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Research Article

Hybrid Steepest Descent Method with Variable Parameters for General Variational Inequalities

Yanrong Yu and Rudong Chen

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We study the strong convergence of a hybrid steepest descent method with variable parameters for the general variational inequality GVI(F,g,C). Consequently, as an application, we obtain some results concerning the constrained generalized pseudoinverse. Our results extend and improve the result of Yao and Noor (2007) and many others.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $F : H \to H$ be an operator such that for some constants $k,\eta > 0$, F is k-Lipschitzian and η-strongly monotone on C; that is, F satisfies the following inequalities: $\|Fx - Fy\| \leq k\|x - y\|$ and $\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2$ for all $x, y \in C$, respectively. Recall that T is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$.

We consider the following variational inequality problem: find a point $u^* \in C$ such that

$$\mathrm{VI}(F,C) : \langle F(u^*), v - v^*\rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

Variational inequalities were introduced and studied by Stampacchia [1] in 1964. It is now well known that a wide class of problems arising in various branches of pure and applied sciences can be studied in the general and unified framework of variational inequalities. Several numerical methods including the projection and its variant forms, Wiener-Hofp equations, auxiliary principle, and descent type have been developed for solving the variational inequalities and related optimization problems. The reader is referred to [1–18] and the references therein.

It is well known that when F is strongly monotone on C, the VI(F,C) has a unique solution and VI(F,C) is equivalent to the fixed point problem

$$u^* = P_C(u^* - \mu F(u^*)), \tag{1.2}$$

where $\mu > 0$ is an arbitrarily fixed constant and P_C is the (nearest point) projection from H onto C. From (1.2), one can suggest a so-called projection method. Using the projection method, one establishes the equivalence between the variational inequalities and fixed-point problem. This alternative equivalence has been used to study the existence theory of the solution and to develop several iterative-type algorithms for solving variational inequalities. Under certain conditions, projection methods and their variant forms can be implemented for solving variational inequalities. However, there are some drawbacks of this method which rule out its problems in applications, for instance, the projection method involves the projection P_C which may not be easily computed due to the complexity of the convex set C.

In order to reduce the complexity probably caused by the projection P_C, Yamada [11] introduced the following hybrid steepest descent method for solving the VI(F,C).

Algorithm 1.1. *For a given $u_0 \in H$, calculate the approximate solution u_n by the iterative scheme*

$$u_{n+1} = Tu_n - \lambda_{n+1}\mu F(Tu_n), \quad n \geq 0, \tag{1.3}$$

where $\mu \in (0, 2\eta/k^2)$ and $\lambda_n \in (0,1)$ satisfy the following conditions:

(1) $\lim_{n\to\infty} \lambda_n = 0$;

(2) $\sum_{n=1}^{\infty} \lambda_n = \infty$;

(3) $\lim_{n\to\infty} (\lambda_n - \lambda_{n+1})/\lambda_{n+1}^2 = 0$.

Yamada [11] proved that the approximate solution $\{u_n\}$, obtained from Algorithm 1.1, converges strongly to the unique solution of the VI(F,C).

Furthermore, Xu and Kim [12] and Zeng et al. [15] considered and studied the convergence of the hybrid steepest descent Algorithm 1.1 and its variant form. For details, please see [12, 15].

Let $F: H \to H$ be a nonlinear operator and let $g: H \to H$ be a continuous mapping. Now, we consider the following general variational inequality problem: find a point $u^* \in H$ such that $g(u^*) \in C$ and

$$\text{GVI}(F,g,C): \langle F(u^*), g(v) - g(u^*)\rangle \geq 0, \quad \forall v \in H,\ g(v) \in C. \tag{1.4}$$

If g is the identity mapping of H, then the GVI(F,g,C) reduces to the VI(F,C).

Although iterative algorithm (1.3) has successfully been applied to finding the unique solution of the VI(F,C). It is clear that it can not be directly applied to computing solution of the GVI(F,g,C) due to the presence of g. Therefore, an important problem is how to apply hybrid steepest descent method to solving GVI(F,g,C). For this purpose, Zeng et al. [13] introduced a hybrid steepest descent method for solving the GVI(F,g,C) as follows.

ALGORITHM 1.2. *Let* $\{\lambda_n\} \subset (0,1)$, $\{\theta_n\} \subset (0,1]$, *and* $\mu \in (0,2\eta/k^2)$. *For a given* $u_0 \in H$, *calculate the approximate solution* u_n *by the iterative scheme*

$$u_{n+1} = (1+\theta_{n+1})Tu_n - \theta_{n+1}g(Tu_n) - \lambda_{n+1}\mu F(Tu_n), \quad n \geq 0, \tag{1.5}$$

where F *is* η*-strongly monotone and* k*-Lipschitzian and* g *is* σ*-Lipschitzian and* δ*-strongly monotone on* C.

They also proved that the approximate solution $\{u_n\}$ obtained from (1.5) converges strongly to the solution of the GVI(F,g,C) under some assumptions on parameters. Consequently, Yao and Noor [7] present a modified iterative algorithm for approximating solution of the GVI(F,g,C). But we note that all of the above work has imposed some additional assumptions on parameters or the iterative sequence $\{u_n\}$. There is a natural question that rises: could we relax it?

