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Research Article

Generalized Variational Inequalities Involving Relaxed Monotone Mappings and Nonexpansive Mappings

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We consider the solvability of generalized variational inequalities involving multivalued relaxed monotone operators and single-valued nonexpansive mappings in the framework of Hilbert spaces. We also study the convergence criteria of iterative methods under some mild conditions. Our results improve and extend the recent ones announced by many others.

1. Introduction and preliminaries

Variational inequalities [1, 2] and hemivariational inequalities [3] have significant applications in various fields of mathematics, physics, economics, and engineering sciences. The associated operator equations are equally essential in the sense that these turn out to be powerful tools to the solvability of variational inequalities. Relaxed monotone operators have applications to constrained hemivariational inequalities. Since in the study of constrained problems in reflexive Banach spaces E the set of all admissible elements is nonconvex but star-shaped, corresponding variational formulations are no longer variational inequalities. Using hemivariational inequalities, one can prove the existence of solutions to the following type of nonconvex constrained problems (P): find u in C such that $\langle Au-g,v\rangle \geq 0$, for all $v \in T_C(u)$, where the admissible set $C \subset E$ is a star-shaped set with respect to a certain ball $B_E(u_0,\rho)$, and $T_C(u)$ denotes Clarke's tangent cone of C at u in C. It is easily seen that when C is convex, (1.1) reduces to the variational inequality of finding u in C such that $\langle Au-g,v\rangle \geq 0$, for all $\in C$.

Example 1.1 (see [3]). Let $A: E\to E^*$ be a maximal monotone operator from a reflexive Banach space E into E^* with strong monotonicity, and let $C \subset E$ be star-shaped with respect to a ball $B_E(u_0,\rho)$. Suppose that $Au_0 - g \neq 0$ and that distance function d_C satisfies the condition of relaxed monotonicity $\langle u^* - v^*, u - v\rangle \geq -c\|u-v\|^2$, for all $u,v \in E$, and for any $u^* \in \partial d_C(u)$ and v^* in $\partial d_C(v)$ with c satisfying $0 < c < 4a^2\rho/\|Au_0 - g\|^2$, where a is the constant for the strong monotonicity of A. Here ∂d_C is a relaxed monotone operator. Then the problem (P) has at least one solution.

Let P_C be the projection of a separable real Hilbert space H onto the nonempty closed convex subset C. We consider the variational inequality problem which is denoted by $\mathrm{VI}(C,A)$: find $u \in C$ such that

$$\langle Au + w, v - u\rangle \geq 0, \quad \forall v \in C,\ w \in Tu, \tag{1.1}$$

where A and T are two nonlinear mappings. Recall the following definitions.

(1) A is called v-strongly monotone if there exists a constant $v > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq v\|x-y\|^2, \quad \forall x,y \in C. \tag{1.2}$$

(2) A is said to be μ-cocoercive if there exists a constant $\mu > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \mu\|Ax-Ay\|^2, \quad \forall x,y \in C. \tag{1.3}$$

(3) A is called relaxed u-cocoercive if there exists a constant $u > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-u)\|Ax-Ay\|^2, \quad \forall x,y \in C. \tag{1.4}$$

(4) A is said to be relaxed (u,v)-cocoercive if there exist two constants $u,v > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-u)\|Ax-Ay\|^2 + v\|x-y\|^2, \quad \forall x,y \in C. \tag{1.5}$$

For $u = 0$, A is v-strongly monotone. This class of mappings is more general than the class of strongly monotone mappings.

(5) $T : H \to 2^H$ is said to be a relaxed monotone operator if there exists a constant $k > 0$ such that $\langle w_1 - w_2, u - v\rangle \geq -k\|u-v\|^2$, where $w_1 \in Tu$ and $w_2 \in Tv$.

(6) A multivalued operator T is Lipschitz continuous if there exists a constant $\lambda > 0$ such that $\|w_1 - w_2\| \leq \lambda\|u - v\|$, where $w_1 \in Tu$ and $w_2 \in Tv$.

(7) $S : C \to C$ is said to be nonexpansive if $\|Sx - Sy\| \leq \|x-y\|$, for all $x,y \in C$. Next we will denote the set of fixed points of S by $F(S)$.

In order to prove our main results, we need the following lemmas and definitions.

