Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2007, Article ID 12324, 14 pages
doi:10.1155/2007/12324

Research Article

Iterative Methods for Generalized von Foerster Equations with Functional Dependence

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Received 4 August 2007; Accepted 13 November 2007

Recommended by Patricia J. Y. Wong

We investigate when a natural iterative method converges to the exact solution of a differential-functional von Foerster-type equation which describes a single population depending on its past time and state densities, and on its total size. On the right-hand side, we assume either Perron comparison conditions or some monotonicity.

1. Introduction

Von Foerster and Volterra-Lotka equations arise in biology, medicine, and chemistry, [1–5]. The independent variables x_j and an unknown function u stand for certain features and densities, respectively. It follows from this natural interpretation that $x_j \geq 0$ and $u \geq 0$. We are interested in the first model, which is essentially nonlocal, because it also contains the total size of population $\int u(t,x)dx$.

Existence results for certain von Foerster type problems has been established by means of the Banach contraction principle, the Schauder fixed point theorem, or iterative methods, see [6–10]. Just because of nonlocal terms, these methods demand very thorough calculations and a proper choice of subspaces of continuous and integrable functions. Sometimes, it may cost some simplifications of the real model. On the other hand, there is a very consistent theory of first-order partial differential-functional equation in [11–13], based on properties of bicharacteristics and on the above-mentioned fixed-point techniques with respect to the uniform norms.

In the present paper, we find natural conditions which guarantee $L^\infty \cap L^1$-convergence of iterative methods. Note that an associate result on fast convergent quasilinearization methods has been published in [14].

Formulation of the differential problem. Let $\tau = (\tau_1,\dots,\tau_n) \in \mathbb{R}_+^n$, $\tau_0 > 0$, where $\mathbb{R}_+ := [0,+\infty)$. Define

$$\begin{gathered} B = [-\tau_0,0] \times [-\tau,\tau], \quad \text{where } [-\tau,\tau] = [-\tau_1,\tau_1] \times \cdots \times [-\tau_n,\tau_n] \\ E_0 = [-\tau_0,0] \times \mathbb{R}^n, \quad E = [0,a] \times \mathbb{R}^n, \quad a > 0. \end{gathered} \tag{1.1}$$

For each function w defined on $[-\tau_0,a]$, we have the Hale functional w_t (see [15]), which is the function defined on $[-\tau_0,0]$ by

$$w_t(s) = w(t+s), \quad (s \in [-\tau_0,0]). \tag{1.2}$$

For each function u defined on $E_0 \cup E$, we similarly write a Hale-type functional $u_{(t,x)}$, defined on B by

$$u_{(t,x)}(s,y) = u(t+s,x+y) \quad \text{for } (s,y) \in B \tag{1.3}$$

(see [11]). Let $\Omega_0 = E \times C([-\tau_0,0],\mathbb{R}_+)$ and $\Omega = E \times C(B,\mathbb{R}_+) \times C([-\tau_0,0],\mathbb{R}_+)$. Take v: $E_0 \to \mathbb{R}_+$ and

$$c_j : \Omega_0 \longrightarrow \mathbb{R}, \quad \lambda : \Omega \longrightarrow \mathbb{R} \quad (j = 1,\dots,n). \tag{1.4}$$

Consider the differential-functional equation

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{n} c_j(t,x,z[u]_t)\frac{\partial u}{\partial x_j} = u(t,x)\lambda(t,x,u_{(t,x)},z[u]_t), \tag{1.5}$$

where

$$z[u](t) := \int_{\mathbb{R}^n} u(t,y)\,dy, \quad t \in [-\tau_0,a], \tag{1.6}$$

with the initial conditions

$$u(t,x) = v(t,x), \quad (t,x) \in E_0, \quad x = (x_1,\dots,x_n) \in \mathbb{R}^n. \tag{1.7}$$

We are looking for Caratheodory solutions to (1.5) and (1.7), see [6, 7, 16]. The functional dependence includes a possible delayed and integral dependence of the Volterra type. The Hale functional $z[u]_t$ takes into consideration the whole population within the time interval $[t-\tau_0,t]$, whereas the Hale-type functional $u_{(t,x)}$ shows the dependence on the density u locally in a neighborhood of (t,x). The functional dependence demands some initial data on a *thick* initial set E_0, which means that a complicated ecological niche must be observed for some time and (perhaps) in some space in order to predict its further evolution.

Example 1.1. The functional dependence in (1.5), represented by the Hale operators, generalizes von Foerster equations with delays, deviations, and integrals, such as the equation with delays:

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{n} \bar{c}_j(t,x,z[u](\beta(t)))\frac{\partial u}{\partial x_j} = u\bar{\lambda}(t,x,u(\bar{\alpha}(t,x)),z[u](\bar{\beta}(t))), \tag{1.8}$$

where $\overline{\alpha}(t,x) = (\overline{\alpha}_0(t,x),\ldots,\overline{\alpha}_n(t,x))$, $\overline{\alpha}_0(t,x) \le t$ and $\beta(t),\overline{\beta}(t) \le t$, and the equation with integrals:

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{n} \overline{c}_j\left(t,x,\int_{t-\tau_0}^{t} z[u](s)ds\right)\frac{\partial u}{\partial x_j} = u\overline{\lambda}\left(t,x,\int_{[x,x+\tau]} u(t,y)dy,\int_{t/3}^{t} z[u](s)ds\right), \tag{1.9}$$

where $\overline{c}_j : E \times \mathbb{R}_+ \to \mathbb{R}$ and $\overline{\lambda} : E \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$.

