PERIODIC SOLUTIONS OF SECOND-ORDER LIÉNARD EQUATIONS WITH p-LAPLACIAN-LIKE OPERATORS

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Received 12 April 2005; Accepted 10 August 2005

The existence of periodic solutions for second-order Liénard equations with p-Laplacian-like operator is studied by applying new generalization of polar coordinates.

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1. Introduction

In recent years, the existence of periodic solutions for second-order Liénard equations

$$u'' + f(u,u')u' + g(u) = e(t,u,u') \tag{1.1}$$

and its special case have been studied by many researchers, we refer the readers to [1, 3, 4, 6, 7, 9–12] and the references therein.

Let us consider the so-called one-dimensional p-Laplacian operator $(\phi_p(u'))'$, where $p > 1$ and $\phi_p : \mathbb{R} \to \mathbb{R}$ is given by $\phi_p(s) = |s|^{p-2}s$ for $s \neq 0$ and $\phi_p(0) = 0$. Periodic boundary conditions containing this operator have been considered in [2, 5].

In [8], Manásevich and Mawhin investigated the existence of periodic solutions to some system cases involving the fairly general vector-valued operator ϕ. They considerd the boundary value problem

$$(\phi(u'))' = f(t,u,u'), \quad u(0) = u(T),\ u'(0) = u'(T), \tag{1.2}$$

where the function $\phi : \mathbb{R}^N \to \mathbb{R}^N$ satisfies some monotonicity conditions which ensure that ϕ is a homeomorphism onto $\mathbb{R}^N$.

Recently, in [16] we studied the existence of periodic solutions for the nonlinear differential equation with a p-Laplacian-like operator

$$(\phi(u'))' + f(t,u,u') = 0. \tag{1.3}$$

Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2006, Article ID 98685, Pages 1–17
DOI 10.1155/JIA/2006/98685

Motivated by the work of [13], in this paper we use new polar coordinates [13] to investigate the existence of periodic solutions for the second-order generalized Liénard equations with p-Laplacian-like operator

$$(\phi(u'))' + f(u,u')u' + g(u) = e(t,u,u'), \quad t \in [0,T]. \tag{1.4}$$

Throughout this paper, we always assume that ϕ, $g \in \mathbb{C}(\mathbb{R},\mathbb{R})$, $f \in \mathbb{C}(\mathbb{R}^2,\mathbb{R})$, $e \in \mathbb{C}([0,T]\times\mathbb{R}^2,\mathbb{R})$. And the following conditions also hold.

(H1) ϕ is continuous and strictly increasing, $y\phi(y) > 0$ for $y \neq 0$, and there exist $p > 2$, $m_2 \geq m_1 > 0$, such that

$$m_1|y|^{p-1} \leq |\phi(y)| \leq m_2|y|^{p-1}. \tag{1.5}$$

(H2) $e \in \mathbb{C}([0,T]\times\mathbb{R}^2,\mathbb{R})$, periodic in t with period T, there exist $\alpha_1,\beta_1,\gamma_1 > 0$, and $p > k > 2$ such that

$$|e(t,x,y)| \leq \alpha_1|x|^{p-1} + \beta_1|y|^{k-1} + \gamma_1 \quad \text{for } (t,x,y) \in [0,T]\times\mathbb{R}^2. \tag{1.6}$$

(H3) $f \in \mathbb{C}(\mathbb{R}^2,\mathbb{R})$, there exist $\alpha_2,\beta_2,\gamma_2 > 0$ such that

$$|f(x,y)| \leq \alpha_2|x|^{p-2} + \beta_2|y|^{k-2} + \gamma_2 \quad \text{for } (x,y) \in \mathbb{R}^2. \tag{1.7}$$

(H4) There exist λ,μ, and $n \geq 0$ such that

$$\begin{aligned}
&\frac{m_2}{m_1}\left(\frac{p'}{p'-1}\right)^{p-1}\left(\frac{2n\pi_p}{T}\right)^p + \frac{\alpha_1}{m_1} + \frac{p-1}{p}\left(\frac{\alpha_2}{m_1}\right)^{p/(p-1)}\left(\frac{m_2}{m_1}\right)^{1/(p-1)^2} < \lambda \\
&\quad \leq \frac{g(x)}{\phi(x)} \leq \mu < \frac{m_1}{m_2}\left(\frac{p'}{p'+1}\right)^{p-1}\left(\frac{2(n+1)\pi_p}{T}\right)^p \\
&\quad - \frac{\alpha_1}{m_2} - \frac{p-1}{p}\left(\frac{\alpha_2}{m_2}\right)^{p/(p-1)}\left(\frac{m_2}{m_1}\right)^{1/(p-1)},
\end{aligned} \tag{1.8}$$

where

$$p' = p(p-1), \qquad \pi_p = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}. \tag{1.9}$$

(H5) Solutions of (1.4) are unique with respect to initial value.

In this paper, we use a new coordinate to estimate the time when a point moves along a trajectory around the origin and then give some sufficient conditions for the existence of periodic solutions of (1.4).

2. Periodic solutions with a Laplacian-like operator

Let $v = \phi(u')$. Then (1.4) is equivalent to the system

$$\begin{aligned}
u' &= \phi^{-1}(v), \\
v' &= -g(u) - f(u,\phi^{-1}(v))\phi^{-1}(v) + e(t,u,\phi^{-1}(v)).
\end{aligned} \tag{2.1}$$

Let $u(t,\xi,\eta)$ denote the solution of (1.4) which satisfies the initial value condition

$$u(0,\xi,\eta)=\xi, \qquad v(0,\xi,\eta)=\eta, \tag{2.2}$$

then we have the following conclusion.

Lemma 2.1. *Suppose (H1)–(H5) hold, then for all $c>0$, there exists constant $A>0$ such that if*

$$\frac{1}{p}|\xi|^p+\frac{p-1}{p}|\eta|^{p/(p-1)}=A^2, \tag{2.3}$$

then

$$\frac{1}{p}\left|u(t,\xi,\eta)\right|^p+\frac{p-1}{p}\left|v(t,\xi,\eta)\right|^{p/(p-1)}\geq c^2 \quad \text{for } t\in[0,T]. \tag{2.4}$$

Proof. Let $(u(t),v(t))$, $t\in[0,T]$, be a solution of (2.1) satisfying $u(0,\xi,\eta)=\xi$, $v(0,\xi,\eta)=\eta$.

