{\mathbf{S}^{n-1}} e^{iP(ry')}\Omega(y')f(x-ry')d\sigma(y')\right|^2 r^{-1}dr\right)^{1/2}. \tag{3.10}$$

Thus, by (3.9), we have

$$\mathcal{S}_{\Omega,P}(f)(x) \le \mathcal{S}_{\Omega,P,0}(f)(x) + \mathcal{S}_{\Omega,P,\infty}(f)(x). \tag{3.11}$$

Now, we estimate $\|\mathcal{S}_{\Omega,P,0}\|_p$.

Let

$$Q(x) = \sum_{|\alpha|\le d} a_\alpha x^\alpha. \tag{3.12}$$

Assume that $\deg(Q) = l$, where $0 \le l \le d$. Define the operators $\mathcal{S}^{(1)}_{\Omega,P,0}$ and $\mathcal{S}^{(2)}_{\Omega,Q,0}$ by

$$\mathcal{S}^{(1)}_{\Omega,P,0}(f)(x) = \left(\int_{0}^{1} \left|\int_{\mathbf{S}^{n-1}} \left(e^{iP(ry')} - e^{iQ(ry')}\right)\Omega(y)f(x-ry')d\sigma(y')\right|^2 r^{-1}dr\right)^{1/2},$$
$$\mathcal{S}^{(2)}_{\Omega,Q,0}(f)(x) = \left(\int_{0}^{1} \left|\int_{\mathbf{S}^{n-1}} e^{iQ(ry')}\Omega(y')f(x-ry')d\sigma(y')\right|^2 r^{-1}dr\right)^{1/2}. \tag{3.13}$$

Now, by the observation that $\eta_0(r) \le 1$ and by Minkowski's inequality, we obtain

$$\mathcal{S}_{\Omega,P,0}(f)(x) \le \mathcal{S}^{(1)}_{\Omega,P,0}(f)(x) + \mathcal{S}^{(2)}_{\Omega,Q,0}(f)(x). \tag{3.14}$$

By induction assumption, it follows that

$$||\mathcal{S}^{(2)}_{\Omega,Q,0}(f)||_p \le \{1+\log^{1/2}(e+\|\Omega\|_q)\}\left\{\frac{2^{1/q'}}{2^{1/q'}-1}\right\}C_p\|f\|_p \tag{3.15}$$

for all $p \ge 2$.

On the other hand, by Cauchy-Schwarz inequality, by the fact that $\|\Omega\|_1 \le 1$, and the inequality

$$\begin{aligned} \left|\left(e^{iP(ry')}-e^{iQ(ry')}\right)\right| &\le r^{d+1}\left|\sum_{|\alpha|=d+1} a_\alpha y'^{\alpha}\right| \\ &\le r^{d+1}, \end{aligned} \tag{3.16}$$

we get

$$\begin{aligned} \mathcal{S}^{(1)}_{\Omega,P,0}(f)(x) &\le \left(\int_0^1\int_{\mathbf{S}^{n-1}}\left|\left(e^{iP(ry')}-e^{iQ(ry')}\right)\right|^2|\Omega(y')|\,|f(x-ry')|^2 d\sigma(y')r^{-1}dr\right)^{1/2} \\ &\le \left(\int_0^1\int_{\mathbf{S}^{n-1}}|\Omega(y')|\,|f(x-ry')|^2 d\sigma(y')r^{2d+1}dr\right)^{1/2} \\ &= \left(\sum_{j=-\infty}^{-1}\int_{2^j}^{2^{j+1}}\int_{\mathbf{S}^{n-1}}|\Omega(y')|\,|f(x-ry')|^2 d\sigma(y)r^{2d+1}dr\right)^{1/2} \\ &\le \left(\sum_{j=-\infty}^{-1}2^{(2d+2)j}\int_{2^j}^{2^{j+1}}\int_{\mathbf{S}^{n-1}}|\Omega(y')|\,|f(x-ry')|^2 d\sigma(y)r^{-1}dr\right)^{1/2} \\ &\le C\left(M_\Omega(|f|^2)\right)^{1/2}(x), \end{aligned} \tag{3.17}$$