The paper is organized as follows:

(i) first, we give key properties of bicharacteristics η and write the solution u of problem (1.5), (1.7) along bicharacteristics for a given function z, which belongs to a priori defined class under natural assumptions on the data;

(ii) considering solutions u along these bicharacteristics η, we get integral fixed-point equations $z = \mathcal{T}[z]$, realized as follows: $z \to \eta[z] \to u[z] \to \mathcal{T}[z]$;

(iii) we define an iterative method of the form

$$z_k \longrightarrow \eta_k \longrightarrow u_k \longrightarrow z_{k+1} = \mathcal{T}[z_k] \tag{1.10}$$

and show its convergence under uniqueness conditions with some uniform Perron comparison functions.

Our convergence result implies the existence and uniqueness. We stress that this existence statement essentially differs from Schauder fixed-point theory: one can find classes of problems, where one of these methods yields the existence, whereas the other one does not.

2. Bicharacteristics

First, for a given function $z \in C([-\tau_0,a],\mathbb{R}_+)$, consider the bicharacteristic equations for problem (1.5), (1.7):

$$\eta'(s) = c(s,\eta(s),z_s), \quad \eta(t) = x. \tag{2.1}$$

Denote by $\eta = \eta[z](\cdot;t,x) = (\eta_1[z](\cdot;t,x),\ldots,\eta_n[z](\cdot;t,x))$ the bicharacteristic curve passing through $(t,x) \in E$, that is, the solution to problem (2.1). Next, we consider the following equation

$$\frac{d}{ds}u(s,\eta[z](s;t,x)) = u(s,\eta[z](s;t,x))\lambda(s,\eta[z](s;t,x),u_{(s,\eta[z](s;t,x))},z_s), \tag{2.2}$$

with the initial condition

$$u(0,\eta[z](0;t,x)) = v(0,\eta[z](0;t,x)). \tag{2.3}$$

For any given function $z \in C([-\tau_0,a],\mathbb{R}_+)$, a solution of (2.2) along bicharacteristics (2.1) is a solution of (1.5). The initial conditions (1.7) and (2.3) correspond to each other.

Assume the following.

(V0) $v \in CB(E_0, \mathbb{R}_+)$ (nonnegative, bounded, and continuous function).

(V1) $z[v] \in C([-\tau_0, 0], \mathbb{R}_+)$, where

$$z[v](t) = \int_{\mathbb{R}^n} v(t,x)\,dx. \tag{2.4}$$

(V2) The function v satisfies the Lipschitz condition

$$|v(t,x) - v(t,\overline{x})| \le L_v \|x - \overline{x}\| \quad \text{on } E_0 \tag{2.5}$$

with some constant $L_v > 0$.

(C0) $c_j : \Omega_0 \to \mathbb{R}$ are continuous in (t,x,q) and

$$\|c(t,x,q) - c(t,\overline{x},\overline{q})\| \le L_c \|x - \overline{x}\| + L_c^* \|q - \overline{q}\|. \tag{2.6}$$

A continuous function $\sigma : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *Perron comparison function* if $\sigma(t,0) \equiv 0$ and the differential problem $y' = \sigma(t,y)$, $y(0) = 0$ has the only zero solution. We call it *uniform* if σ, multiplied by any positive constant, is also a Perron comparison function. We call it *monotone* if σ is nondecreasing in the second variable.

(Λ0) $\lambda : \Omega \to \mathbb{R}$ is continuous in (t,x,w,q) and

$$|\lambda(t,x,w,q) - \lambda(t,\overline{x},\overline{w},\overline{q})| \le M_\lambda \sigma(t, \|x - \overline{x}\| + \|w - \overline{w}\| + \|q - \overline{q}\|), \tag{2.7}$$

where $\sigma : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a monotone uniform Perron comparison function.

(Λ1) There exists a function $L_\lambda \in L^1([0,a], \mathbb{R}_+)$ such that

$$\lambda(t,x,w,q) \le L_\lambda(t) \tag{2.8}$$

for $(t,x) \in E, w \in C(B, \mathbb{R}_+)$, $q \in C([-\tau_0, 0], \mathbb{R}_+)$.

Denote

$$W(t,x,w,q) = \lambda(t,x,w,q) + \operatorname{tr}\partial_x c(t,x,q) \tag{2.9}$$

for $(t,x) \in E, w \in C(B, \mathbb{R}_+)$, $q \in C([-\tau_0, 0], \mathbb{R}_+)$, where $\operatorname{tr}\partial_x c$ stands for the trace of the matrix $\partial_x c = [\partial_{x_k} c_j]_{j,k=1,\dots,n}$.

(W0) There exists $M_W \in \mathbb{R}_+$ such that

$$|W(t,x,w,q) - W(t,\overline{x},\overline{w},\overline{q})| \le M_W\, \sigma(t, \|x - \overline{x}\| + \|w - \overline{w}\| + \|q - \overline{q}\|), \tag{2.10}$$

where $\sigma : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a monotone uniform Perron comparison function.

(W1) There exists a function $L_W \in L^1([0,a], \mathbb{R}_+)$ such that

$$W(t,x,w,q) \le L_W(t) \tag{2.11}$$

for $(t,x) \in E$, $w \in C(B, \mathbb{R}_+)$, $q \in C([-\tau_0, 0], \mathbb{R}_+)$.