Let

$$r^2(t)=\frac{1}{p}\left|u(t)\right|^p+\frac{p-1}{p}\left|v(t)\right|^{p/(p-1)}. \tag{2.5}$$

It is clear that (H1) implies

$$\left(\frac{|v|}{m_2}\right)^{1/(p-1)}\leq\left|\phi^{-1}(v)\right|\leq\left(\frac{|v|}{m_1}\right)^{1/(p-1)}. \tag{2.6}$$

So we have

$$\begin{aligned}
\left|\frac{dr^2(t)}{dt}\right| &= \left|\left|u(t)\right|^{p-2}u(t)u'(t)+\left|v(t)\right|^{(2-p)/(p-1)}v(t)v'(t)\right|\\
&\leq |u|^{p-1}\left|\phi^{-1}(v)\right|+|v|^{1/(p-1)}\left|-g(u)-f(u,\phi^{-1}(v))\phi^{-1}(v)+e(t,u,\phi^{-1}(v))\right|\\
&\leq |u|^{p-1}\left|\phi^{-1}(v)\right|+\mu|v|^{1/(p-1)}\left|\phi(u)\right|\\
&\quad+|v|^{1/(p-1)}\left(\alpha_2|u|^{p-2}+\beta_2\left|\phi^{-1}(v)\right|^{k-2}+\gamma_2\right)\left|\phi^{-1}(v)\right|\\
&\quad+|v|^{1/(p-1)}\left(\alpha_1|u|^{p-1}+\beta_1\left|\phi^{-1}(v)\right|^{k-1}+\gamma_1\right)\\
&\leq |u|^{p-1}\left(\frac{|v|}{m_1}\right)^{1/(p-1)}+\mu m_2|v|^{1/(p-1)}|u|^{p-1}\\
&\quad+\alpha_2 m_1^{-1/(p-1)}|v|^{2/(p-1)}|u|^{p-2}+\beta_2 m_1^{(1-k)/(p-1)}|v|^{k/(p-1)}\\
&\quad+\gamma_2 m_1^{-1/(p-1)}|v|^{2/(p-1)}+\alpha_1|v|^{1/(p-1)}|u|^{p-1}\\
&\quad+\beta_1 m_1^{(1-k)/(p-1)}|v|^{k/(p-1)}+\gamma_1|v|^{1/(p-1)}\\
&= l_1|u|^{p-1}|v|^{1/(p-1)}+l_2|v|^{k/(p-1)}+l_3|v|^{2/(p-1)}|u|^{p-2}+l_4|v|^{2/(p-1)}+\gamma_1|v|^{1/(p-1)},
\end{aligned} \tag{2.7}$$

where

$$l_1 = m_1^{-1/(p-1)} + \mu m_2 + \alpha_1, \qquad l_2 = \beta_1 m_1^{(1-k)/(p-1)} + \beta_2 m_1^{(1-k)/(p-1)},$$
$$l_3 = \alpha_2 m_1^{-1/(p-1)}, \qquad l_4 = \gamma_2 m_1^{-1/(p-1)}, \tag{2.8}$$

while

$$\begin{aligned}
l_1|u|^{p-1}|v|^{1/(p-1)} &\leq l_1\left(\frac{1}{p}|v|^{p/(p-1)} + \frac{p-1}{p}|u|^p\right)\\
&\leq l_1 \max\left\{p-1, \frac{1}{p-1}\right\}\left(\frac{1}{p}|u|^p + \frac{p-1}{p}|v|^{p/(p-1)}\right)\\
&= l_1 \max\left\{p-1, \frac{1}{p-1}\right\} r^2,\\
l_2|v|^{k/(p-1)} &\leq \frac{k}{p}|v|^{p/(p-1)} + \frac{p-k}{p} l_2^{p/(p-k)} \leq \frac{k}{p-1} r^2 + \frac{p-k}{p} l_2^{p/(p-k)}\\
l_3|v|^{2/(p-1)}|u|^{p-2} &\leq l_3\left(\frac{2}{p}|v|^{p/(p-1)} + \frac{p-2}{p}|u|^p\right) \leq l_3\left(\frac{2}{p-1} + p - 2\right) r^2,\\
l_4|v|^{2/(p-1)} &\leq \frac{2}{p}|v|^{p/(p-1)} + \frac{p-2}{p} l_4^{p/(p-2)} \leq \frac{2}{p-1} r^2 + \frac{p-2}{p} l_4^{p/(p-2)},\\
\gamma_1|v|^{1/(p-1)} &\leq \frac{1}{p}|v|^{p/(p-1)} + \frac{p-1}{p}\gamma_1^{p/(p-1)} \leq \frac{1}{p-1} r^2 + \frac{p-1}{p}\gamma_1^{p/(p-1)}.
\end{aligned} \tag{2.9}$$

So,

$$\left|\frac{dr^2(t)}{dt}\right| \leq br^2(t) + a, \tag{2.10}$$

where

$$a = \frac{p-k}{p} l_2^{p/(p-k)} + \frac{p-2}{p} l_4^{p/(p-2)} + \frac{p-1}{p}\gamma_1^{p/(p-1)},$$
$$b = l_1 \max\left\{p-1, \frac{1}{p-1}\right\} + l_3\left(\frac{2}{p-1} + p - 2\right) + \frac{k+3}{p-1}. \tag{2.11}$$

It follows that

$$\begin{aligned}
\left(r^2(0) + \frac{a}{b}\right)e^{-bT} &\leq \left(r^2(0) + \frac{a}{b}\right)e^{-bt} \leq \left(r^2(t) + \frac{a}{b}\right)\\
&\leq \left(r^2(0) + \frac{a}{b}\right)e^{bt} \leq \left(r^2(0) + \frac{a}{b}\right)e^{bT}, \quad 0 \leq t \leq T.
\end{aligned} \tag{2.12}$$

Let $A = [(c^2 + a/b)e^{bT} - a/b]^{1/2}$, then $r(0) = A$ implies $r(t) \geq c$. □