Our purpose in this paper is to suggest and analyze a hybrid steepest descent method with variable parameters for solving general variational inequalities. It is shown that the convergence of the proposed method can be proved under some mild conditions on parameters. We also give an application of the proposed method for solving constrained generalized pseudoinverse problem.

2. Preliminaries

In the sequel, we will make use of the following results.

LEMMA 2.1 [12]. *Let* $\{s_n\}$ *be a sequence of nonnegative numbers satisfying the condition*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\beta_n, \quad n \geq 0, \tag{2.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *are sequences of real numbers such that*

(i) $\{\alpha_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$,

(ii) $\limsup_{n\to\infty}\beta_n \leq 0$ *or* $\sum_{n=0}^{\infty}\alpha_n\beta_n$ *is convergent.*

Then, $\lim_{n\to\infty} s_n = 0$.

LEMMA 2.2 [19]. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. *Suppose* $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then,* $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

LEMMA 2.3 [20] (demiclosedness principle). *Assume that* T *is a nonexpansive self-mapping of a closed convex subset* C *of a Hilbert space* H. *If* T *has a fixed point, then* $I - T$ *is demiclosed. That is, whenever* $\{x_n\}$ *is a sequence in* C *weakly converging to some* $x \in C$ *and the sequence* $\{(I-T)x_n\}$ *strongly converges to some* y, *it follows that* $(I-T)x = y$. *Here,* I *is the identity operator of* H.

The following lemma is an immediate consequence of an inner product.

LEMMA 2.4. *In a real Hilbert space* H, *there holds the inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x,y \in H. \tag{2.2}$$

3. Modified hybrid steepest descent method

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $F: H \to H$ be k-Lipschitzian and η-strongly monotone mapping on C and let $g: H \to H$ be σ-Lipschitzian and δ-strongly monotone mapping on C for some constants $\sigma > 0$ and $\delta > 1$. Assume also that the unique solution u^* of the VI(F,C) is a fixed point of g.

Denote by P_C the projection of H onto C. Namely, for each $x \in H$, $P_C x$ is the unique element in C satisfying

$$||x - P_C x|| = \min\{\|x - y\| : y \in C\}. \tag{3.1}$$

It is known that the projection P_C is characterized by inequality

$$\langle x - P_C x, y - P_C x\rangle \le 0, \quad \forall y \in C. \tag{3.2}$$

Thus, it follows that the GVI(F,g,C) is equivalent to the fixed point problem $g(u^*) = P_C(I - \mu F)g(u^*)$, where $\mu > 0$ is an arbitrary constant.

In this section, assume that $T_i : H \to H$ is a nonexpansive mapping for each $1 \le i \le N$ with $\bigcap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset$. Let $\delta_{n1}, \delta_{n2}, \dots, \delta_{nN} \in (0,1]$, $n \ge 1$. We define, for each $n \ge 1$, mappings $U_{n1}, U_{n2}, \dots, U_{nN}$ by

$$\begin{aligned} U_{n1} &= \delta_{n1} T_1 + (1 - \delta_{n1}) I, \\ U_{n2} &= \delta_{n2} T_2 U_{n1} + (1 - \delta_{n2}) I, \\ &\vdots \\ U_{n,N-1} &= \delta_{n,N-1} T_{N-1} U_{n,N-2} + (1 - \delta_{n,N-1}) I, \\ W_n := U_{nN} &= \delta_{nN} T_N U_{n,N-1} + (1 - \delta_{nN}) I. \end{aligned} \tag{3.3}$$

Such a mapping W_n is called the W-mapping generated by $T_1, \dots, T_N$ and $\delta_{n1}, \delta_{n2}, \dots, \delta_{nN}$. Nonexpansivity of T_i yields the nonexpansivity of W_n. Moreover, [21, Lemma 3.1] shows that

$$\mathrm{Fix}(W_n) = F. \tag{3.4}$$

Such property of W_n will be crucial in the proof on our result.

Now we suggest the following iterative algorithm for solving GVI(F,g,C).

ALGORITHM 3.1. *Let $\{\alpha_n\} \subset [a,b] \subset (0,1)$, $\{\lambda_n\} \subset (0,1)$, $\{\theta_n\} \subset (0,1]$, and $\{\mu_n\} \subset (0, 2\eta/k^2)$. For a given $u_0 \in H$, compute the approximate solution $\{u_n\}$ by the iterative scheme*

$$\begin{aligned} u_{n+1} = {} & W_n u_n - \lambda_{n+1}\mu_{n+1} F(W_n u_n) + \alpha_{n+1}(u_n - W_n u_n) \\ & + \theta_{n+1}(W_n u_n - g(W_n u_n)), \quad n \ge 0. \end{aligned} \tag{3.5}$$

At this point, we state and prove our main result.