Lemma 2.1. *If the conditions (V0) and (Λ1) are satisfied, then any solution u of (2.2) has the estimate*

$$0 \le u(t,x) \le \|v(0,\cdot)\|_\infty \exp\Big(\int_0^t L_\lambda(s)ds\Big) \quad \text{on } E. \tag{2.12}$$

2.1. The fixed-point equation. Let

$$Z(t) = \max_{-\tau_0 \le s \le 0} \|v(s,\cdot)\|_1 \exp\Big(\int_0^t L_W(s)ds\Big), \tag{2.13}$$

where we put $L_W(s) = 0$ for $s \in [-\tau_0, 0]$, and

$$\mathcal{X} = \{z \in C([-\tau_0, a], \mathbb{R}_+) : z(t) \le Z(t)\}. \tag{2.14}$$

Consider the operator $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ given by the formula

$$\mathcal{T}[z](t) = \int_{\mathbf{R}^n} u[z](t,x)\,dx \quad \text{for } t \ge 0, \tag{2.15}$$

where $u = u[z] \in C^1(E, \mathbf{R}_+)$ is the solution of (2.2) and (2.3) with the initial condition $u[z](t,x) = v(t,x)$ on E_0. The function $u = u[z]$ has the following representation on E:

$$u[z](t,x) = v(0,\eta(0))\exp\Big(\int_0^t \lambda(s,\eta(s),u_{(s,\eta(s))},z_s)\,ds\Big), \tag{2.16}$$

where $\eta(s) = \eta[z](s;t,x)$. By Lemma 2.1, we write (2.15) in the following way:

$$\mathcal{T}[z](t) = \int_{\mathbb{R}^n} v(0,\eta(0))\exp\Big(\int_0^t \lambda(s,\eta(s),u_{(s,\eta(s))},z_s)\,ds\Big)dx \tag{2.17}$$

for $t \ge 0$. The bicharacteristics admit the following group property:

$$y = \eta[z](0;t,x) \Longleftrightarrow \eta[z](s;t,x) = \eta[z](s;0,y), \tag{2.18}$$

that is, any bicharacteristic curve passing through the points $(0,y)$ and (t,x) has the same value at $s \in [0,a]$.

If we change the variables $y = \eta[z](0;t,x)$, then by the Liouville theorem, the Jacobian $J = \det[\partial c/\partial x]$ is given by the formula

$$J(0;t,x) = \exp\Big(-\int_0^t \operatorname{tr}\partial_x c(s,\eta[z](s;0,y),z_s)\,ds\Big). \tag{2.19}$$

Hence (2.17) can be written in the form

$$\mathcal{T}[z](t) = \int_{\mathbb{R}^n} v(0,y)\exp\Big(\int_0^t W(s,\eta(s),u_{(s,\eta(s))},z_s)\,ds\Big)dy, \tag{2.20}$$

where $\eta(s) = \eta[z](s;0,y)$.

Lemma 2.2. *If the conditions (V0), (V1), and (W1) are satisfied, then*

$$0 \le \mathcal{T}[z](t) \le Z(t) < +\infty \quad \text{for } t \in [0,a], \tag{2.21}$$

where Z is given by (2.13).

Proof. This assertion follows from (2.20) and Assumptions (V0), (V1), and (W1). □

The respective fixed-point equation for bicharacteristics $\eta = \eta[z]$ has the form

$$\eta(s;t,x) = x - \int_s^t c(\zeta, \eta(\zeta;t,x), z_\zeta)\, d\zeta. \tag{2.22}$$

Lemma 2.3. *If Assumption (C0) is satisfied and $z, \bar{z} \in \mathcal{Z}$, then*

$$\|\eta[z](s;t,x) - \eta[\bar{z}](s;t,x)\| \le \int_s^t L_c^* \|z_\zeta - \bar{z}_\zeta\| e^{L_c(\zeta - s)}\, d\zeta. \tag{2.23}$$

3. The iterative method

Define the iterative method by $z^{(k+1)} = \mathcal{T}[z^{(k)}]$ with an arbitrary function $z^{(0)} \in \mathcal{Z}$, where the class $\mathcal{Z}$ is defined by (2.14). We prove its uniform convergence under natural assumptions on the given functions. The algorithm splits into three stages:

(1) finding bicharacteristics $\eta^{(k)} = \eta[z^{(k)}]$, given by (2.22);
(2) finding $u^{(k)} = u[z^{(k)}]$ as a solution of (2.16);
(3) calculating $z^{(k+1)} = \mathcal{T}[z^{(k)}]$ by means of (2.17) or (2.20). In this way, there are given the integ ral equations

$$\begin{aligned} \eta^{(k)}(s;t,x) &= x - \int_s^t c\Big(\zeta, \eta^{(k)}(\zeta;t,x), z_\zeta^{(k)}\Big)\, d\zeta, \\ u^{(k)}(t,x) &= v(0, \eta^{(k)}(0;t,x)) \exp\Big(\int_0^t \lambda(Q^{(k)}(s))\, ds\Big), \\ z^{(k+1)}(t) &= \int_{\mathbb{R}^n} v(0,y) \exp\Big(\int_0^t W(R^{(k)}(s))\, ds\Big)\, dy, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} Q^{(k)}(s) &= \Big(s, \eta^{(k)}(s;t,x), u^{(k)}_{(s,\eta^{(k)}(s;t,x))}, z_s^{(k)}\Big), \\ R^{(k)}(s) &= \Big(s, \eta^{(k)}(s;0,y), u^{(k)}_{(s,\eta^{(k)}(s;0,y))}, z_s^{(k)}\Big). \end{aligned} \tag{3.2}$$

Theorem 3.1. *If $z^{(0)} \in \mathcal{Z}$ and Assumptions (V0)–(V2), (C0), (Λ0), (Λ1), (W0), and (W1) are satisfied, and there are $K \in \mathbb{R}_+$, $\theta \in (0,1]$ such that*

$$\sigma(t,r) \le K t^{\theta - 1} p r^{1-1/p} \quad \textit{for } p \ge 2, \tag{3.3}$$

then the iterative method $z^{(k+1)} = \mathcal{T}[z^{(k)}]$ is well defined in $\mathcal{Z}$ and uniformly converges to the unique fixed point $z = \mathcal{T}[z]$ on a sufficiently small $[0,a]$ (locally).