Lemma 2.2. *Let $(u(t),v(t))$ be a solution of (2.1). Suppose the conditions of (H1)–(H5) are satisfied. Then there is R such that under the generalized polar coordinates, $r(0)\geq R$ implies that*

$$\frac{d\theta(t)}{dt}\leq 0,\quad t\in[0,T]. \tag{2.13}$$

Proof. Applying generalized polar coordinates,

$$\begin{aligned} u&=p^{1/p}r^{2/p}|\cos\theta|^{(2-p)/p}\cos\theta,\\ v&=\left(\frac{p}{p-1}\right)^{(p-1)/p}r^{2(p-1)/p}|\sin\theta|^{(p-2)/p}\sin\theta, \end{aligned} \tag{2.14}$$

or

$$\begin{aligned} r\cos\theta&=\frac{1}{\sqrt{p}}|u|^{(p-2)/2}u,\\ r\sin\theta&=\sqrt{\frac{p-1}{p}}|v|^{(2-p)/2(p-1)}v. \end{aligned} \tag{2.15}$$

Then $\theta=\tan^{-1}[\sqrt{p-1}(|v|^{((2-p)/2(p-1))}v/|u|^{((p-2)/2)}u)]$. So we have

$$\begin{aligned} \theta'&=\frac{|u|^{((p-2)/2)}|v|^{((2-p)/2(p-1))}}{2\sqrt{p-1}r^2}[uv'-(p-1)u'v]\\ &=-\frac{|u|^{((p-2)/2)}|v|^{((2-p)/2(p-1))}}{2\sqrt{p-1}r^2}\Big[ug(u)+uf(u,\phi^{-1}(v))\phi^{-1}(v)\\ &\qquad+(p-1)v\phi^{-1}(v)-ue(t,u,\phi^{-1}(v))\Big] \end{aligned} \tag{2.16}$$

as

$$\begin{aligned} &ug(u)+uf(u,\phi^{-1}(v))\phi^{-1}(v)+(p-1)v\phi^{-1}(v)-ue(t,u,\phi^{-1}(v))\\ &\quad\geq\lambda u\phi(u)+(p-1)v\phi^{-1}(v)-|u|\Big(\alpha_2|u|^{p-2}+\beta_2\left|\phi^{-1}(v)\right|^{k-2}+\gamma_2\Big)\left|\phi^{-1}(v)\right|\\ &\qquad-|u|\Big(\alpha_1|u|^{p-1}+\beta_1\left|\phi^{-1}(v)\right|^{k-1}+\gamma_1\Big)\\ &\quad\geq\lambda m_1|u|^p+(p-1)m_2^{-1/(p-1)}|v|^{p/(p-1)}-\alpha_2m_1^{-1/(p-1)}|u|^{p-1}|v|^{1/(p-1)}\\ &\qquad-\gamma_2m_1^{-1/(p-1)}|u||v|^{1/(p-1)}-\alpha_1|u|^p-(\beta_1+\beta_2)m_1^{(1-k)/(p-1)}|u||v|^{(k-1)/(p-1)}-\gamma_1|u|\\ &\quad=(\lambda m_1-\alpha_1)|u|^p+(p-1)m_2^{-1/(p-1)}|v|^{p/(p-1)}-\alpha_2m_1^{-1/(p-1)}|u|^{p-1}|v|^{1/(p-1)}\\ &\qquad-\gamma_2m_1^{-1/(p-1)}|u||v|^{1/(p-1)}-(\beta_1+\beta_2)m_1^{(1-k)/(p-1)}|u||v|^{(k-1)/(p-1)}-\gamma_1|u|. \end{aligned} \tag{2.17}$$

Let

$$\tau = \frac{p(p-1)}{4(k-1)} m_2^{-1/(p-1)}, \qquad \beta' = \frac{4(\beta_1+\beta_2)(k-1)}{p(p-1)} m_1^{(1-k)/(p-1)} m_2^{1/(p-1)}, \tag{2.18}$$

so we have

$$\begin{aligned}
(\beta_1+\beta_2)&m_1^{(1-k)/(p-1)}|u||v|^{(k-1)/(p-1)} \\
&= \tau|u|\left(|v|^{(k-1)/(p-1)}\beta'\right) \le \tau|u|\left(\frac{k-1}{p-1}|v| + \frac{p-k}{p-1}\beta'^{(p-1)/(p-k)}\right) \\
&= \frac{1}{4}pm_2^{-1/(p-1)}|u||v| + \frac{p(p-k)}{4(k-1)}m_2^{-1/(p-1)}\beta'^{(p-1)/(p-k)}|u| \\
&\le \frac{1}{4}pm_2^{-1/(p-1)}\left(\frac{1}{p}|u|^p + \frac{p-1}{p}|v|^{p/(p-1)}\right) + \frac{p(p-k)}{4(k-1)}m_2^{-1/(p-1)}\beta'^{(p-1)/(p-k)}|u|.
\end{aligned} \tag{2.19}$$

Let

$$\tau_1 = \frac{1}{4}p(p-1)m_2^{-1/(p-1)}, \qquad \beta_1' = \frac{4\gamma_2}{p(p-1)}\left(\frac{m_2}{m_1}\right)^{1/(p-1)}, \tag{2.20}$$

then

$$\begin{aligned}
\gamma_2 m_1^{-1/(p-1)}|u||v|^{1/(p-1)} &= \tau_1|u|\left(|v|^{1/(p-1)}\beta_1'\right) \\
&\le \tau_1|u|\left(\frac{1}{p-1}|v| + \frac{p-2}{p-1}\beta_1'^{(p-1)/(p-2)}\right) \\
&= \frac{1}{4}pm_2^{-1/(p-1)}|u||v| + \frac{p(p-2)}{4}m_2^{-1/(p-1)}\beta_1'^{(p-1)/(p-2)}|u| \\
&\le \frac{1}{4}pm_2^{-1/(p-1)}\left(\frac{1}{p}|u|^p + \frac{p-1}{p}|v|^{p/(p-1)}\right) \\
&\quad + \frac{p(p-2)}{4}m_2^{-1/(p-1)}\beta_1'^{(p-1)/(p-2)}|u|.
\end{aligned} \tag{2.21}$$

Let

$$\tau_2 = \frac{1}{4}p(p-1)m_2^{-1/(p-1)}, \qquad \beta_2' = \frac{4\alpha_2}{p(p-1)}\left(\frac{m_2}{m_1}\right)^{1/(p-1)} \tag{2.22}$$

then

$$\begin{aligned}\alpha_2 m_1^{-1/(p-1)}|u|^{p-1}|v|^{1/(p-1)} &= \tau_2\left(|v|^{1/(p-1)}\beta_2'|u|^{p-1}\right) \le \tau_2\left(\frac{1}{p}|v|^{p/(p-1)} + \frac{p-1}{p}\left(\beta_2'|u|^{p-1}\right)^{p/(p-1)}\right) \\ &\le \frac{1}{4}pm_2^{-1/(p-1)}\left(\frac{1}{p}|u|^p + \frac{p-1}{p}|v|^{p/(p-1)}\right) + \frac{p-1}{p}\tau_2\beta_2'^{p/(p-1)}|u|^p.\end{aligned} \tag{2.23}$$