THEOREM 3.2. *Assume that $0 < a \le \alpha_n \le b < 1$, $0 < \mu_n < 2\eta/k^2$, and $u^* \in Fix(g)$. Let δ_{n1}, $\delta_{n2}, \dots, \delta_{nN}$ be real numbers such that $\lim_{n\to\infty}(\delta_{n+1,i} - \delta_{n,i}) = 0$ for all $i = 1, 2, \dots, N$. Assume*

$\{\lambda_n\}$ and $\{\theta_n\}$ satisfy the follwoing conditions:

(i) $\lim_{n\to\infty}\lambda_n = 0$, $\sum_{n=1}^{\infty}\lambda_n = \infty$;

(ii) $\theta_n \in (0, 2(1-a)(\delta-1)/(\sigma^2-1)]$;

(iii) $\lim_{n\to\infty}\theta_n = 0, \lim_{n\to\infty}\lambda_n/\theta_n = 0$.

Then the sequence $\{u_n\}$ generated by Algorithm 3.1 converges strongly to u^* which is a solution of the GVI(F, g, C).

Proof. Now we divide our proof into the following steps.

Step 1. First, we prove that $\{u_n\}$ is bounded. From (3.5), we have

$$\begin{aligned}||u_{n+1}-u^*|| &= ||(1-\alpha_{n+1}+\theta_{n+1})W_nu_n+\alpha_{n+1}u_n-\theta_{n+1}g(W_nu_n)\\ &\quad -\lambda_{n+1}\mu_{n+1}F(W_nu_n)-u^*||\\ &= ||(1-\alpha_{n+1})(W_nu_n-u^*)-\theta_{n+1}(g(W_nu_n)-u^*)\\ &\quad +\alpha_{n+1}(u_n-u^*)+\theta_{n+1}(W_nu_n-u^*)\\ &\quad -\lambda_{n+1}\mu_{n+1}(F(W_nu_n)-F(u^*))+\lambda_{n+1}\mu_{n+1}F(u^*)||\\ &\le ||(1-\alpha_{n+1})(W_nu_n-u^*)-\theta_{n+1}(g(W_nu_n)-u^*)||\\ &\quad +||\theta_{n+1}(W_nu_n-u^*)-\lambda_{n+1}\mu_{n+1}(F(W_nu_n)-F(u^*))||\\ &\quad +\alpha_{n+1}||u_n-u^*||+\lambda_{n+1}\mu_{n+1}||F(u^*)||.\end{aligned} \tag{3.6}$$

Observe that

$$\begin{aligned}&||(1-\alpha_{n+1})(W_nu_n-u^*)-\theta_{n+1}(g(W_nu_n)-u^*)||^2\\ &\quad=(1-\alpha_{n+1})^2||W_nu_n-u^*||^2\\ &\qquad-2(1-\alpha_{n+1})\theta_{n+1}\langle g(W_nu_n)-g(u^*),W_nu_n-u^*\rangle+\theta_{n+1}^2||g(W_nu_n)-u^*||^2\\ &\quad\le[(1-\alpha_{n+1})^2-2(1-\alpha_{n+1})\delta\theta_{n+1}+\sigma^2\theta_{n+1}^2]||W_nu_n-u^*||^2\\ &\quad\le[(1-\alpha_{n+1})^2-2(1-\alpha_{n+1})\delta\theta_{n+1}+\sigma^2\theta_{n+1}^2]||u_n-u^*||^2,\\ &[8pt]||\theta_{n+1}(W_nu_n-u^*)-\lambda_{n+1}\mu_{n+1}(F(W_nu_n)-F(u^*))||^2\\ &\quad=\theta_{n+1}^2||W_nu_n-u^*||^2-2\theta_{n+1}\lambda_{n+1}\mu_{n+1}\langle F(W_nu_n)-F(u^*),W_nu_n-u^*\rangle\\ &\qquad+\lambda_{n+1}^2\mu_{n+1}^2||F(W_nu_n)-F(u^*)||^2\\ &\quad\le(\theta_{n+1}^2-2\mu_{n+1}\eta\theta_{n+1}\lambda_{n+1}+\mu_{n+1}^2k^2\lambda_{n+1})||W_nu_n-u^*||^2\\ &\quad\le(\theta_{n+1}^2-2\mu_{n+1}\eta\theta_{n+1}\lambda_{n+1}+\mu_{n+1}^2k^2\lambda_{n+1})||u_n-u^*||^2\\ &\quad=\theta_{n+1}^2\left[\left(1-\frac{\lambda_{n+1}}{\theta_{n+1}}\mu_{n+1}k\right)^2+\frac{2\lambda_{n+1}\mu_{n+1}(k-\eta)}{\theta_{n+1}}\right]||u_n-u^*||^2.\end{aligned} \tag{3.7}$$