where M_Ω is the operator given by (2.1) with $\mathcal{P}(y) = y$. Thus, by (3.17), by the fact that $\|\Omega\|_1 \le 1$, and Lemma 2.1, we obtain

$$||\mathcal{S}^{(1)}_{\Omega,P,0}(f)||_p \le C_p\|f\|_p \tag{3.18}$$

for all $p \ge 2$ with constant C_p independent of the function Ω and the coefficients of the polynomial P. Therefore, by (3.14), by Minkowski's inequality, by (3.15), and (3.18), we obtain

$$||\mathcal{S}_{\Omega,P,0}(f)||_p \le \{1+\log^{1/2}(e+\|\Omega\|_q)\}\left\{\frac{2^{1/q'}}{2^{1/q'}-1}\right\}C_p\|f\|_p \tag{3.19}$$

for all $p \ge 2$.

Finally, we prove the L^p boundedness of $\mathcal{S}_{\Omega,P,\infty}$. By generalized Minkowski's inequality, we can write $\mathcal{S}_{\Omega,P,\infty}$ as

$$\begin{aligned}\mathcal{S}_{\Omega,P,\infty}(f)(x) &= \left(\int_{2^{-\log(e+\|\Omega\|_q)}}^{\infty}\left|\eta_\infty(r)\int_{\mathbf{S}^{n-1}} e^{iP(ry')}\Omega(y')f(x-ry')d\sigma(y')\right|^2 r^{-1}dr\right)^{1/2}\\ &= \left(\int_0^{\infty}\left|\sum_{k=-\infty}^{0}\psi_k(r)\int_{\mathbf{S}^{n-1}} e^{iP(ry')}\Omega(y')f(x-ry')d\sigma(y')\right|^2 \frac{1}{r}dr\right)^{1/2}\\ &\le \sum_{k=-\infty}^{0}\mathcal{S}_{\Omega,P,\infty,k}(f)(x),\end{aligned} \tag{3.20}$$

where

$$\mathcal{S}_{\Omega,P,\infty,k}(f)(x) = \left(\int_{2^{-\log(e+\|\Omega\|_q)(k+1)}}^{2^{-\log(e+\|\Omega\|_q)(k-1)}}\left|\int_{\mathbf{S}^{n-1}} e^{iP(ry')}\Omega(y')f(x-ry')d\sigma(y')\right|^2 r^{-1}dr\right)^{1/2}. \tag{3.21}$$

By Plancherel's theorem, Fubini's theorem, and Lemma 2.3, we have

$$\left|\left|\mathcal{S}_{\Omega,P,\infty,k}(f)\right|\right|_2^2 = \int_{\mathbb{R}^n}\left|\hat{f}(\xi)\right|^2 \mathbf{J}_{k,\Omega}(\xi)d\xi \le 2^{(k+1)/4q'}\log\left(e+\|\Omega\|_q\right)\|f\|_2^2. \tag{3.22}$$

Thus,

$$\left|\left|\mathcal{S}_{\Omega,P,\infty,k}(f)\right|\right|_2 \le 2^{(k+1)/8q'}\log^{1/2}\left(e+\|\Omega\|_q\right)\|f\|_2. \tag{3.23}$$