We select λ large enough such that

$$\delta = \lambda m_1 - \alpha_1 - \frac{p-1}{p}\tau_2\beta_2'^{p/(p-1)} - m_2^{-1/(p-1)} > 0, \tag{2.24}$$

Let $d = \gamma_1 + (p(p-k)/4(k-1))m_2^{-1/(p-1)}\beta'^{(p-1)/(p-k)} + (p(p-2)/4)m_2^{-1/(p-1)}\beta_1'^{(p-1)/(p-2)}$, we also have

$$d|u| = \delta p|u|\left(\frac{d}{\delta p}\right) \le \delta|u|^p + (p-1)\delta\left(\frac{d}{p\delta}\right)^{p/(p-1)}, \tag{2.25}$$

therefore

$$\begin{aligned}&ug(u) + uf\left(u,\phi^{-1}(v)\right)\phi^{-1}(v) + (p-1)v\phi^{-1}(v) - ue\left(t,u,\phi^{-1}(v)\right) \\ &\quad\ge \frac{1}{4}pm_2^{-1/(p-1)}\left[\frac{1}{p}|u|^P p + \frac{p-1}{p}|v|^{p/(p-1)}\right] - (p-1)\delta\left(\frac{d}{p\delta}\right)^{p/(p-1)} \\ &\quad= \frac{1}{4}pm_2^{-1/(p-1)}r^2(t) - (p-1)\delta\left(\frac{d}{p\delta}\right)^{p/(p-1)}.\end{aligned} \tag{2.26}$$

Lemma 2.1 implies that there is $\mathbb{R} > 0$, such that

$$\frac{1}{4}pm_2^{-1/(p-1)}r^2(t) > (p-1)\delta\left(\frac{d}{p\delta}\right)^{p/(p-1)} \tag{2.27}$$

when $r(0) > \mathbb{R}$, then our assertion is verified. □

LEMMA 2.3. *Assume that (H1)–(H5) hold, and*

$$\frac{1}{p}|\xi|^p + \frac{p-1}{p}|\eta|^{p/(p-1)} = A^2 \quad (A \gg 1) \tag{2.28}$$

then

$$(u(T,\xi,\eta),v(T,\xi,\eta)) \neq (\lambda^{2/p}\xi,\lambda^{2(p-1)/p}\eta), \tag{2.29}$$

where λ is an arbitrary positive number.

Proof. It follows from Lemma 2.1 that if

$$\frac{1}{p}|\xi|^p + \frac{p-1}{p}|\eta|^{p/(p-1)} = A^2, \tag{2.30}$$

then

$$\frac{1}{p}\left|u(t,\xi,\eta)\right|^p + \frac{p-1}{p}\left|v(t,\xi,\eta)\right|^{p/(p-1)} \geq c^2 \quad \text{for } t \in [0,T]. \tag{2.31}$$

According to the generalized polar coordinates (2.14), we have

$$r(t) \geq c \quad \text{for } t \in [0,T] \text{ if } r(0) = A. \tag{2.32}$$

On the other hand, when $r(0) \to \infty$, it holds uniformly from (H1)–(H3) that

$$\begin{aligned}
-\theta' &= \frac{|u|^{(p-2)/2}|v|^{(2-p)/2(p-1)}}{2\sqrt{p-1}r^2}\Big[ug(u) + uf(u,\phi^{-1}(v))\phi^{-1}(v) \\
&\qquad + (p-1)v\phi^{-1}(v) - ue(t,u,\phi^{-1}(v))\Big] \\
&\geq \frac{|u|^{(p-2)/2}|v|^{(2-p)/2(p-1)}}{2\sqrt{p-1}r^2}\Big[(\lambda m_1 - \alpha_1)|u|^p + (p-1)m_2^{-1/(p-1)}|v|^{p/(p-1)} \\
&\qquad - \alpha_2 m_1^{-1/(p-1)}|u|^{p-1}|v|^{1/(p-1)} - \gamma_2 m_1^{-1/(p-1)}|u||v|^{1/(p-1)} \\
&\qquad - (\beta_1+\beta_2)m_1^{(1-k)/(p-1)}|u||v|^{(k-1)/(p-1)} - \gamma_1|u|\Big]
\end{aligned} \tag{2.33}$$

as

$$\begin{aligned}
&\alpha_2 m_1^{-1/(p-1)}|u|^{p-1}|v|^{1/(p-1)} \\
&\quad = m_2^{-1/(p-1)}\left(|v|^{1/(p-1)}\right)\left[\alpha_2\left(\frac{m_2}{m_1}\right)^{1/(p-1)}|u|^{p-1}\right] \\
&\quad \leq m_2^{-1/(p-1)}\left[\frac{1}{p}|v|^{p/(p-1)} + \frac{p-1}{p}\alpha_2^{p/(p-1)}\left(\frac{m_2}{m_1}\right)^{p/(p-1)^2}|u|^p\right] \\
&\quad = \frac{1}{p}m_2^{-1/(p-1)}|v|^{p/(p-1)} + \frac{p-1}{p}\alpha_2^{p/(p-1)}m_1^{-p/(p-1)^2}m_2^{1/(p-1)^2}|u|^p.
\end{aligned} \tag{2.34}$$