From (3.7), we have

$$\begin{aligned}
&\|u_{n+1}-u^*\| \\
&\quad\le \left(\sqrt{(1-\alpha_{n+1})^2-2(1-\alpha_{n+1})\delta\theta_{n+1}+\sigma^2\theta_{n+1}^2}+\alpha_{n+1}\right)\|u_n-u^*\| \\
&\qquad+\theta_{n+1}\sqrt{\left(1-\frac{\lambda_{n+1}\mu_{n+1}k}{\theta_{n+1}}\right)^2+\frac{2\lambda_{n+1}\mu_{n+1}(k-\eta)}{\theta_{n+1}}}\|u_n-u^*\|+\lambda_{n+1}\mu_{n+1}\|F(u^*)\| \\
&\quad\le \left(\sqrt{(1-\alpha_{n+1})^2-2(1-\alpha_{n+1})\delta\theta_{n+1}+\sigma^2\theta_{n+1}^2}+\alpha_{n+1}\right)\|u_n-u^*\| \\
&\qquad+\theta_{n+1}\left|1-\frac{\lambda_{n+1}\mu_{n+1}k}{\theta_{n+1}}\right|\sqrt{1+\left(\frac{2\lambda_{n+1}\mu_{n+1}(k-\eta)}{\theta_{n+1}}\right)\Big/\left(1-\frac{\lambda_{n+1}\mu_{n+1}k}{\theta_{n+1}}\right)^2} \\
&\qquad\times\|u_n-u^*\|+\lambda_{n+1}\mu_{n+1}\|F(u^*)\|.
\end{aligned}\tag{3.8}$$

Now we can see that (iii) yields

$$\lim_{n\to\infty}\left(\frac{\lambda_{n+1}\mu_{n+1}k}{\theta_{n+1}}-\frac{\eta}{k}\right)\Big/\left(1-\frac{\lambda_{n+1}\mu_{n+1}k}{\theta_{n+1}}\right)=-\frac{\eta}{k}.\tag{3.9}$$

Hence, we infer that there exists an integer $N_0\ge 0$ such that for all $n\ge N_0$, $(1/2)\lambda_{n+1}\mu_{n+1}\eta<1$, and $(\lambda_{n+1}\mu_{n+1}k/\theta_{n+1}-\eta/k)/(1-\lambda_{n+1}\mu_{n+1}k/\theta_{n+1})<-\eta/2k$. Thus we deduce that for all $n\ge N_0$,

$$\begin{aligned}
&\theta_{n+1}\left|1-\frac{\lambda_{n+1}\mu_{n+1}k}{\theta_{n+1}}\right|\sqrt{1+\left(\frac{2\lambda_{n+1}\mu_{n+1}(k-\eta)}{\theta_{n+1}}\right)\Big/\left(1-\frac{\lambda_{n+1}\mu_{n+1}k}{\theta_{n+1}}\right)^2} \\
&\quad\le \theta_{n+1}\left(1-\frac{\lambda_{n+1}\mu_{n+1}k}{\theta_{n+1}}\right)\left(1+\left(\frac{\lambda_{n+1}\mu_{n+1}(k-\eta)}{\theta_{n+1}}\right)\Big/\left(1-\frac{\lambda_{n+1}\mu_{n+1}k}{\theta_{n+1}}\right)^2\right) \\
&\quad=\theta_{n+1}-\lambda_{n+1}\mu_{n+1}k+\frac{\lambda_{n+1}\mu_{n+1}(k-\eta)}{1-\lambda_{n+1}\mu_{n+1}k/\theta_{n+1}} \\
&\quad=\theta_{n+1}+\frac{-\lambda_{n+1}\mu_{n+1}k+(\lambda_{n+1}\mu_{n+1}k)^2/\theta_{n+1}+\lambda_{n+1}\mu_{n+1}k-\lambda_{n+1}\mu_{n+1}\eta}{1-\lambda_{n+1}\mu_{n+1}k/\theta_{n+1}} \\
&\quad=\theta_{n+1}+\lambda_{n+1}\mu_{n+1}k\left[\left(\frac{\lambda_{n+1}\mu_{n+1}k}{\theta_{n+1}}-\frac{\eta}{k}\right)\Big/\left(1-\frac{\lambda_{n+1}\mu_{n+1}k}{\theta_{n+1}}\right)\right] \\
&\quad\le\theta_{n+1}-\frac{1}{2}\lambda_{n+1}\mu_{n+1}\eta.
\end{aligned}\tag{3.10}$$

From (ii) and (iii), we can choose sufficient small θ_{n+1} such that

$$\begin{aligned}
& 0<\theta_{n+1} \leq \frac{2(1-\alpha_{n+1})(\delta-1)}{\sigma^2-1} \\
& \Longrightarrow \theta_{n+1}(\sigma^2-1) \leq 2(1-\alpha_{n+1})(\delta-1) \\
& \Longrightarrow \sigma^2\theta_{n+1}-2(1-\alpha_{n+1})\delta \leq \theta_{n+1}-2(1-\alpha_{n+1}) \\
& \Longrightarrow \sigma^2\theta_{n+1}^2-2(1-\alpha_{n+1})\delta\theta_{n+1} \\
& \leq \theta_{n+1}^2-2\theta_{n+1}(1-\alpha_{n+1}) \\
& \Longrightarrow (1-\alpha_{n+1})^2-2(1-\alpha_{n+1})\delta\theta_{n+1}+\sigma^2\theta_{n+1}^2 \\
& \leq (1-\alpha_{n+1})^2-2\theta_{n+1}(1-\alpha_{n+1})+\theta_{n+1}^2 \\
& \Longrightarrow \sqrt{(1-\alpha_{n+1})^2-2(1-\alpha_{n+1})\delta\theta_{n+1}+\sigma^2\theta_{n+1}^2} \\
& \leq 1-\alpha_{n+1}-\theta_{n+1} \\
& \Longrightarrow \sqrt{(1-\alpha_{n+1})^2-2(1-\alpha_{n+1})\delta\theta_{n+1}+\sigma^2\theta_{n+1}^2} \\
& \quad +\alpha_{n+1}+\theta_{n+1} \leq 1.
\end{aligned} \tag{3.11}$$