Now, for $p>2$, choose $g\in L^{(p/2)'}$ with $\|g\|_{(p/2)'}=1$ such that

$$\begin{aligned}&\left|\left|\mathcal{S}_{\Omega,P,\infty,k}(f)\right|\right|_p^2\\ &= \int_{\mathbb{R}^n}\int_1^{2^{2\log(e+\|\Omega\|_q)}}\left|\int_{\mathbf{S}^{n-1}} E_{k,\Omega}(r,P(y'),0)\Omega(y')f(x-2^{-\gamma_{k,\Omega}}ry')d\sigma(y')\right|^2 r^{-1}dr\left|g(x)\right|dx\\ &\le \int_{\mathbb{R}^n}\left|f(z)\right|^2\int_1^{2^{2\log(e+\|\Omega\|_q)}}\int_{\mathbf{S}^{n-1}}\left|\Omega(y')\right|\left|g(z+2^{-\gamma_{k,\Omega}}ry')\right|\frac{d\sigma(y')dr}{r}dz\\ &\le C\log\left(e+\|\Omega\|_q\right)\|f\|_p^2\left|\left|M_\Omega\tilde{g}(z)\right|\right|_{(p/2)'},\end{aligned} \tag{3.24}$$

where M_Ω is the operator given by (2.1) with $\mathcal{P}(y)=y$. Thus, Lemma 2.1 and (3.24) imply that

$$\left|\left|\mathcal{S}_{\Omega,P,\infty,k}(f)\right|\right|_p \le \log^{1/2}\left(e+\|\Omega\|_q\right)C\|f\|_p, \tag{3.25}$$

which when combined with (3.23) implies

$$\left|\left|\mathcal{S}_{\Omega,P,\infty,k}(f)\right|\right|_p \le 2^{(k+1)\delta/8q'}\log^{1/2}\left(e+\|\Omega\|_q\right)C\|f\|_p, \tag{3.26}$$

where δ is a constant that is independent of the essential variables. Thus, by (3.20), (3.26), and Minkowski's inequality, we get

$$||\mathcal{S}_{\Omega,P,\infty}(f)||_p \le C\log^{1/2}(e+\|\Omega\|_q)\left\{\frac{2^{1/q'}}{2^{1/q'}-1}\right\}C_p\|f\|_p \tag{3.27}$$

for all $p \ge 2$. Hence, by Minkowski's inequality, (3.11), (3.19), and (3.27), we obtain (1.4) for the given polynomial P. This completes the proof. □

4. Proof of results concerning $L(\log L)^{1/2}(\mathbf{S}^{n-1})$

Proof of Theorem 1.2. Given $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$, then we decompose Ω as a sum of functions in $L^2(\mathbf{S}^{n-1})$. More precisely, there exists a sequence $\{\Omega_m : m = 0,1,2,\dots\}$ of functions in $L^1(\mathbf{S}^{n-1})$ with

$$\Omega = \sum_{m=0}^{\infty}\Omega_m \tag{4.1}$$

such that

$$\begin{gathered}\int_{\mathbf{S}^{n-1}}\Omega_m(y')d\sigma(y') = 0, \qquad ||\Omega_m||_1 \le C, \qquad \Omega_0 \in L^2(\mathbf{S}^{n-1}),\\ ||\Omega_m||_\infty \le 2^{4m}C \quad \text{for } m = 1,2,3,\dots,\end{gathered} \tag{4.2}$$

$$\sum_{m=1}^{\infty}\sqrt{m}||\Omega_m||_1 \le \|\Omega\|_{L(\log L)^{1/2}(\mathbf{S}^{n-1})}C. \tag{4.3}$$

For a detailed proof of the existence of the decomposition (4.1), one might look into [2, 5].

Now, by (4.1), we have the following:

$$\mathcal{M}_{\Omega,P}(f)(x) \le \mathcal{M}_{\Omega_0,P}(f)(x) + \sum_{m=1}^{\infty}||\Omega_m||_1\mathcal{M}_{\Omega_m,P}(f)(x). \tag{4.4}$$

By Lemma 2.4, we have

$$||\mathcal{M}_{\Omega_0,P}(f)||_p \le \{1+\log^{1/2}(e+||\Omega_0||_2)\}C_p\|f\|_p \tag{4.5}$$

for all $p \ge 2$.