Remark 3.2. The technical condition (3.3) is fulfilled in the Lipschitz case ($\sigma(t,r) = Lr$) as well as the simplest nonlinear Perron comparison functions such as $\sigma(t,r) = Lr\ln(1+1/r)$. Its formulation also includes weak singularities, that is, $\sigma(t,r) = t^{-1/2}Lr$ and $\sigma(t,r) = t^{-1/2}Lr\ln(1+1/r)$.

Proof of Theorem 3.1. Denote

$$\Delta z^{(k)} = z^{(k+1)} - z^{(k)}, \qquad \Delta\eta^{(k)} = \eta^{(k+1)} - \eta^{(k)}, \qquad \Delta u^{(k)} = u^{(k+1)} - u^{(k)}. \tag{3.4}$$

Then we have the estimates

$$\begin{aligned}
||\Delta\eta^{(k)}(s;t,x)|| &\le \int_s^t L_c^* \left|\left|\Delta z_\zeta^{(k)}\right|\right| e^{L_c(\zeta-s)}d\zeta, \\
|\Delta u^{(k)}(t,x)| &\le L_v||\Delta\eta^{(k)}(0;t,x)||\exp\left(\int_0^t L_\lambda(s)ds\right) \\
&\quad + \|v\|_\infty \exp\left(\int_0^t L_\lambda(s)ds\right)\int_0^t M_\lambda\sigma(s,P^{(k)}(s;t,x))ds, \\
|\Delta z^{(k+1)}(t)| &\le Z(t)\int_0^t M_W\sigma(s,P^{(k)}(s;t,x))ds,
\end{aligned} \tag{3.5}$$

where $P^{(k)}(s;t,x) = \|\Delta\eta^{(k)}(s;t,x)\| + \|\Delta u^{(k)}\|_s + \|\Delta z^{(k)}\|_s$. Denote $\widehat{L}_\lambda = \int_0^a L_\lambda(s)ds$ and

$$\Psi^{(k)}(s,t) = \overline{\psi}^{(k)}(s) + \psi^{(k)}(s) + \int_s^t L_c^* e^{L_c a}\psi^{(k)}(\zeta)d\zeta. \tag{3.6}$$

Consider the comparison equations

$$\begin{aligned}
\overline{\psi}^{(k)}(t) &= L_v\int_0^t L_c^* e^{L_c a+\hat{L}_\lambda}\psi^{(k)}(s)ds + \|v\|_\infty e^{\hat{L}_\lambda}\int_0^t M_\lambda\sigma(s,\Psi^{(k)}(s,t))ds, \\
\psi^{(k+1)}(t) &= Z(t)\int_0^t M_W\sigma(s,\Psi^{(k)}(s,t))ds
\end{aligned} \tag{3.7}$$

with $\psi^{(0)}(t) = Z(t)$ and

$$\begin{aligned}
\overline{\psi}^{(0)}(t) &= \|v\|_\infty \exp\left(\int_0^t L_W(s)ds\right) + L_v\int_0^t L_c^* e^{L_c a+\hat{L}_\lambda}Z(s)ds \\
&\quad + \|v\|_\infty e^{\hat{L}_\lambda}\int_0^t M_\lambda\sigma\Big(s,\overline{\psi}^{(0)}(s) + Z(s) + \int_s^t L_c^* e^{L_c a}Z(\zeta)d\zeta\Big)ds.
\end{aligned} \tag{3.8}$$

The remaining part of the proof is split into several auxiliary lemmas. □

Lemma 3.3. *Under the assumptions of Theorem 3.1, there is $a_0 \in (0,a]$ such that $|\Delta u^{(k)}(t,x)| \le \overline{\psi}^{(k)}(t)$, $|\Delta z^{(k)}(t)| \le \psi^{(k)}(t)$,*

$$||\Delta\eta^{(k)}(s;t,x)|| \le \int_s^t L_c^* e^{L_c a}\psi^{(k)}(\zeta)d\zeta \tag{3.9}$$

on $[0,a_0]\times\mathbb{R}_+^n$, and the sequences $\{\psi^{(k)}\}$ and $\{\overline{\psi}^{(k)}\}$ are nondecreasing in k.

Lemma 3.4. *Under the assumptions of Theorem 3.1, the estimate*

$$\int_0^t \sigma(s, As^l + Bt^{l+1})\,ds \le t^{l+\theta-l/p} pK\theta^{-1}\left[\frac{A}{\theta+l} + \frac{Ba}{\theta}\right]^{1-1/p} \tag{3.10}$$

holds.

Proof. By the Hölder inequality, we have

$$\begin{aligned}
&\int_0^t \sigma(s, As^l + Bt^{l+1})\,ds \\
&\qquad \le pK\int_0^t s^{\theta-1}\left[As^l + Bt^{l+1}\right]^{1-1/p} ds \\
&\qquad \le pK\left\{\int_0^t s^{\theta-1} ds\right\}^{1/p}\left\{\int_0^t s^{\theta-1}\left[As^l + Bt^{l+1}\right] ds\right\}^{1-1/p} \\
&\qquad \le pK\theta^{-1}t^{\theta/p}\left[\frac{At^{\theta+l}}{\theta+l} + \frac{Bt^{\theta+l+1}}{\theta}\right]^{1-1/p}.
\end{aligned} \tag{3.11}$$

□

Lemma 3.5. *Under the assumptions of Theorem 3.1, the sequences* $\{\psi^{(k)}\}$ *and* $\{\overline{\psi}^{(k)}\}$ *tend uniformly to* 0 *as* $k \to +\infty$.