Lemma 1.2 (see [4]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\lambda_n)a_n + b_n, \quad \forall n \geq n_0, \tag{1.6}$$

where n_0 is some nonnegative integer and $\{\lambda_n\}$ is a sequence in $(0,1)$ with $\sum_{n=1}^{\infty}\lambda_n = \infty$, $b_n = o(\lambda)$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.3. *For any $z \in H$, $u \in C$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in C, \tag{1.7}$$

if and only if $u = P_C z$.

From Lemma 1.3, one can easily get the following results.

Lemma 1.4. *$u \in C$ is a solution of the* VI(C,A) *if and only if u satisfies*

$$u = P_C[u - \rho(Au + w)], \tag{1.8}$$

where w is in Tu and $\rho > 0$ is a constant.

If $u \in F(S) \cap \mathrm{VI}(C,A)$, one can easily see that

$$u = Su = P_C[u - \rho(Au + w)] = SP_C[u - \rho(Au + w)], \tag{1.9}$$

where $\rho > 0$ is a constant.

This formulation is used to suggest the following iterative methods for finding a common element of two different sets of fixed points of a nonexpansive mapping as well as the solutions of the general variational inequalities involving multivalued relaxed monotone mappings.

2. Algorithms

Algorithm 2.1. For any $u_0 \in C$ and $w_0 \in Tu_0$, compute the sequence $\{u_n\}$ by the iterative processes:

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n SP_C[u_n - \rho(Au_n + w_n)], \tag{2.1}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$, for all $n \geq 0$, and S is a nonexpansive mapping.

(I) If $S = I$ in Algorithm 2.1, then we have the following algorithm.

Algorithm 2.2. For any $u_0 \in C$ and $w_0 \in Tu_0$, compute the sequence $\{u_n\}$ by the iterative processes:

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n P_C[u_n - \rho(Au_n + w_n)], \tag{2.2}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$, for all $n \geq 0$.

(II) If $S = I$ and $\{\alpha_n\} = 1$ in Algorithm 2.1, then we have the following algorithm.

Algorithm 2.3. For any $u_0 \in C$ and $w_0 \in Tu_0$, compute the sequence $\{u_n\}$ by the iterative processes:

$$u_{n+1} = P_C[u_n - \rho(Au_n + w_n)], \tag{2.3}$$

which was mainly considered by Verma [5].

3. Main results

Theorem 3.1. *Let C be a closed convex subset of a separable real Hilbert space H. Let $A: C \to H$ be a relaxed (u,v)-cocoercive and μ-Lipschitz continuous mapping, and let S be a nonexpansive mapping from C into itself such that $F(S) \cap \mathrm{VI}(C,A) \neq \varnothing$. Let $T: H \to 2^H$ be a multivalued relaxed monotone and Lipschitz continuous operator with corresponding constants $k > 0$ and $m > 0$. Let $\{u_n\}$ be a sequence generated by Algorithm 2.1. $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $0 < \rho < 2(r - \gamma\mu - k)/(\mu + m)^2$, $r > \gamma\mu + k$.

Then the sequence $\{u_n\}$ converges strongly to $u^* \in F(S) \cap \mathrm{VI}(C,A)$.

Proof. Let $u \in C$ be the common element of $F(S) \cap \mathrm{VI}(C,A)$, then we have

$$u^* = (1 - \alpha_n)u^* + \alpha_n S P_C[u^* - \rho(Au^* + w^*)], \tag{3.1}$$

where $w^* \in Tu^*$. Observing (2.1), we obtain

$$\begin{aligned} \|u_{n+1} - u^*\| &= \|(1-\alpha_n)u_n + \alpha_n S P_C[u_n - \rho(Au_n + w_n)] - u^*\| \\ &= \|(1-\alpha_n)u_n + \alpha_n S P_C[u_n - \rho(Au_n + w_n)] \\ &\qquad - (1-\alpha)u^* + \alpha S P_C[u^* - \rho(Au^* + w^*)]\| \\ &= (1-\alpha_n)\|u_n - u^*\| + \alpha_n \|[u_n - \rho(Au_n + w_n)] - [u^* - \rho(Au^* + w^*)]\|. \end{aligned} \tag{3.2}$$