Next, by observing that

$$1+\log^{1/2}(e+||\Omega_m||_\infty) \le 1+\log^{1/2}(e+2^{4m}) \le 4\sqrt{m} \tag{4.6}$$

for all $m \ge 1$, Theorem 1.1 implies that

$$||\mathcal{M}_{\Omega_m,P}(f)||_p \le 4\sqrt{m}C_p\|f\|_p \tag{4.7}$$

for all $p \ge 2$ with constant C_p independent of m.

Thus, by Minkowski's inequality, (4.4), (4.5), (4.7), and (4.2), we obtain

$$\left\|\mathcal{M}_{\Omega,P}(f)\right\|_p \le C_p\|f\|_p \tag{4.8}$$

for all $p \ge 2$. This completes the proof. □

Proof of Corollary 1.5. By the inequality (1.5) and the decomposition (4.1), we have

$$\left|T_{\Omega,P,h}(f)(x)\right| \le \left|T_{\Omega_0,P,h}(f)(x)\right| + \sum_{m=1}^{\infty}\left\|\Omega_m\right\|_1\left|T_{\Omega_m,P,h}(f)(x)\right|. \tag{4.9}$$

Thus, by Theorem 1.4, (4.9), and a similar argument as in the proof of Theorem 1.2, the proof is complete. □

5. Proof of results concerning block spaces

We start this section by recalling the definition of block spaces introduced by Jiang and Lu (see [16]).

Definition 5.1. (1) For $x_0' \in \mathbf{S}^{n-1}$ and $0 < \theta_0 \le 2$, the set $B(x_0',\theta_0) = \{x' \in \mathbf{S}^{n-1} : |x' - x_0'| < \theta_0\}$ is called a cap on $\mathbf{S}^{n-1}$.

(2) For $1 < q \le \infty$, a measurable function b is called a $q-$block on $\mathbf{S}^{n-1}$ if b is a function supported on some cap $I = B(x_0',\theta_0)$ with $\|b\|_{L^q} \le |I|^{-1/q'}$, where $|I| = \sigma(I)$ and $1/q + 1/q' = 1$.

(3) $B_q^{\kappa,v}(\mathbf{S}^{n-1}) = \{\Omega \in L^1(\mathbf{S}^{n-1}) : \Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu$, where each c_μ is a complex number; each b_μ is a q-block supported on a cap I_μ on $\mathbf{S}^{n-1}$; and $M_q^{\kappa,v}(\{c_\mu\},\{I_\mu\}) = \sum_{\mu=1}^{\infty} |c_\mu|(1 + \phi_{\kappa,v}(|I_\mu|)) < \infty$, where $\phi_{\kappa,v}(t) = \int_t^1 u^{-1-\kappa}\log^v(u^{-1})du$ if $0 < t < 1$ and $\phi_{\kappa,v}(t) = 0$ if $t \ge 1\}$.

Notice that $\phi_{\kappa,v}(t) \sim t^{-\kappa}\log^v(t^{-1})$ as $t \to 0$ for $\kappa > 0$, $v \in \mathbb{R}$, and $\phi_{0,v}(t) \sim \log^{v+1}(t^{-1})$ as $t \to 0$ for $v > -1$. Moreover, among many properties of block spaces [17], we cite the following which are closely related to our work:

$$\begin{gathered}
B_q^{0,0} \subset B_q^{0,-1/2} \quad (q > 1),\\
B_{q_2}^{0,v} \subset B_{q_1}^{0,v} \quad (1 < q_1 < q_2),\\
L^q(\mathbf{S}^{n-1}) \subseteq B_q^{0,v}(\mathbf{S}^{n-1}) \quad (\text{for } v > -1),\\
\bigcup_{q>1} B_q^{0,v}(\mathbf{S}^{n-1}) \nsubseteq \bigcup_{p>1} L^p(\mathbf{S}^{n-1}) \quad \text{for any } v > -1.
\end{gathered} \tag{5.1}$$