Consequently it follows from (3.6) and (3.8)–(3.11), for all $n \geq N_0$, that

$$\|u_{n+1}-u^*\| \leq \left(1-\frac{1}{2}\lambda_{n+1}\mu_{n+1}\eta\right)\|u_n-u^*\|+\lambda_{n+1}\mu_{n+1}\|F(u^*)\|. \tag{3.12}$$

By induction, it easy to see that

$$\|u_n-u^*\| \leq \max\left\{\max_{0\leq i\leq N_0}\|u_i-u^*\|, \frac{2}{\eta}\|F(u^*)\|\right\}, \quad n \geq 0. \tag{3.13}$$

Hence, $\{x_n\}$ is bounded, so are $\{W_nu_n\}$, $\{g(u_n)\}$, and $\{F(W_nu_n)\}$. We will use M to denote the possible different constants appearing in the following reasoning.

Define

$$u_{n+1}=\alpha_{n+1}u_n+(1-\alpha_{n+1})y_n. \tag{3.14}$$

From the definition of y_n, we obtain

$$
\begin{aligned}
y_{n+1}-y_n &= \frac{u_{n+2}-\alpha_{n+2}u_{n+1}}{1-\alpha_{n+2}}-\frac{u_{n+1}-\alpha_{n+1}u_n}{1-\alpha_{n+1}} \\
&= \frac{(1-\alpha_{n+2}+\theta_{n+2})W_{n+1}u_{n+1}-\theta_{n+2}g(W_{n+1}u_{n+1})}{1-\alpha_{n+2}} \\
&\quad -\frac{\lambda_{n+2}\mu_{n+2}F(W_{n+1}u_{n+1})}{1-\alpha_{n+2}}+\frac{\lambda_{n+1}\mu_{n+1}F(W_nu_n)}{1-\alpha_{n+1}} \\
&\quad -\frac{(1-\alpha_{n+1}+\theta_{n+1})W_nu_n-\theta_{n+1}g(W_nu_n)}{1-\alpha_{n+1}} \\
&= W_{n+1}u_{n+1}-W_nu_n+\frac{\theta_{n+2}}{1-\alpha_{n+2}}W_{n+1}u_{n+1}-\frac{\theta_{n+1}}{1-\alpha_{n+1}}W_nu_n \\
&\quad +\frac{\theta_{n+1}}{1-\alpha_{n+1}}g(W_nu_n)-\frac{\theta_{n+2}}{1-\alpha_{n+2}}g(W_{n+1}u_{n+1}) \\
&\quad +\frac{\lambda_{n+1}\mu_{n+1}}{1-\alpha_{n+1}}F(W_nu_n)-\frac{\lambda_{n+2}\mu_{n+2}}{1-\alpha_{n+2}}F(W_{n+1}u_{n+1}) \\
&= W_{n+1}u_{n+1}-W_{n+1}u_n+W_{n+1}u_n-W_nu_n \\
&\quad +\frac{\theta_{n+2}}{1-\alpha_{n+2}}W_{n+1}u_{n+1}-\frac{\theta_{n+1}}{1-\alpha_{n+1}}W_nu_n \\
&\quad +\frac{\theta_{n+1}}{1-\alpha_{n+1}}g(W_nu_n)-\frac{\theta_{n+2}}{1-\alpha_{n+2}}g(Wn+1u_{n+1}) \\
&\quad +\frac{\lambda_{n+1}\mu_{n+1}}{1-\alpha_{n+1}}F(W_nu_n)-\frac{\lambda_{n+2}\mu_{n+2}}{1-\alpha_{n+2}}F(W_{n+1}u_{n+1}).
\end{aligned}
\tag{3.15}
$$

It follows that

$$
\begin{aligned}
&\|y_{n+1}-y_n\|-\|u_{n+1}-u_n\| \\
&\quad\le \|W_{n+1}u_n-W_nu_n\|+\frac{\theta_{n+2}}{1-\alpha_{n+2}}\|W_{n+1}u_{n+1}\| \\
&\qquad +\frac{\theta_{n+1}}{1-\alpha_{n+1}}\|W_nu_n\|+\frac{\theta_{n+1}}{1-\alpha_{n+1}}\|g(W_nu_n)\|+\frac{\theta_{n+2}}{1-\alpha_{n+2}}\|g(W_{n+1}u_{n+1})\| \\
&\qquad +\frac{\lambda_{n+1}\mu_{n+1}}{1-\alpha_{n+1}}\|F(W_nu_n)\|+\frac{\lambda_{n+2}\mu_{n+2}}{1-\alpha_{n+2}}\|F(W_{n+1}u_{n+1})\|.
\end{aligned}
\tag{3.16}
$$