So

$$
\begin{aligned}
-\theta' \geq{}& \frac{|u|^{(p-2)/2}|v|^{(2-p)/2(p-1)}}{2\sqrt{p-1}r^2}\Bigg[(\lambda m_1-\alpha_1-\tilde{\alpha})|u|^p+\frac{p'-1}{p'}(p-1)m_2^{-1/(p-1)}|v|^{p/(p-1)} \\
&\qquad -\gamma_2 m_1^{-1/(p-1)}|u||v|^{1/(p-1)} \\
&\qquad -(\beta_1+\beta_2)m_1^{(1-k)/(p-1)}|u||v|^{(k-1)/(p-1)}-\gamma_1|u|\Bigg] \\
={}& \frac{p|\sin\theta|^{(2-p)/p}|\cos\theta|^{(p-2)/p}}{2(p-1)^{1/p}}\left[(\lambda m_1-\alpha_1-\tilde{\alpha})\cos^2\theta+\frac{p'-1}{p'}m_2^{-1/(p-1)}\sin^2\theta\right] \\
&-\frac{\gamma_2 m_1^{-1/(p-1)}p^{2/p}}{2(p-1)^{2/p}r^{2(p-2)/p}}|\cos\theta||\sin\theta|^{(4-p)/p} \\
&-\frac{(\beta_1+\beta_2)m_1^{(1-k)/(p-1)}p^{k/p}}{2(p-1)^{k/p}r^{2(p-k)/p}}|\cos\theta||\sin\theta|^{(2k-p)/p} \\
&-\frac{\gamma_1 p^{1/p}}{2(p-1)^{1/p}r^{2(p-1)/p}}|\cos\theta||\sin\theta|^{(2-p)/p} \\
={}& a_1\left(b_1\cos^2\theta+\sin^2\theta\right)|\sin\theta|^{(2-p)/p}|\cos\theta|^{(p-2)/p} \\
&-\frac{\gamma_2 m_1^{-1/(p-1)}p^{2/p}}{2(p-1)^{2/p}r^{2(p-2)/p}}|\cos\theta||\sin\theta|^{(4-p)/p} \\
&-\frac{(\beta_1+\beta_2)m_1^{(1-k)/(p-1)}p^{k/p}}{2(p-1)^{k/p}r^{2(p-k)/p}}|\cos\theta||\sin\theta|^{(2k-p)/p} \\
&-\frac{\gamma_1 p^{1/p}}{2(p-1)^{1/p}r^{2(p-1)/p}}|\cos\theta||\sin\theta|^{(2-p)/p},
\end{aligned}
\tag{2.35}
$$

where

$$
\begin{gathered}
\tilde{\alpha}=\frac{p-1}{p}\alpha_2^{p/(p-1)}m_1^{-p/(p-1)^2}m_2^{1/(p-1)^2},\qquad p'=p(p-1), \\
a_1=\frac{p(p'-1)}{2p'(p-1)^{1/p}m_2^{1/(p-1)}},\qquad b_1=\frac{p'}{p'-1}(\lambda m_1-\alpha_1-\tilde{\alpha})m_2^{1/(p-1)}.
\end{gathered}
\tag{2.36}
$$

Denote $\hat{b}=\min\{b_1,1\}$, then we have

$$
\begin{aligned}
-\theta' \geq{}& a_1\left(b_1\cos^2\theta+\sin^2\theta\right)|\sin\theta|^{(2-p)/p}|\cos\theta|^{(p-2)/p} \\
&-\frac{\gamma_2 m_1^{-1/(p-1)}p^{2/p}}{2\hat{b}(p-1)^{2/p}r^{2(p-2)/p}}\left(b_1\cos^2\theta+\sin^2\theta\right)|\cos\theta||\sin\theta|^{(4-p)/p}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{(\beta_1+\beta_2)m_1^{(1-k)/(p-1)}p^{k/p}}{2\widehat{b}(p-1)^{k/p}r^{2(p-k)/p}}\left(b_1\cos^2\theta+\sin^2\theta\right)|\sin\theta|^{(2-p)/p}|\cos\theta|^{(p-2)/p}\\
&-\frac{\gamma_1 p^{1/p}}{2\widehat{b}(p-1)^{1/p}r^{2(p-1)/p}}\left(b_1\cos^2\theta+\sin^2\theta\right)|\sin\theta|^{(2-p)/p}|\cos\theta|^{(p-2)/p}\\
&=\widehat{a}_1\left(b_1\cos^2\theta+\sin^2\theta\right)|\sin\theta|^{(2-p)/p}|\cos\theta|^{(p-2)/p},
\end{aligned}
\tag{2.37}
$$

where

$$
\widehat{a}_1=a_1-\frac{\gamma_2 m_1^{-1/(p-1)}p^{2/p}}{2\widehat{b}(p-1)^{2/p}r^{2(p-2)/p}}-\frac{(\beta_1+\beta_2)m_2^{(1-k)/(p-1)}p^{k/p}}{2\widehat{b}(p-1)^{k/p}r^{2(p-k)/p}}-\frac{\gamma_1 p^{1/p}}{2\widehat{b}(p-1)^{1/p}r^{2(p-1)/p}}.
\tag{2.38}
$$

Assume that it takes time Δt for the motion $(r(t),\theta(t))(r(0)=A,\ \theta(0)=\theta_0)$ to complete one cycle around the origin. It follows from the above inequality that

$$
\begin{aligned}
\Delta t&<\int_{\theta_0}^{\theta_0+2\pi}\frac{d\theta}{\widehat{a}_1\left(b_1\cos^2\theta+\sin^2\theta\right)|\sin\theta|^{(2-p)/p}|\cos\theta|^{(p-2)/p}}\\
&=\frac{4}{\widehat{a}_1}\int_0^{\pi/2}\frac{d\theta}{\left(b_1\cos^2\theta+\sin^2\theta\right)|\sin\theta|^{(2-p)/p}|\cos\theta|^{(p-2)/p}}.
\end{aligned}
\tag{2.39}
$$

Let

$$
\eta=\tan^{-1}\frac{1}{\sqrt{b_1}}\tan\theta,
\tag{2.40}
$$

then

$$
\Delta t<\frac{4}{\widehat{a}_1 b_1^{1/p}}\int_0^{\pi/2}\frac{d\eta}{|\tan\eta|^{(2-p)/p}}=\frac{2}{\widehat{a}_1 b_1^{1/p}}B\left(\frac{1}{p},\frac{p-1}{p}\right)=\frac{2\pi}{\widehat{a}_1 b_1^{1/p}\sin(\pi/p)},
\tag{2.41}
$$

from (H4), we have

$$
a_1 b_1^{1/p}\sin\frac{\pi}{p}=\frac{\pi}{\pi_p}\left(\frac{p'-1}{p'}\right)^{(p-1)/p}\left(\frac{\lambda m_1-\alpha_1-\widetilde{\alpha}}{m_2}\right)^{1/p}>\frac{2n\pi}{T}.
\tag{2.42}
$$