Proof of Theorem 1.3. Assume that $\Omega \in B_q^{0,-1/2}(\mathbf{S}^{n-1})$, $q > 1$. Then

$$\Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu, \tag{5.2}$$

where each c_μ is a complex number; each b_μ is a q-block supported on a cap I_μ on $\mathbf{S}^{n-1}$; and

$$M_q^{0,-1/2}(\{c_\mu\},\{I_\mu\}) = \sum_{\mu=1}^{\infty} |c_\mu| \left(1+\log^{1/2}\left(|I_\mu|^{-1}\right)\right) < \infty. \tag{5.3}$$

Without loss of generality, we may assume that $|I_\mu| < 1$. For each μ, let

$$\bar{b}_\mu(x) = b_\mu(x) - \int_{\mathbf{S}^{n-1}} b_\mu(u)du. \tag{5.4}$$

Then, it follows that

$$\|\bar{b}_\mu\|_q \le C|I|^{-1/q'}, \qquad \|\bar{b}_\mu\|_1 \le C. \tag{5.5}$$

By the decomposition (5.3), we have

$$\mathcal{M}_{\Omega,P}(f)(x) \le \sum_{\mu=1}^{\infty} |c_\mu| \mathcal{M}_{\bar{b}_\mu,P}(f)(x). \tag{5.6}$$

Thus, by Minkowski's inequality, (5.5), and Theorem 1.1, we have

$$\begin{aligned} \|\mathcal{M}_{\Omega,P}(f)\|_p &\le C_p \sum_{\mu=1}^{\infty} |c_\mu| \{1+\log^{1/2}(e+|I|^{-1/q'})\} \|f\|_p \\ &\le C_{p,q} \sum_{\mu=1}^{\infty} |c_\mu| \left(1+\log^{1/2}\left(|I_\mu|^{-1}\right)\right) \|f\|_p \\ &\le \widetilde{C}_{p,q} \|f\|_p \end{aligned} \tag{5.7}$$

for all $p \ge 2$, where the last inequality follows by (5.3). This completes the proof. □

A proof of Corollary 1.6 can be obtained by a similar argument as in the proof of Corollary 1.5. We omit the details.

6. Further applications

This section is devoted to present some results that follow by applying our results in Section 1.

Parametric Marcinkiewicz integral operators. The parametric Marcinkiewicz integral operator related to the operator $\mathcal{M}_{\Omega,P}$ is defined by

$$\mu_{\Omega,P}^{\rho} f(x) = \left(\int_{-\infty}^{\infty} \left| 2^{-\rho t} \int_{|y|\le 2^t} e^{iP(y)} f(x-y)|y|^{-n+\rho}\Omega(y)dy \right|^2 dt\right)^{1/2}, \tag{6.1}$$

where ρ is a positive real number. Clearly, when $P=0$, the operator $\mu_{\Omega}^{\rho} = \mu_{\Omega,0}^{\rho}$ is the well-known parametric Marcinkiewicz integral operator introduced by Hörmander [15].

Now, it is straightforward to see that

$$\mu^{\rho}_{\Omega,P} f(x) \leq C(\rho)\mathcal{M}_{\Omega,P} f(x). \tag{6.2}$$

Therefore, by (6.2), Theorem 1.1, and the decompositions (4.1) and (5.2), we can easily obtain the following theorem.