Now we consider the second term of the right side of (3.2). By the assumption that A is relaxed (γ, r)-cocoercive and μ-Lipschitz continuous and T is relaxed monotone and m-Lipschitz continuous, we obtain

$$\begin{aligned} &\|u_n - u^* - \rho[(Au_n + w_n) - (Au^* + w^*)]\|^2 \\ &\quad = \|u_n - u^*\|^2 - 2\rho\langle (Au_n + w_n) - (Au^* + w^*), u_n - u^*\rangle \\ &\qquad + \rho^2 \|(Au_n + w_n) - (Au^* + w^*)\|^2 \\ &\quad = \|u_n - u^*\|^2 - 2\rho\langle Au_n - Au^*, u_n - u^*\rangle - 2\rho\langle w_n - w^*, u_n - u^*\rangle \\ &\qquad + \rho^2 \|(Au_n + w_n) - (Au^* + w^*)\|^2 \\ &\quad \le \|u_n - u^*\|^2 - 2\rho(-\gamma\|Au_n - Au^*\| + r\|u_n - u^*\|) + 2\rho k\|u_n - u^*\| \\ &\qquad + \rho^2 \|(Au_n + w_n) - (Au^* + w^*)\|^2 \\ &\quad \le \|u_n - u^*\|^2 + 2\rho(\gamma\mu - r + k)\|u_n - u^*\| + \rho^2 \|(Au_n + w_n) - (Au^* + w^*)\|^2. \end{aligned} \tag{3.3}$$

Next we consider the second term of the right side of (3.3):

$$\begin{aligned}
&\|(Au_n + w_n) - (Au^* + w^*)\| \\
&\quad = \|(Au_n - Au^*) + (w_n - w^*)\| \le \|Au_n - Au^*\| + \|w_n - w^*\| \le (\mu + m)\|u_n - u^*\|.
\end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.3) yields

$$\begin{aligned}
&\|u_n - u^* - \rho[(Au_n + w_n) - (Au^* + w^*)]\|^2 \\
&\quad \le \|u_n - u^*\|^2 + 2\rho(\gamma\mu - r + k)\|u_n - u^*\| + \rho^2(\mu + m)^2\|u_n - u^*\|^2 \\
&\quad = [1 + 2\rho(\gamma\mu - r + k) + \rho^2(\mu + m)^2]\|u_n - u^*\|^2 = \theta^2\|u_n - u^*\|^2,
\end{aligned} \tag{3.5}$$

where $\theta = \sqrt{1 + 2\rho(\gamma\mu - r + k) + \rho^2(\mu + m)^2}$. From condition (ii), we have $\theta < 1$. Substituting (3.5) into (3.2), we have

$$\|u_{n+1} - u^*\| \le (1 - \alpha_n)\|u_n - u^*\| + \alpha_n\theta\|u_n - u^*\| \le [1 - \alpha_n(1 - \theta)]\|u_n - u^*\|. \tag{3.6}$$

Observing condition (i) and applying Lemma 1.2 into (3.6), we can get $\lim_{n\to\infty}\|u_n - u^*\| = 0$. This completes the proof. □

From Theorem 3.1, we have the following theorems immediately.

Theorem 3.2. *Let C be a closed convex subset of a separable real Hilbert space H. Let $A : C \to H$ be a relaxed (u,v)-cocoercive and μ-Lipschitz continuous mapping such that $\mathrm{VI}(C,A) \neq \emptyset$. Let $T : H \to 2^H$ be a multivalued relaxed monotone and Lipschitz continuous operator with corresponding constants $k > 0$ and $m > 0$. Let $\{u_n\}$ be a sequence generated by Algorithm 2.2. $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty}\alpha_n = \infty$,

(ii) $0 < \rho < 2(r - \gamma\mu - k)/(\mu + m)^2$, $r > \gamma\mu + k$.

Then the sequence $\{u_n\}$ converges strongly to $u^* \in \mathrm{VI}(C,A)$.

Theorem 3.3. *Let C be a closed convex subset of a separable real Hilbert space H. Let $A : C \to H$ be a relaxed (u,v)-cocoercive and μ-Lipschitz continuous mapping such that $\mathrm{VI}(C,A) \neq \emptyset$. Let $T : H \to 2^H$ be a multivalued relaxed monotone and Lipschitz continuous operator with corresponding constants $k > 0$ and $m > 0$. Let $\{u_n\}$ be a sequence generated by Algorithm 2.3. Assume that the following condition is satisfied: $0 < \rho < 2(r - \gamma\mu - k)/(\mu + m)^2$, $r > \gamma\mu + k$, then the sequence $\{u_n\}$ converges strongly to $u^* \in \mathrm{VI}(C,A)$.*

Remark 3.4. Theorem 3.3 includes [5] as a special case when A collapses to a strong monotone mapping.

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