So there exists $\sigma>0$ such that $(a_1-\sigma)b_1^{1/p}\sin(\pi/p)>2n\pi/T$. For the $\sigma>0$, there exists $\mathbb{R}'>0$ such that

$$
0<\frac{\gamma_2 m_1^{-1/(p-1)}p^{2/p}}{2\widehat{b}(p-1)^{2/p}r^{2(p-2)/p}}+\frac{(\beta_1+\beta_2)m_2^{(1-k)/(p-1)}p^{k/p}}{2\widehat{b}(p-1)^{k/p}r^{2(p-k)/p}}+\frac{\gamma p^{1/p}}{2\widehat{b}(p-1)^{1/p}r^{2(p-1)/p}}<\sigma
\tag{2.43}
$$

for $A > \mathbb{R}'$ large enough. So we have

$$\hat{a}_1 b_1^{1/p} \sin\frac{\pi}{p} = \left(a_1 - \frac{\gamma_2 m_1^{-1/(p-1)} p^{2/p}}{2\hat{b}(p-1)^{2/p} r^{2(p-2)/p}} - \frac{(\beta_1+\beta_2) m_2^{(1-k)/(p-1)} p^{k/p}}{2\hat{b}(p-1)^{k/p} r^{2(p-k)/p}} - \frac{\gamma p^{1/p}}{2\hat{b}(p-1)^{1/p} r^{2(p-1)/p}}\right) b_1^{1/p} \sin\frac{\pi}{p} > (a_1 - \sigma) b_1^{1/p} \sin\frac{\pi}{p} > \frac{2n\pi}{T}. \tag{2.44}$$

Therefore

$$\frac{T}{\Delta t} > n \tag{2.45}$$

as

$$\begin{aligned} \alpha_2 m_1^{-1/(p-1)} |u|^{p-1} |v|^{1/(p-1)} &= m_1^{-1/(p-1)} \left(|v|^{1/(p-1)}\right)\left(\alpha_2 |u|^{p-1}\right) \\ &\le m_1^{-1/(p-1)} \left[\frac{1}{p} |v|^{p/(p-1)} + \frac{p-1}{p} \alpha_2^{p/(p-1)} |u|^p\right] \\ &= \frac{1}{p} m_1^{-1/(p-1)} |v|^{p/(p-1)} + \frac{p-1}{p} \alpha_2^{p/(p-1)} m_1^{-1/(p-1)} |u|^p. \end{aligned} \tag{2.46}$$

Similarly, we have

$$\begin{aligned} 0 < -\theta' = {} & \frac{|u|^{(p-2)/2} |v|^{(2-p)/2(p-1)}}{2\sqrt{p-1} r^2} \Big[ug(u) + uf\left(u, \phi^{-1}(v)\right)\phi^{-1}(v) + (p-1)v\phi^{-1}(v) \\ & \quad - ue\left(t, u, \phi^{-1}(v)\right)\Big] \\ \le {} & \frac{|u|^{(p-2)/2} |v|^{(2-p)/2(p-1)}}{2\sqrt{p-1} r^2} \Big[(\mu m_2 + \alpha_1)|u|^p + (p-1) m_1^{-1/(p-1)} |v|^{p/(p-1)} \\ & \quad + \alpha_2 m_1^{-1/(p-1)} |u|^{p-1} |v|^{1/(p-1)} + \gamma_2 m_1^{-1/(p-1)} |u||v|^{1/(p-1)} \\ & \quad + (\beta_1+\beta_2) m_1^{(1-k)/(p-1)} |u||v|^{(k-1)/(p-1)} + \gamma_1 |u|\Big] \\ \le {} & \frac{|u|^{(p-2)/2} |v|^{(2-p)/2(p-1)}}{2\sqrt{p-1} r^2} \Big[(\mu m_2 + \alpha_1 + \tilde{\alpha}')|u|^p + \frac{p'+1}{p'}(p-1) m_1^{-1/(p-1)} |v|^{p/(p-1)} \\ & \quad + \gamma_2 m_1^{-1/(p-1)} |u||v|^{1/(p-1)} \\ & \quad + (\beta_1+\beta_2) m_1^{(1-k)/(p-1)} |u||v|^{(k-1)/(p-1)} + \gamma_1 |u|\Big] \end{aligned}$$

$$
\begin{aligned}
&= \frac{p|\sin\theta|^{(2-p)/p}|\cos\theta|^{(p-2)/p}}{2(p-1)^{1/p}}\left[(\mu m_2+\alpha_1+\widetilde{\alpha'})\cos^2\theta+\frac{p'+1}{p'}m_1^{-1/(p-1)}\sin^2\theta\right]\\
&\quad+\frac{\gamma_2 m_1^{-1/(p-1)}p^{2/p}}{2(p-1)^{2/p}r^{2(p-2)/p}}|\cos\theta||\sin\theta|^{(4-p)/p}\\
&\quad+\frac{(\beta_1+\beta_2)m_1^{(1-k)/(p-1)}p^{k/p}}{2(p-1)^{k/p}r^{2(p-k)/p}}|\cos\theta||\sin\theta|^{(2k-P)/p}\\
&\quad+\frac{\gamma_1 p^{1/p}}{2(p-1)^{1/p}r^{2(p-1)/p}}|\cos\theta||\sin\theta|^{(2-p)/p}\\
&= a_2\left(b_2\cos^2\theta+\sin^2\theta\right)|\sin\theta|^{(2-p)/p}|\cos\theta|^{(p-2)/p}\\
&\quad+\frac{\gamma_2 m_1^{-1/(p-1)}p^{2/p}}{2(p-1)^{2/p}r^{2(p-2)/p}}|\cos\theta||\sin\theta|^{(4-p)/p}\\
&\quad+\frac{(\beta_1+\beta_2)m_1^{(1-k)/(p-1)}p^{k/p}}{2(p-1)^{k/p}r^{2(p-k)/p}}|\cos\theta||\sin\theta|^{(2k-p)/p}\\
&\quad+\frac{\gamma_1 p^{1/p}}{2(p-1)^{1/p}r^{2(p-1)/p}}|\cos\theta||\sin\theta|^{(2-p)/p},
\end{aligned}
\tag{2.47}
$$

where

$$
\begin{gathered}
\widetilde{\alpha'}=\frac{p-1}{p}\alpha_2^{p/(p-1)}m_1^{-1/(p-1)},\qquad a_2=\frac{p(p'+1)}{2p'(p-1)^{1/p}m_1^{1/(p-1)}},\\
b_2=\frac{p'}{p'+1}(\mu m_2+\alpha_1+\widetilde{\alpha'})m_1^{1/(p-1)},
\end{gathered}
\tag{2.48}
$$