Proof. Denote $M = L_v L_c^* e^{L_c a}$, $M^* = \|v\|_\infty e^{\hat{L}_\lambda} M_\lambda + Z(a) M_W$, and $c_a = L_c^* e^{L_c a}$. Then the equation

$$\hat{\psi}(t) = M\int_0^t \hat{\psi}(s)\,ds + M^*\int_0^t \sigma\left(s, \hat{\psi}(s) + c_a\int_s^t \hat{\psi}(\zeta)\,d\zeta\right) ds \tag{3.12}$$

describes a comparison function $\hat{\psi}$ with respect to $\psi + \overline{\psi}$, where

$$\psi(t) = \lim_{k\to\infty} \psi^{(k)}(t), \qquad \overline{\psi}(t) = \lim_{k\to\infty} \overline{\psi}^{(k)}(t). \tag{3.13}$$

One can prove, by induction on k, that $\hat{\psi}(t) \le \hat{C}_k t^{\theta/2}$ and $\hat{C}_k a^{\theta/2} \to 0$ as $k \to +\infty$, provided that the interval $[0, a]$ is sufficiently small. Take an arbitrary $\hat{C}_0$ which estimates $\hat{\psi}(t)$. Applying Lemma 3.4 with $p = 2$ to (3.12), we get

$$\hat{\psi}(t) \le Mt\hat{C}_0 + M^* t^\theta 2K\theta^{-1}\left[\frac{\hat{C}_0(1 + c_a a)}{\theta}\right]^{1/2} \le t^{\theta/2}\hat{C}_1, \tag{3.14}$$

where

$$\hat{C}_1 = Ma^{1-\theta/2}\hat{C}_0 + a^{\theta/2} 2K\theta^{-1}\left[\frac{\hat{C}_0(1 + c_a a)}{\theta}\right]^{1/2}. \tag{3.15}$$

Suppose that the desired estimate holds for some $k \geq 1$. Applying Lemma 3.4 with $p = 2k$ to (3.12), we get

$$\hat{\psi}(t) \leq M\frac{t^{1+k\theta/2}}{1+k\theta/2}\hat{C}_k + M^* t^{(k+1)\theta/2} 2K\theta^{-1}\left[\frac{\hat{C}_k}{\theta+k\theta/2} + \frac{c_a\hat{C}_k}{\theta(1+k\theta/2)}\right]^{1-1/(2k)}, \tag{3.16}$$

hence $\hat{\psi}(t) \leq t^{(k+1)\theta/2}\hat{C}_{k+1}$, where

$$\hat{C}_{k+1} = M\hat{C}_k\frac{a^{1-\theta/2}}{1+k\theta/2} + M^* 2K\theta^{-1}\left[\frac{\hat{C}_k}{\theta+k\theta/2} + \frac{c_a\hat{C}_k}{\theta(1+k\theta/2)}\right]^{1-1/(2k)}. \tag{3.17}$$

The constants $\hat{C}_k$ have an upper estimate of the form AQ^k, thus $\hat{\psi}(t) \equiv 0$ in a neighborhood of 0 (because $\hat{\psi}(t) \leq AQ^k t^{k\theta/2}$). □

Lemma 3.6. *Under the assumptions of Theorem 3.1, the sequences $\{z^{(k)}\}$, $\{u^{(k)}\}$, and $\{\eta^{(k)}\}$ tend uniformly to $z, u[z], \eta[z]$ such that $z = \mathcal{T}[z]$.*

Proof. We intend to find the following estimates:

$$\psi^{(k)}(t) \leq C_k t^{l_k}, \qquad \overline{\psi}^{(k)}(t) \leq \overline{C}_k t^{l_k}, \tag{3.18}$$

where the series $\sum_k C_k t^{l_k}$ is convergent in a neighborhood of 0. The assertion can be seen if we replace the comparison equation (3.7) by the inequalities

$$\begin{aligned} \overline{C}_k t^{l_k} &\geq L_v \int_0^t L_c^* e^{L_c a + \hat{L}_\lambda} C_k s^{l_k} ds \\ &\quad + \|v\|_\infty e^{\hat{L}_\lambda} \int_0^t M_\lambda \sigma\left(s, (C_k + \overline{C}_k)s^{l_k} + L_c^* e^{L_c a} C_k t^{l_k+1}/(l_k+1)\right) ds, \\ C_{k+1} t^{l_{k+1}} &\geq Z(a) \int_0^t M_W \sigma\left(s, (C_k + \overline{C}_k)s^{l_k} + L_c^* e^{L_c a} C_k t^{l_k+1}/(l_k+1)\right) ds, \end{aligned} \tag{3.19}$$

with $C_0 t^{l_0} = Z(a)$ and some

$$\overline{C}_0 \geq a L_v L_c^* e^{L_c a + \hat{L}_\lambda} Z(a) + \|v\|_\infty e^{\hat{L}_\lambda} M_\lambda \textstyle\int_0^a \sigma\left(s, \overline{C}_0 + Z(a) + a L_c^* e^{L_c a} Z(a)\right) ds . \tag{3.20}$$