From (3.3), since T_i and $U_{n,i}$ for all $i=1,2,\dots,N$ are nonexpansive,

$$
\begin{aligned}
&\|W_{n+1}u_n-W_nu_n\| \\
&\quad=\|\delta_{n+1,N}T_NU_{n+1,N-1}u_n+(1-\delta_{n+1,N})u_n-\delta_{n,N}T_NU_{n,N-1}u_n-(1-\delta_{n,N})u_n\| \\
&\quad\le |\delta_{n+1,N}-\delta_{n,N}|\,\|u_n\|+\|\delta_{n+1,N}T_NU_{n+1,N-1}u_n-\delta_{n,N}T_NU_{n,N-1}u_n\| \\
&\quad\le |\delta_{n+1,N}-\delta_{n,N}|\,\|u_n\|+\|\delta_{n+1,N}(T_NU_{n+1,N-1}u_n-T_NU_{n,N-1}u_n)\| \\
&\qquad +|\delta_{n+1,N}-\delta_{n,N}|\,\|T_NU_{n,N-1}u_n\| \\
&\quad\le 2M|\delta_{n+1,N}-\delta_{n,N}|+\delta_{n+1,N}\|U_{n+1,N-1}u_n-U_{n,N-1}u_n\|.
\end{aligned}
\tag{3.17}
$$

Again, from (3.3),

$$\begin{aligned}
&\|U_{n+1,N-1}u_n - U_{n,N-1}u_n\| \\
&\quad = \|\delta_{n+1,N-1}T_{N-1}U_{n+1,N-2}u_n + (1-\delta_{n+1,N-1})u_n \\
&\qquad - \delta_{n,N-1}T_{N-1}U_{n,N-2}u_n - (1-\delta_{n,N-1})u_n\| \\
&\quad \le |\delta_{n+1,N-1} - \delta_{n,N-1}|\,\|u_n\| \\
&\qquad + \|\delta_{n+1,N-1}T_{N-1}U_{n+1,N-2}u_n - \delta_{n,N-1}T_{N-1}U_{n,N-2}u_n\| \\
&\quad \le |\delta_{n+1,N-1} - \delta_{n,N-1}|\,\|u_n\| \\
&\qquad + \delta_{n+1,N-1}\|T_{N-1}U_{n+1,N-2}u_n - T_{N-1}U_{n,N-2}u_n\| \\
&\qquad + |\delta_{n+1,N-1} - \delta_{n,N-1}|M \\
&\quad \le 2M|\delta_{n+1,N-1} - \delta_{n,N-1}| + \delta_{n+1,N-1}\|U_{n+1,N-2}u_n - U_{n,N-2}u_n\| \\
&\quad \le 2M|\delta_{n+1,N-1} - \delta_{n,N-1}| + \|U_{n+1,N-2}u_n - U_{n,N-2}u_n\|.
\end{aligned} \tag{3.18}$$

Therefore, we have

$$\begin{aligned}
&\|U_{n+1,N-1}u_n - U_{n,N-1}u_n\| \\
&\quad \le 2M|\delta_{n+1,N-1} - \delta_{n,N-1}| + 2M|\delta_{n+1,N-2} - \delta_{n,N-2}| \\
&\qquad + \|U_{n+1,N-3}u_n - U_{n,N-3}u_n\| \\
&\quad \le 2M\sum_{i=2}^{N-1}|\delta_{n+1,i} - \delta_{n,i}| + \|U_{n+1,1}u_n - U_{n,1}u_n\| \\
&\quad = \|\delta_{n+1,1}T_1u_n + (1-\delta_{n+1,1})u_n - \delta_{n,1}T_1u_n - (1-\delta_{n,1})u_n\| \\
&\qquad + 2M\sum_{i=2}^{N-1}|\delta_{n+1,i} - \delta_{n,i}|,
\end{aligned} \tag{3.19}$$

then

$$\begin{aligned}
&\|U_{n+1,N-1}u_n - U_{n,N-1}u_n\| \\
&\quad \le |\delta_{n+1,1} - \delta_{n,1}|\,\|u_n\| + \|\delta_{n+1,1}T_1u_n - \delta_{n,1}T_1u_n\| \\
&\qquad + 2M\sum_{i=2}^{N-1}|\delta_{n+1,i} - \delta_{n,i}| \le 2M\sum_{i=1}^{N-1}|\delta_{n+1,i} - \delta_{n,i}|.
\end{aligned} \tag{3.20}$$