with the similar argument, we also get

$$
\frac{T}{\Delta t}<n+1. \tag{2.49}
$$

Then it holds that

$$
n<\frac{T}{\Delta t}<n+1. \tag{2.50}
$$

To finish the proof, we claim that If $n<T/\Delta t<n+1$, then $(u(T,\xi,\eta),v(T,\xi,\eta))\neq(\lambda^{2/p}\xi,\lambda^{2(p-1)/p}\eta)$. If there is $\lambda>0$ such that $(u(T,\xi,\eta),v(T,\xi,\eta))=(\lambda^{2/p}\xi,\lambda^{2(p-1)/p}\eta)$, then

$$\left(p^{1/p}r(T)^{2/p}\,|\cos\theta(T)|^{(2-p)/p}\cos\theta(T),\left(\frac{p}{p-1}\right)^{(p-1)/p}\times r(T)^{2(p-1)/p}\,|\sin\theta(T)|^{(p-2)/p}\sin\theta(T)\right)$$
$$=\left(\lambda^{2/p}p^{1/p}r(0)^{2/p}\,|\cos\theta(0)|^{(2-p)/p}\cos\theta(0),\lambda^{2(p-1)/p}\left(\frac{p}{p-1}\right)^{(p-1)/p}\times r(0)^{2(p-1)/p}\,|\sin\theta(0)|^{(p-2)/p}\sin\theta(0)\right). \tag{2.51}$$

So

$$r(T)^{2/p}\,|\cos\theta(T)|^{(2-p)/p}\cos\theta(T)=\lambda^{2/p}r(0)^{2/p}\,|\cos\theta(0)|^{(2-p)/p}\cos\theta(0), \tag{2.52}$$

$$r(T)^{2(p-1)/p}\,|\sin\theta(T)|^{(p-2/p}\sin\theta(T)=\lambda^{2(p-1)/p}r(0)^{2(p-1)/p}\,|\sin\theta(0)|^{(p-2/p}\sin\theta(0). \tag{2.53}$$

From (2.52) we have

$$r(T)^{2/p}\,|\cos\theta(T)|^{2/p}\operatorname{sgn}\cos\theta(T)=(\lambda r(0))^{2/p}\,|\cos\theta(0)|^{2/p}\operatorname{sgn}\cos\theta(0), \tag{2.54}$$

so, $\operatorname{sgn}\cos\theta(T)=\operatorname{sgn}\cos\theta(0)$, therefore, $r(T)^{2/p}|\cos\theta(T)|^{2/p}=(\lambda r(0))^{2/p}|\cos\theta(0)|^{2/p}$, moreover,

$$r(T)\cos\theta(T)=\lambda r(0)\cos\theta(0). \tag{2.55}$$

Similarly from (2.53) one has

$$r(T)\sin\theta(T)=\lambda r(0)\sin\theta(0). \tag{2.56}$$

So, from (2.55) and (2.56), we have

$$r(T)=\lambda r(0),\qquad (\cos\theta(T),\sin\theta(T))=(\cos\theta(0),\sin\theta(0)). \tag{2.57}$$

Therefore,

$$\theta(T)=\theta(0)+2k\pi\qquad\text{or}\qquad\theta(T)-\theta(0)=2k\pi. \tag{2.58}$$

However, from $n\Delta t<T<(n+1)\Delta t$, we have

$$\theta(T)-\theta(0)<\theta(n\Delta t)-\theta(0)=-2n\pi, \tag{2.59}$$

$$\theta(T)-\theta(0)>\theta((n+1)\Delta t)-\theta(0)=-2(n+1)\pi, \tag{2.60}$$

since $\theta'<0$. So there is no integer k such that $\theta(T)-\theta(0)=2k\pi$.

Therefore, the conclusion follows. □

Theorem 2.4. *Suppose (H1)–(H5) hold. Then (1.4) has at least one T-periodic solution $u(t)$.*

Proof. By Lemma 2.3, we know that there exists $A > 0$ ($A \gg 1$) such that if

$$\frac{1}{p}|\xi|^p + \frac{p-1}{p}|\eta|^{p/(p-1)} = A^2, \tag{2.61}$$

then

$$(u(T,\xi,\eta),v(T,\xi,\eta)) \neq (\lambda^{2/p}\xi,\lambda^{2(p-1)/p}\eta) \quad \text{for } \lambda > 0. \tag{2.62}$$

Assume that

$$\xi_1 = u(T,\xi,\eta), \qquad \eta_1 = v(T,\xi,\eta). \tag{2.63}$$

Consider a two-dimensional open region D_A bounded by

$$D_A = \left\{(\xi,\eta) : \frac{1}{p}|\xi|^p + \frac{p-1}{p}|\eta|^{p/(p-1)} = A^2\right\}, \tag{2.64}$$

then we define a topological mapping

$$H : D_A \longmapsto \mathbb{R}^2, \qquad (\xi,\eta) \longmapsto (\xi_1,\eta_1). \tag{2.65}$$

It follows from Lemma 2.3 that

$$(\xi_1,\eta_1) \neq (\lambda^{2/p}\xi,\lambda^{2(p-1)/p}\eta), \quad (\xi,\eta) \in \partial D_A. \tag{2.66}$$

Now we define a homotopy $h : \overline{D}_A \times [0,1] \to R^2$ by

$$\begin{aligned} h(\xi,\eta,\mu) &= -(\mu^{2/p}\xi,\mu^{2(p-1)/p}\eta) + ((1-\mu)^{2/p}\xi_1,(1-\mu)^{2(p-1)/p}\eta_1) \\ &= -\begin{pmatrix} \mu^{2/p} & 0 \\ 0 & \mu^{2(p-1)/p} \end{pmatrix} I(\xi,\eta) + \begin{pmatrix} (1-\mu)^{2/p} & 0 \\ 0 & (1-\mu)^{2(p-1)/p} \end{pmatrix} H(\xi,\eta), \end{aligned} \tag{2.67}$$

for $\mu \in [0,1]$. It is easy to see that $h(\xi,\eta,0), h(\xi,\eta,1) \neq 0$ for $(\xi,\eta) \in \partial D_A$. Then we show that $h(\xi,\eta,\mu) \neq 0$ for $(\xi,\eta) \in \partial D_A$, where $\mu \in (0,1)$. If not, there is $\mu_0 \in (0,1)$, $(\xi,\eta) \in \partial D_A$ such that $h(\xi,\eta,\mu_0) = 0$, that is,

$$(\xi_1,\eta_1) = \left(\left(\frac{\mu}{1-\mu}\right)^{2/p}\xi, \left(\frac{\mu}{1-\mu}\right)^{2(p-1)/p}\eta\right), \tag{2.68}$$

which is impossible. So $h(\xi,\eta,\mu) \neq 0$ for $\mu \in [0,1]$.