THEOREM 6.1. *Suppose that $\rho > 0$ and that $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ satisfying (1.1). Then the parametric Marcinkiewicz integral operator $\mu^{\rho}_{\Omega,P}$ is bounded on L^p for all $p \geq 2$ with L^p bounds that may depend on the degree of the polynomial P but they are independent of the function Ω and the coefficients of the polynomial P.*

THEOREM 6.2. *Suppose that $\rho > 0$ and that $\Omega \in B^{0,-1/2}_{q}(\mathbf{S}^{n-1})$, $q > 1$, satisfying (1.1). Then the parametric Marcinkiewicz integral operator $\mu^{\rho}_{\Omega,P}$ is bounded on L^p for all $p \geq 2$ with L^p bounds that may depend on the degree of the polynomial P but they are independent of the function Ω and the coefficients of the polynomial P.*

We remark here that by specializing to the case $P = 0$ and $\rho = 1$, the resulting operator $\mu_{\Omega} = \mu^{1}_{\Omega,0}$ is the classical Marcinkiewicz integral operator introduced by Stein [25]. Thus, Theorems 6.1 and 6.2 generalize as well as improve the result in (see [25]). Furthermore, Theorems 6.1 and 6.2 generalize the corresponding results in [2, 3, 8]. For more background information and related results about Marcinkiewicz integral operators, we refer the readers to consult [6, 8, 15, 25], and the references therein.

Morrey spaces. In [20], Mizuhara introduced the following generalized Morrey spaces.

Definition 6.3. Let $\phi : (0,\infty) \to (0,\infty)$ be an increasing function that satisfies $\phi(2r) \leq D\phi(r)$ for any $r > 0$, where D is a constant independent of r. Let $1 \leq p < \infty$. A locally integrable function $f \in L^{p,\phi}$ if

$$\int_{B_r(x_0)} |f(x)|^p dx \leq C^p \phi(r) \tag{6.3}$$

for all $x_0 \in \mathbb{R}^n$ and $r > 0$, where $B_r(x_0)$ is the ball with center x_0 and radius r.

It is worth pointing out here that Morrey spaces have been used to study several problems in harmonic analysis, such as studying the local behavior of solutions to second-order elliptic partial differential equations and measuring the regularity of the solution to an elliptic second-order equation with discontinuous coefficients; see [13, 21], and references therein.

By Theorem 1.1, the decompositions (4.1) and (5.2), and following a similar argument as in the proof of Theorem 5 in [13], we obtain the following theorem.

THEOREM 6.4. *Suppose that $\Omega \in L(\log^{+} L)^{1/2}(\mathbf{S}^{n-1}) \cup B^{0,-1/2}_{q}(\mathbf{S}^{n-1})$, $q > 1$, satisfying (1.1). Let $P : \mathbb{R}^n \to \mathbb{R}$ be a real-valued polynomial. Then $\mathcal{M}_{\Omega,P}$ is bounded on $L^{p,\phi}(\mathbb{R}^n)$ for all $p \geq 2$ with L^p bounds independent of the coefficients of the polynomial P.*

Hence, by (6.2) and Theorem 6.4, we obtain that the operator $\mu^{\rho}_{\Omega,P}$ is bounded on $L^{p,\phi}(\mathbb{R}^n)$ for all $p \geq 2$ with L^p bounds independent of the coefficients of the polynomial

P. Moreover, by (1.5) and Theorem 6.4, it follows that the operator $T_{\Omega,P,h}$ is bounded on $L^{p,\phi}$ for all $1 < p < \infty$ and $h \in L^2(\mathbb{R}_+, r^{-1}dr)$.

L^p estimates with radial weights. The results in this paper can be easily extended to the radial weights setting introduced by Duoandikoetxea [9]. In order to state our weighted L^p estimates, we recall the definition of the radial weights [9, 13].

Definition 6.5. Suppose that $\omega(t) \geq 0$ and $\omega \in L^1_{\text{loc}}(\mathbb{R}^+)$. For $1 < p < \infty$, $\omega \in A_p(\mathbb{R}^+)$ if there is a constant $C > 0$ such that for any interval $I \subseteq \mathbb{R}^+$,

$$\left(|I|^{-1}\int_I \omega(t)dt\right)\left(|I|^{-1}\int_I \omega(t)^{-1/(p-1)}dt\right) \leq C < \infty. \tag{6.4}$$

If there is a constant $C > 0$ such that

$$\omega^*(t) \leq C\omega(t) \quad \text{for a.e. } t \in \mathbb{R}^+, \tag{6.5}$$

where ω^* is the Hardy-Littlewood maximal function of ω on $\mathbb{R}^+$, then $\omega \in A_1(\mathbb{R}^+)$.