If we put

$$l_0 = 0, \qquad p_0 = 2/\theta, \qquad l_k = k\theta/2, \quad p_k = 4k \quad \text{for } k = 1, 2, \ldots \tag{3.21}$$

and exploit Lemma 3.4, then $C_k, \overline{C}_k$ can be defined as follows:

$$\begin{aligned} \overline{C}_k t^{l_k} &\geq L_v L_c^* e^{L_c a + \hat{L}_\lambda} C_k t^{l_k+1}/(l_k+1) \\ &\quad + \|v\|_\infty e^{\hat{L}_\lambda} M_\lambda p_k K \theta^{-1} t^{l_k+\theta/2}\left[\frac{C_k + \overline{C}_k}{\theta + l_k} + \frac{a L_c^* e^{L_c a} C_k}{\theta(l_k+1)}\right]^{1-1/p_k}, \\ C_{k+1} t^{l_{k+1}} &\geq Z(a) M_W p_k K \theta^{-1} t^{l_k+\theta/2}\left[\frac{C_k + \overline{C}_k}{\theta + l_k} + \frac{a L_c^* e^{L_c a} C_k}{\theta(l_k+1)}\right]^{1-1/p_k}. \end{aligned} \tag{3.22}$$

These inequalities reduce to the system of algebraic equations

$$\begin{aligned}\overline{C}_k &= L_v L_c^* e^{L_c a+\hat{L}_\lambda} C_k a/(l_k+1)\\ &\quad + \|v\|_\infty e^{\hat{L}_\lambda} M_\lambda p_k K \theta^{-1} a^{\theta/2}\left[\frac{C_k+\overline{C}_k}{\theta+l_k}+\frac{aL_c^* e^{L_c a}C_k}{\theta(l_k+1)}\right]^{1-1/p_k},\\ C_{k+1} &= Z(a)M_W p_k K\theta^{-1}\left[\frac{C_k+\overline{C}_k}{\theta+l_k}+\frac{aL_c^* e^{L_c a}C_k}{\theta(l_k+1)}\right]^{1-1/p_k}.\end{aligned} \tag{3.23}$$

A simple separation of variables yields

$$\begin{aligned}\overline{C}_k &= L_v L_c^* e^{L_c a+\hat{L}_\lambda} C_k a/(l_k+1) + C_{k+1}\frac{\|v\|_\infty e^{\hat{L}_\lambda}M_\lambda}{Z(a)M_W},\\ C_{k+1} &= Z(a)M_W p_k K\theta^{-1}\Bigg[C_{k+1}\frac{\|v\|_\infty e^{\hat{L}_\lambda}M_\lambda}{Z(a)M_W(\theta+l_k)}\\ &\quad + \frac{C_k(1+L_v L_c^* e^{L_c a+\hat{L}_\lambda}C_k a/(l_k+1))}{\theta+l_k}+\frac{aL_c^* e^{L_c a}C_k}{\theta(l_k+1)}\Bigg]^{1-1/p_k}.\end{aligned} \tag{3.24}$$

From the last equation, it follows that one can find positive constants A,Q such that $AQ^k \geq C_k$. Thus the series $\sum_k C_k t^{l_k}$ is convergent on a sufficiently small interval $[0,a]$, hence the series $\psi^{(0)}+\psi^{(2)}+\cdots$ uniformly converges, and $z^{(k)}$ has a limit, which is continuous. □

Corollary 3.7. *If Assumptions (V0)–(V2), (C0), (Λ0), (Λ1), (W0), and (W1) are satisfied, then there exists the unique solution of problem (1.5)–(1.7), locally with respect to t.*

4. The iterative method: global convergence

In this section, we prove the global convergence of our iterative method, that is, on the whole interval $[0,a]$. We deal with the problem of global convergence of the iterative method in two ways. The first case is based on the method used in the previous section under strengthened assumptions (Λ0) and (W0). Namely, we replace nonlinear Perron comparison functions by the Lipschitz condition with a function $L \in L^1([0,a],\mathbb{R}_+)$ or with a positive Lipschitz constant $\overline{L}$. We also discuss another case which leads to global convergence results, that is, the monotone iterations with respect to the function $z^{(k)}$, $u^{(k)}$. This approach demands some monotonicity of the functions λ and W.

4.1. The Lipschitz case. Suppose that Assumptions (V0)–(V2), (C0), (Λ1), and (W1), formulated in Section 2, are valid. We modify some assumptions on the functions λ and W as follows:

($\widetilde{\Lambda}$0) $\lambda:\Omega\to\mathbb{R}$ is continuous in (t,x,w,q) and there exists a function $L\in L^1([0,a],\mathbb{R}_+)$ such that

$$\left|\lambda(t,x,w,q)-\lambda(t,\overline{x},\overline{w},\overline{q})\right| \leq L(t)\left(\|x-\overline{x}\|+\|w-\overline{w}\|+\|q-\overline{q}\|\right); \tag{4.1}$$

($\widetilde{\mathrm{W}}0$) $W:\Omega\to\mathbb{R}$ and there exists a function $L\in L^1([0,a],\mathbb{R}_+)$ such that

$$|W(t,x,w,q)-W(t,\overline{x},\overline{w},\overline{q})|\le L(t)\big(\|x-\overline{x}\|+\|w-\overline{w}\|+\|q-\overline{q}\|\big). \tag{4.2}$$

Using the same notation as in the proof of Theorem 3.1, we have the estimates

$$\begin{aligned}
||\Delta\eta^{(k)}(s;t,x)|| &\le \int_s^t L_c^*\left|\left|\Delta z_\zeta^{(k)}\right|\right|e^{L_c(\zeta-s)}d\zeta,\\
|\Delta u^{(k)}(t,x)| &\le L_v||\Delta\eta^{(k)}(0;t,x)||\exp\left(\int_0^t L_\lambda(s)ds\right)\\
&\quad+\|v\|_\infty\exp\left(\int_0^t L_\lambda(s)ds\right)\int_0^t L(s)P^{(k)}(s;t,x)ds,\\
|\Delta z^{(k+1)}(t)| &\le Z(a)\int_0^t L(s)P^{(k)}(s;t,x)ds,
\end{aligned} \tag{4.3}$$

where $P^{(k)}(s;t,x)=\|\Delta\eta^{(k)}(s;t,x)\|+\|\Delta u^{(k)}\|_s+\|\Delta z^{(k)}\|_s$.