Then, $\deg\{h(\xi,\eta,0),D_A,0\} = \deg\{h(\xi,\eta,1),D_A,0\}$, that is,

$$\deg\{H,D_A,0\} = \deg\{-I,D_A,0\} \neq 0. \tag{2.69}$$

Therefore, H has at least one fixed point $(\xi^*,\eta^*) \in D_A$. It is easy to see that $u(t) = u(t,\xi^*,\eta^*)$ is a T-periodic solution of (1.4). □

If we let $\phi(u) = \varphi_p(u) = |u|^{p-2}u$, $p > 2$, then we have the following special cases of (1.4):

$$(\varphi_p(u'))' + f(u,u')u' + g(u) = p(t,u,u') \quad t \in [0,T], \tag{2.70}$$

so we can easy get the following results.

Theorem 2.5. *Assume (H2) and (H3) hold and solutions of (2.70) are unique with respect to initial value, moreover suppose that there exist λ, μ, and n such that*

$$\begin{gathered}\left(\frac{p'}{p'-1}\right)^{p-1}\left(\frac{2n\pi_p}{T}\right)^{p} + \alpha_1 + \frac{p-1}{p}\alpha_2{}^{p/p-1} \\ < \lambda \le \frac{g(x)}{\phi_p(x)} \le \mu < \left(\frac{p'}{p'+1}\right)^{p-1}\left(\frac{2(n+1)\pi_p}{T}\right)^{p} - \alpha_1 - \frac{p-1}{p}\alpha_2{}^{p/p-1},\end{gathered} \tag{2.71}$$

then (2.70) has at least one T-periodic solution.

3. Example

In this section, we present an example to illustrate our main results. Consider the following differential equation:

$$(\phi(u'))' + f(u,u')u' + g(u) = e(t,u,u'), \quad t \in [0,T], \tag{3.1}$$

where

$$\begin{gathered}\phi(x) = |x|(x + \sin x), \qquad f(x,y) = |y|^{3/4} + a, \quad a > 0, \qquad g(x) = 2\phi(x), \\ e(t,x,y) = -\frac{2}{3}|x|x - |y|^{3/4}y + b\cos 2\pi t, \quad b > 0.\end{gathered} \tag{3.2}$$

We claim that

$$\frac{2}{3}|x|^2 \le |\phi(x)| \le 2|x|^2. \tag{3.3}$$

In fact, if $x \neq 0$, we have

$$|\phi(x)| = |x|^2\left|1 + \frac{\sin x}{x}\right| > |x|^2\left(1 - \frac{1}{\pi}\right) > \frac{2}{3}|x|^2, \tag{3.4}$$

so (3.3) holds. Therefore, $p = 3$, $m_1 = 2/3$, $m_2 = 2$. Also, we can get $\alpha_1 = 2/3$, $\beta_1 = 1$, $\gamma_1 = b$, $\alpha_2 = 0$, $\beta_2 = 1$, $\gamma_2 = a$, $k = 11/4$.

Let $n = 0$ and $T = 1$, then conditions (H1)–(H4) are satisfied.

Now, we check that condition (H5) is satisfied.

Suppose that $x_1(t)$ and $x_2(t)$ are two different solutions to (3.1) satisfying

$$x_1(t_0) = x_2(t_0) = x_0, \qquad x_1'(t_0) = x_2'(t_0) = x_0'. \tag{3.5}$$

Let $y = \phi(x')$, then $(x_i(t), y_i(t)) = (x_i(t), \phi(x_i'(t)))$ $(i = 1,2)$ are two different solutions to the system

$$x' = \phi^{-1}(y),$$
$$y' = -g(x) - f(x, \phi^{-1}(y))\phi^{-1}(y) + e(t, x, \phi^{-1}(y)), \tag{3.6}$$

satisfying $(x_i(t_0), y_i(t_0)) = (x_0, \phi(x'(t_0)))$ $(i = 1,2)$.

Without loss of generality, we assume that there exists $t_1 > t_0$ such that

$$x_2(t) > x_1(t), \quad t \in (t_0, t_1]. \tag{3.7}$$

As $x_1(t_0) = x_2(t_0) = x_0$, $x_1'(t_0) = x_2'(t_0) = x_0'$, and $x_i \in \mathbb{C}^2[t_0, t_1]$, so there exists $t^* \in (t_0, t_1)$ such that

$$x_2'(t) > x_1'(t), \quad t \in (t_0, t^*]. \tag{3.8}$$

Therefore, for $t \in (t_0, t^*]$, we have

$$\begin{aligned} y_2(t) - y_1(t) = &-\int_{t_0}^{t} \Big\{[g(x_2(s)) - g(x_1(s))] + [f(x_2(s), x_2'(s))x_2'(s) - f(x_1(s), x_1'(s))x_1'(s)] \\ &\quad - [e(s, x_2(s), x_2'(s)) - e(s, x_1(s), x_1'(s))]\Big\} ds \\ = &-\int_{t_0}^{t} \Big\{2[\phi(x_2(s)) - \phi(x_1(s))] + 2\Big[|x_2'(s)|^{3/4}x_2'(s) - |x_1'(s)|^{3/4}x_1'(s)\Big] \\ &\quad + a(x_2'(s) - x_1'(s)) + \frac{2}{3}\Big[|x_2'(s)|x_2'(s) - |x_1'(s)|x_1'(s)\Big]\Big\} ds < 0. \end{aligned} \tag{3.9}$$

That is,

$$\phi(x_2'(t)) - \phi(x_1'(t)) < 0, \quad t \in (t_0, t^*]. \tag{3.10}$$

So, $x_2'(t) < x_1'(t), t \in (t_0, t^*]$, this is a contradiction.

Therefore, by Theorem 2.4, we can conclude that (3.1) has at least one 1-periodic solution.

Acknowledgments

The authors of this paper wish to thank the referee for his (or her) valuable suggestions regarding the original manuscript. The project is supported by the National Natural Science Foundation of China (10371006).

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