Substituting (3.20) into (3.17), we have

$$\begin{aligned}
\|W_{n+1}u_n - W_nu_n\| &\le 2M|\delta_{n+1,N} - \delta_{n,N}| + 2\delta_{n+1,N}M\sum_{i=1}^{N-1}|\delta_{n+1,i} - \delta_{n,i}| \\
&\le 2M\sum_{i=1}^{N}|\delta_{n+1,i} - \delta_{n,i}|.
\end{aligned} \tag{3.21}$$

Since $\{u_n\}$, $\{F(W_nu_n)\}$, $\{g(W_nu_n)\}$ are all bounded, it follows from (3.16), (3.21), (i), and (iii) that

$$\limsup_{n\to\infty}(||y_{n+1}-y_n||-||u_{n+1}-u_n||)\leq 0. \tag{3.22}$$

Hence, by Lemma 2.2, we know

$$\lim_{n\to\infty}||y_n-u_n||=0. \tag{3.23}$$

Consequently,

$$\lim_{n\to\infty}||u_{n+1}-u_n||=\lim_{n\to\infty}(1-\alpha_{n+1})||y_n-u_n||=0. \tag{3.24}$$

On the other hand,

$$\begin{aligned}||u_n-W_nu_n|| &\leq ||u_{n+1}-W_nu_n||+||u_{n+1}-u_n||\\ &\leq \alpha_{n+1}||u_n-W_nu_n||+\theta_{n+1}||W_nu_n||\\ &\quad+\theta_{n+1}||g(W_nu_n)||+\lambda_{n+1}\mu_{n+1}||F(W_nu_n)||\\ &\quad+||u_{n+1}-u_n||,\end{aligned} \tag{3.25}$$

this together with conditions (i), (iii), and (3.24) implies

$$\lim_{n\to\infty}||u_n-W_nu_n||=0. \tag{3.26}$$

We next show that

$$\limsup_{n\to\infty}\langle -F(x^*),u_n-x^*\rangle\leq 0. \tag{3.27}$$

To prove this, we pick a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\limsup_{n\to\infty}\langle -F(x^*),u_n-x^*\rangle=\lim_{i\to\infty}\langle -F(x^*),u_{n_i}-x^*\rangle. \tag{3.28}$$

Without loss of generality, we may further assume that $u_{n_i}\to z$ weakly for some $z\in H$.
By Lemma 2.3 and (3.26), we have

$$z\in \operatorname{Fix}(W_n), \tag{3.29}$$

this imply that

$$z\in\bigcap_{i=1}^{N}\operatorname{Fix}(T_i). \tag{3.30}$$

Since x^* solves $\mathrm{VI}(F,C)$. Then we obtain

$$\limsup_{n\to\infty}\langle -F(x^*),u_n-x^*\rangle=\langle -F(x^*),z-x^*\rangle\leq 0. \tag{3.31}$$

Finally, we show that $u_n \to u^*$ in norm. From (3.7)–(3.10) and Lemma 2.4, we have

$$\begin{aligned} ||u_{n+1} - u^*||^2 &= ||(1-\alpha_{n+1})(W_n u_n - u^*) - \theta_{n+1}(g(W_n u_n) - u^*) \\ &\quad + \alpha_{n+1}(u_n - u^*) + \theta_{n+1}(W_n u_n - u^*) \\ &\quad - \lambda_{n+1}\mu_{n+1}(F(W_n u_n) - F(u^*)) + \lambda_{n+1}\mu_{n+1}F(u^*)|| \\ &\le ||(1-\alpha_{n+1})(W_n u_n - u^*) - \theta_{n+1}(g(W_n u_n) - u^*) \\ &\quad + \alpha_{n+1}(u_n - u^*) + \theta_{n+1}(W_n u_n - u^*) \\ &\quad - \lambda_{n+1}\mu_{n+1}(F(W_n u_n) - F(u^*))||^2 \\ &\quad + 2\lambda_{n+1}\mu_{n+1}\langle -F(u^*), u_{n+1} - u^*\rangle \\ &\le \left(1 - \frac{1}{2}\lambda_{n+1}\mu_{n+1}\eta\right)||u_n - u^*||^2 \\ &\quad + 2\lambda_{n+1}\mu_{n+1}\langle -F(u^*), u_{n+1} - u^*\rangle. \end{aligned} \tag{3.32}$$

An application of Lemma 2.1 combined with (3.31) yields that $\|u_n - u^*\| \to 0$. This completes the proof. □

4. Application to constrained generalized pseudoinverse

Let K be a nonempty closed convex subset of a real Hilbert space H. Let A be a bounded linear operator on H. Given an element $b \in H$, consider the minimization problem

$$\min_{x\in K} \|Ax - b\|^2. \tag{4.1}$$

Let S_b denote the solution set. Then, S_b is closed and convex. It is known that S_b is nonempty if and only if $P_{\overline{A(K)}}(b) \in A(K)$. In this case, S_b has a unique element with minimum norm; that is, there exists a unique point $\hat{x} \in S_b$ satisfying

$$\|\hat{x}\|^2 = \min\{\|x\|^2 : x \in S_b\}. \tag{4.2}$$

Definition 4.1 [22]. The K-constrained pseudoinverse of A (symbol $\hat{A}_K$) is defined as

$$D(\hat{A}_K) = \{b \in H : P_{\overline{A(K)}}(b) \in A(K)\}, \quad \hat{A}_K(b) = \hat{x}, \quad b \in D(\hat{A}_k), \tag{4.3}$$

where $\hat{x} \in S_b$ is the unique solution of (4.2).