Denote $\widehat{L}_\lambda=\int_0^a L_\lambda(s)ds$ and

$$\Psi^{(k)}(s,t)=\overline{\psi}^{(k)}(s)+\psi^{(k)}(s)+\int_s^t L_c^*e^{L_ca}\psi^{(k)}(\zeta)d\zeta. \tag{4.4}$$

Similarly, as in the previous section, consider the comparison equations

$$\begin{aligned}
\overline{\psi}^{(k)}(t)&=L_v\int_0^t L_c^*e^{L_ca+\widehat{L}_\lambda}\psi^{(k)}(s)ds+\|v\|_\infty e^{\widehat{L}_\lambda}\int_0^t L(s)\Psi^{(k)}(s,t)ds,\\
\psi^{(k+1)}(t)&=Z(a)\int_0^t L(s)\Psi^{(k)}(s,t)ds
\end{aligned} \tag{4.5}$$

with $\psi^{(0)}(t)=Z(a)$ and

$$\begin{aligned}
\overline{\psi}^{(0)}(t)&=\|v\|_\infty\exp\left(\int_0^t L_W(s)ds\right)+L_vL_c^*e^{L_ca+\widehat{L}_\lambda}Z(a)t\\
&\quad+\|v\|_\infty e^{\widehat{L}_\lambda}\int_0^t L(s)\big(\overline{\psi}^{(0)}(s)+Z(a)+(t-s)L_c^*e^{L_ca}Z(a)\big)ds.
\end{aligned} \tag{4.6}$$

Lemma 4.1. *Under the assumptions (V0)–(V2), (C0), ($\widetilde{\Lambda}$0), ($\widetilde{W}$0), (Λ1), and (W1) the following estimates hold:* $|\Delta u^{(k)}(t,x)|\le\overline{\psi}^{(k)}(t)$, $|\Delta z^{(k)}(t)|\le\psi^{(k)}(t)$,

$$||\Delta\eta^{(k)}(s;t,x)||\le\int_s^t L_c^*e^{L_ca}\psi^{(k)}(\zeta)d\zeta \tag{4.7}$$

on $[0,a]\times\mathbf{R}_+^n$, *and the sequences* $\{\psi^{(k)}\}$ *and* $\{\overline{\psi}^{(k)}\}$ *are nondecreasing in k.*

Lemma 4.2. *Under the assumptions of Lemma 4.1, the sequences* $\{\psi^{(k)}\}$ *and* $\{\overline{\psi}^{(k)}\}$ *tend uniformly to* 0 *as* $k\to+\infty$.

Proof. Denote $M = L_v L_c^* e^{L_c a + \hat{L}_\lambda}$, $M^* = \|v\|_\infty e^{\hat{L}_\lambda}, c_a = L_c^* e^{L_c a}$, and $\hat{L} = \int_0^a L(s)ds$. From (4.5), we have the estimates

$$\begin{aligned} \overline{\psi}^{(k)}(t) &\leq \Gamma(a) \int_0^t M^* L(s) \psi^{(k)}(s) ds, \\ \psi^{(k+1)}(t) &\leq Z(a) \int_0^t \Delta(s) \psi^{(k)}(s) ds, \end{aligned} \tag{4.8}$$

where $\Gamma(t) = \exp\left(\int_0^t (M + M^* L(s)) ds\right)$ and $\Delta(t) = \hat{L}\Gamma(a) M^* L(t) + L(t) + c_a \hat{L}$. A simple calculation shows that

$$\psi^{(k)}(t) \leq \frac{Z(a)^{k+1} \left(\int_0^t \Delta(s) ds\right)^k}{k!}, \quad t \in [0,a]. \tag{4.9}$$

Hence the sequences $\{\psi^{(k)}\}$ and $\{\overline{\psi}^{(k)}\}$ tend uniformly to 0 as $k \to +\infty$. □

Theorem 4.3. *Under the assumptions of Lemma 4.1, the sequences $\{z^{(k)}\}$, $\{u^{(k)}\}$, and $\{\eta^{(k)}\}$ tend uniformly to z, $u[z]$, $\eta[z]$ such that $z = \mathcal{T}[z]$.*

Proof. The assertion follows from Lemma 4.2. □

Remark 4.4. The assertion of Theorem 4.3 holds if the integrable function $L(\cdot)$ in Assumptions $(\widetilde{\Lambda}0)$ and $(\widetilde{W}0)$ is constant: $L(t) = \overline{L}$. The increments $z^{(k+1)} - z^{(k)}$, $u^{(k+1)} - u^{(k)}$ tend to zero very fast since they are estimated by the sequences from Lemma 4.2.