We let $\widetilde{A}_p(\mathbb{R}^+)$ be the class of functions ω that can be written as follows: $\omega(x) = \nu_1(|x|)\nu_2(|x|)^{1-p}$, where either $\nu_i \in A_1(\mathbb{R}^+)$ is decreasing or $\nu_i^2 \in A_1(\mathbb{R}^+)$, $i = 1,2$. Also, for $1 < p < \infty$, we let

$$\bar{A}_p(\mathbb{R}^+) = \{\omega(x) = \omega(|x|) : \omega(t) > 0,\ \omega(t) \in L^1_{\text{loc}}(\mathbb{R}^+),\ \omega^2(t) \in A_p(\mathbb{R}^+)\} \tag{6.6}$$

and let $A_p^I(\mathbb{R}^n)$ be the weighted class defined by using all n-dimensional intervals with sides parallel to coordinate axes. The weighted L^p space $L^p(\mathbb{R}^n, \omega(x)dx)$ associated to the weight ω is defined to be the class of all measurable functions f with $\|f\|_{L^p(\omega)} < \infty$, where

$$\|f\|_{L^p(\omega)} = \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx\right)^{1/p}. \tag{6.7}$$

It is known that $\bar{A}_p(\mathbb{R}^+) \subseteq \widetilde{A}_p(\mathbb{R}^+)$; see [13]. Moreover, if $\omega(t) \in \bar{A}_p(\mathbb{R}^+)$, then $\omega(|x|)$ is in Muckenhoupt weighted class $A_p(\mathbb{R}^n)$ whose definition can be found in [14].

By the same argument in this paper with minor modifications, it can be easily shown that the weighted version of all L^p estimates obtained in this paper holds. In particular, we have the following theorem.

Theorem 6.6. *Suppose that $\rho > 0$ and that $\Omega \in L(\log^+ L)^{1/2}(\mathbf{S}^{n-1}) \cup B_q^{0,-1/2}(\mathbf{S}^{n-1})$, $q > 1$, satisfying (1.1). Let $P : \mathbb{R}^n \to \mathbb{R}$ be a real-valued polynomial. If $\omega \in \widetilde{A}_{p/2} \cap A_{p/2}^I$, $2 \leq p < \infty$, then the operators $\mathcal{M}_{\Omega,P}$ and $\mu^{\rho}_{\Omega,P}$ are bounded on $L^p(\omega)$ with L^p bounds independent of the coefficients of the polynomial P.*

A special class of radial weights that have received a considerable amount of attention is the class of power weights $|x|^{\alpha}$. For background information and related results on power weights, we refer the readers to consult [9, 13], among others. By the observation that $|x|^{\alpha} \in \widetilde{A}_{p/2} \cap A_{p/2}^I$ if $\alpha \in (-1, p/2 - 1)$, it follows from Theorem 6.6 that the following holds.

Corollary 6.7. *Suppose that* $\rho > 0$ *and that* $\Omega \in L(\log^+ L)^{1/2}(\mathbf{S}^{n-1}) \cup B_q^{0,-1/2}(\mathbf{S}^{n-1})$, $q > 1$, *satisfying (1.1). Let* $P : \mathbb{R}^n \to \mathbb{R}$ *be a real-valued polynomial. Then the operators* $\mathcal{M}_{\Omega,P}$ *and* $\mu^{\rho}_{\Omega,P}$ *are bounded on* $L^p(|x|^\alpha)$ *if* $\alpha \in (-1, p/2 - 1)$ *with* $L^p(|x|^\alpha)$ *bounds independent of the coefficients of the polynomial* P.

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