Now we recall the K-constrained generalized pseudoinverse of A.

Let $\theta : H \to R$ be a differentiable convex function such that θ' is a k-Lipschitzian and η-strongly monotone operator for some $k > 0$ and $\eta > 0$. Under these assumptions, there exists a unique point $\hat{x}_0 \in S_b$ for $b \in D(\hat{A}_K)$ such that

$$\theta(\hat{x}_0) = \min\{\theta(x) : x \in S_b\}. \tag{4.4}$$

Definition 4.2. The K-constrained generalized pseudoinverse of A associated with θ (symbol $\hat{A}_{K,\theta}$) is defined as $D(\hat{A}_{K,\theta}) = D(\hat{A}_K)$, $\hat{A}_{K,\theta}(b) = \hat{x}_0$, and $b \in D(\hat{A}_{K,\theta})$, where

$\hat{x}_0 \in S_b$ is the unique solution to (4.4). Note that if $\theta(x) = \|x\|^2/2$, then the K-constrained generalized pseudoinverse $\widehat{A}_{K,\theta}$ of A associated with θ reduces to the K-constrained pseudoinverse $\widehat{A}_K$ of A in Definition 4.1.

We now apply the result in Section 3 to construct the K-constrained generalized pseudoinverse $\widehat{A}_{K,\theta}$ of A. First observe that $\tilde{x} \in K$ satisfies the minimization problem (4.1) if and only if there holds the following optimality condition: $\langle A^*(A\tilde{x} - b), x - \tilde{x}\rangle \geq 0$, $x \in K$, where A^* is the adjoint of A. This for each $\lambda > 0$, is equivalent to,

$$\begin{aligned} \langle [\lambda A^* b + (I - \lambda A^* A)\tilde{x}] - \tilde{x}, \tilde{x} - x\rangle \geq 0, \quad x \in K, \\ P_K(\lambda A^* b + (I - \lambda A^* A)\tilde{x}) = \tilde{x}. \end{aligned} \tag{4.5}$$

Define a mapping $T : H \to H$ by

$$Tx = P_K(A^* b + (I - \lambda A^* A)x), \quad x \in H. \tag{4.6}$$

Lemma 4.3 [12]. *If $\lambda \in (0, 2\|A\|^{-2})$ and if $b \in D(\widehat{A}_K)$, then T is attracting nonexpansive and Fix$(T) = S_b$.*

The proofs of the following Theorems 4.4 and 4.5 are obtained easily; we omit them.

Theorem 4.4. *Assume that $0 < \mu_n < 2\eta/k^2$. Assume $\{\lambda_n\}$ and $\{\theta_n\}$ satisfy the following conditions:*

(i) $\lim_{n\to\infty} \lambda_n = 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$;
(ii) $\theta_n \in (0, 2(1-a)(\delta - 1)/(\sigma^2 - 1)]$;
(iii) $\lim_{n\to\infty} \theta_n = 0, \lim_{n\to\infty} \lambda_n/\theta_n = 0$.

Given an initial guess $u_0 \in H$, let $\{u_n\}$ be the sequence generated by the algorithm

$$\begin{aligned} u_{n+1} = {} & Tu_n - \lambda_{n+1}\mu_{n+1}\theta'(Tu_n) + \alpha_{n+1}(u_n - Tu_n) \\ & - \theta_{n+1}(g(Tu_n) - Tu_n), \quad n \geq 0, \end{aligned} \tag{4.7}$$

where T is given in (4.6). Suppose that the unique solution $\hat{u}_0$ of (4.4) is also a fixed point of g. Then $\{u_n\}$ strongly converges to $\widehat{A}_{K,\theta}(b)$.

Theorem 4.5. *Assume that $0 < \mu_n < 2\eta/k^2$. Assume that the restrictions (ii) and (iii) hold for $\{\theta_n\}$ and also that the control condition (i) holds for $\{\lambda_n\}$. Given an initial guess $u_0 \in H$, suppose that the unique solution $\widehat{u_0}$ of (4.4) is also a fixed point of g. Then the sequence $\{u_n\}$ generated by the algorithm*

$$\begin{aligned} u_{n+1} = {} & W_n u_n - \lambda_{n+1}\mu_{n+1}\theta'(W_n u_n) + \alpha_{n+1}(u_n - W_n u_n) \\ & - \theta_{n+1}(g(W_n u_n) - W_n u_n), \quad n \geq 0, \end{aligned} \tag{4.8}$$

converges to $\widehat{A}_{K,\theta}(b)$.

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Yanrong Yu: Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China
Email address: tjcrd@yahoo.com.cn

Rudong Chen: Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China
Email address: chenrd@tjpu.edu.cn