4.2. Monotone iterations. In the sequel, assume that $\overline{c}_j : E \to \mathbb{R}$ is given by the formula $\overline{c}_j(t,x) = c_j(t,x,0)$ for $j = 1,\ldots,n$, which means that it does not depend on z. Consider the differential-functional equation

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{n} \overline{c}_j(t,x) \frac{\partial u}{\partial x_j} = u(t,x) \lambda(t,x,u_{(t,x)}, z[u]_t), \tag{4.10}$$

with the function z given by (1.6) and with the initial condition (1.7). Similarly, as in Section 3, define the iterative method by means of integral equations

$$\eta(s;t,x) = x - \int_s^t \overline{c}(\zeta, \eta(\zeta;t,x)) d\zeta, \tag{4.11}$$

$$u^{(k)}(t,x) = v(0, \eta(0;t,x)) \exp\left(\int_0^t \lambda(Q^{(k)}(s)) ds\right), \tag{4.12}$$

$$z^{(k+1)}(t) = \int_{\mathbb{R}^n} v(0,y) \exp\left(\int_0^t W(R^{(k)}(s)) ds\right) dy, \tag{4.13}$$

where

$$\begin{aligned} Q^{(k)}(s) &= \left(s, \eta(s;t,x), u^{(k)}_{(s,\eta(s;t,x))}, z_s^{(k)}\right), \\ R^{(k)}(s) &= \left(s, \eta(s;0,y), u^{(k)}_{(s,\eta(s;0,y))}, z_s^{(k)}\right). \end{aligned} \tag{4.14}$$

Since the functions $\overline{c}_j$, $j = 1,\ldots,n$, do not depend on z, bicharacteristics at each stage of iterations remain the same. The iterations start with the functions $z(t) = 0, t \in [0,a]$ or $z(t) = Z(t), t \in [0,a]$, where Z is given by (2.13). Both cases demand some monotonicity of the functions λ and W. We modify some assumptions on the functions $\overline{c}$, λ, and W as follows:

($\overline{\mathrm{C}}0$) $\overline{c}_j : E \to \mathbb{R}$ are continuous in (t,x) and

$$\left\|\overline{c}(t,x) - \overline{c}(t,\overline{x})\right\| \leq \sigma_c(t, \|x - \overline{x}\|), \tag{4.15}$$

where σ_c is a Perron comparison function;

($\overline{\Lambda}0$) $\lambda : \Omega \to \mathbb{R}$ is continuous in (t,x,w,q), quasimonotone with respect to the last two variables and

$$\left|\lambda(t,x,w,q) - \lambda(t,\overline{x},\overline{w},\overline{q})\right| \leq \sigma(t, \|x - \overline{x}\| + \|w - \overline{w}\| + \|q - \overline{q}\|), \tag{4.16}$$

where $\sigma : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a Perron comparison function.

Denote

$$W(t,x,w,q) = \lambda(t,x,w,q) + \operatorname{tr}\partial_x \overline{c}(t,x). \tag{4.17}$$

Remark 4.5. The monotonicity of the function W with respect to the last two variables follows from the monotonicity of the function λ.

Theorem 4.6. *If assumptions (V0), (V1), ($\overline{\mathrm{C}}$0), ($\overline{\Lambda}$0), (Λ1), and (W1) are satisfied and* $z^{(0)}(t) = 0$ ($t \in [0,a]$), *then the sequences* $\{z^{(k)}\}$ *and* $\{u^{(k)}\}$ *are nondecreasing and tend to* $z, u[z]$ *such that* $z = \mathcal{T}[z]$.

Proof. For a given function $z^{(0)}$, by (4.12), we find the function $u^{(0)}$, which is the solution of (1.5). The functions $z^{(1)}$, $u^{(1)}$ are computed by (4.11)–(4.13). Clearly, $z^{(0)}(t) \leq z^{(1)}(t)$, $t \in [0,a]$, and the functions $u^{(k)}$, $k \geq 0$, are solutions of (1.5) for a given function $z = z^{(k)} \in \mathfrak{X}$. From the monotonicity of λ with respect to the last variable, we have inequalities

$$\partial_t u^{(k)}(t,x) - F[u^{(k)}, z^{(k)}](t,x) \leq \partial_t u^{(k+1)}(t,x) - F[u^{(k+1)}, z^{(k+1)}](t,x) \tag{4.18}$$

on E, where F is Niemycki operator corresponding to (4.10), that is,

$$F[u,z](t,x) = u(t,x)\lambda(t,x,u_{(t,x)}, z[u]_t) - \sum_{j=1}^{n} \overline{c}_j(t,x)\frac{\partial u}{\partial x_j}(t,x). \tag{4.19}$$

The initial condition for the functions $u^{(k)}$ ($k = 0,1,\ldots$) is given by (1.7). The theorem on functional differential inequalities yields $u^{(k)} \leq u^{(k+1)}$ on E (see [11, pp. 142–145, Theorems 5.5 and 5.10]). The monotonicity of the sequence $\{z^{(k)}\}$ follows from (4.13), the monotonicity of λ, and Remark 4.5. □

Now, we discus the case when the iteration starts with the function $z^{(0)} = Z(t), t \in [0,a]$, where Z is given by (2.13). Under the respective monotonicity assumptions on λ and W, we prove that the sequences $\{u^{(k)}\}$ and $\{z^{(k)}\}$ are nonincreasing and tend to the unique solution of problem (1.5)–(1.7). The only difficulty is to choose an integrable function $u^{(0)}$ which estimates all the solutions obtained in the iterative process. We state it as follows.

THEOREM 4.7. *If assumptions (V0), (V1), ($\overline{C}$0), ($\overline{\Lambda}$0), (Λ1), and (W1) are satisfied,* $z^{(0)}(t) = Z(t)$ *and* $u^{(0)}(t,x) = v(0,\eta(0;t,x))\exp\left(\int_0^t L_\lambda(s)ds\right)$, *then* $\{z^{(k)}\}$ *and* $\{u^{(k)}\}$ *are nonincreasing and tend to* $z, u[z]$ *such that* $z = \mathcal{T}[z]$.

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