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Research Article

A Convexity Property for an Integral Operator on the Class $S_P(\beta)$

Daniel Breaz

Department of Mathematics, "1 Decembrie 1918" University, Alba Iulia 510009, Romania

Correspondence should be addressed to Daniel Breaz, dbreaz@uab.ro

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We consider an integral operator, $F_n(z)$, for analytic functions, $f_i(z)$, in the open unit disk, U. The object of this paper is to prove the convexity properties for the integral operator $F_n(z)$, on the class $S_p(\beta)$.

1. Introduction

Let $U = \{z \in C, |z| < 1\}$ be the unit disc of the complex plane and denote by $H(U)$ the class of the holomorphic functions in U. Let $A = \{f \in H(U),\ f(z) = z + a_2z^2 + a_3z^3 + \cdots,\ z \in U\}$ be the class of analytic functions in U and $S = \{f \in A : f \text{ is univalent in } U\}$.

Denote with K the class of convex functions in U, defined by

$$K = \left\{ f \in A : \mathbf{Re}\left\{ \frac{zf''(z)}{f'(z)} + 1 \right\} > 0,\ z \in U \right\}. \tag{1.1}$$

A function $f \in S$ is the convex function of order α, $0 \leq \alpha < 1$, and denote this class by $K(\alpha)$ if f verifies the inequality

$$\mathbf{Re}\left\{ \frac{zf''(z)}{f'(z)} + 1 > \alpha,\ z \in U \right\}. \tag{1.2}$$

Consider the class $S_p(\beta)$, which was introduced by Ronning [1] and which is defined by

$$f \in S_p(\beta) \Longleftrightarrow \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \mathbf{Re}\left\{ \frac{zf'(z)}{f(z)} - \beta \right\}, \tag{1.3}$$

where β is a real number with the property $-1 \leq \beta < 1$.

For $f_i(z) \in A$ and $\alpha_i > 0,\ i \in \{1, \dots, n\}$, we define the integral operator $F_n(z)$ given by

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot \dots \cdot \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt. \tag{1.4}$$

This integral operator was first defined by B. Breaz and N. Breaz [2]. It is easy to see that $F_n(z) \in A$.

2. Main results

Theorem 2.1. *Let $\alpha_i > 0$, for $i \in \{1, \dots, n\}$, let β_i be real numbers with the property $-1 \le \beta_i < 1$, and let $f_i \in S_p(\beta_i)$ for $i \in \{1, \dots, n\}$.*

If

$$0 < \sum_{i=1}^{n} \alpha_i (1 - \beta_i) \le 1, \tag{2.1}$$

then the function F_n given by (1.4) *is convex of order $1 + \sum_{i=1}^{n} \alpha_i (\beta_i - 1)$.*

Proof. We calculate for F_n the derivatives of first and second orders.

From (1.4) we obtain

$$F_n'(z) = \left(\frac{f_1(z)}{z}\right)^{\alpha_1} \cdot \dots \cdot \left(\frac{f_n(z)}{z}\right)^{\alpha_n},$$
$$F_n''(z) = \sum_{i=1}^{n} \alpha_i \left(\frac{f_i(z)}{z}\right)^{\alpha_i} \left(\frac{z f_i'(z) - f_i(z)}{z f_i(z)}\right) \prod_{\substack{j=1 \\ j \ne i}}^{n} \left(\frac{f_j(z)}{z}\right)^{\alpha_j}. \tag{2.2}$$

After some calculus, we obtain that

$$\frac{F_n''(z)}{F_n'(z)} = \alpha_1 \left(\frac{z f_1'(z) - f_1(z)}{z f_1(z)}\right) + \cdots + \alpha_n \left(\frac{z f_n'(z) - f_n(z)}{z f_n(z)}\right). \tag{2.3}$$

This relation is equivalent to

$$\frac{F_n''(z)}{F_n'(z)} = \alpha_1 \left(\frac{f_1'(z)}{f_1(z)} - \frac{1}{z}\right) + \cdots + \alpha_n \left(\frac{f_n'(z)}{f_n(z)} - \frac{1}{z}\right). \tag{2.4}$$

If we multiply the relation (2.4) with z, then we obtain

$$\frac{z F_n''(z)}{F_n'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{z f_i'(z)}{f_i(z)} - 1\right) = \sum_{i=1}^{n} \alpha_i \frac{z f_i'(z)}{f_i(z)} - \sum_{i=1}^{n} \alpha_i. \tag{2.5}$$

The relation (2.5) is equivalent to

$$\frac{z F_n''(z)}{F_n'(z)} + 1 = \sum_{i=1}^{n} \alpha_i \frac{z f_i'(z)}{f_i(z)} - \sum_{i=1}^{n} \alpha_i + 1. \tag{2.6}$$

This relation is equivalent to

$$\frac{zF_n''(z)}{F_n'(z)} + 1 = \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - \beta_i \right) + \sum_{i=1}^{n} \alpha_i \beta_i - \sum_{i=1}^{n} \alpha_i + 1. \tag{2.7}$$

We calculate the real part from both terms of the above equality and obtain

$$\mathbf{Re}\left(\frac{zF_n''(z)}{F_n'(z)} + 1 \right) = \sum_{i=1}^{n} \alpha_i \mathbf{Re}\left(\frac{zf_i'(z)}{f_i(z)} - \beta_i \right) + \sum_{i=1}^{n} \alpha_i \beta_i - \sum_{i=1}^{n} \alpha_i + 1. \tag{2.8}$$

Because $f_i \in S_p(\beta_i)$ for $i = \{1, \dots, n\}$, we apply in the above relation inequality (1.3) and obtain

$$\mathbf{Re}\left(\frac{zF_n''(z)}{F_n'(z)} + 1 \right) > \sum_{i=1}^{n} \alpha_i \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| + \sum_{i=1}^{n} \alpha_i (\beta_i - 1) + 1. \tag{2.9}$$

Since $\alpha_i |zf_i'(z)/f_i(z) - 1| > 0$ for all $i \in \{1, \dots, n\}$, we obtain that

$$\mathbf{Re}\left(\frac{zF_n''(z)}{F_n'(z)} + 1 \right) > \sum_{i=1}^{n} \alpha_i (\beta_i - 1) + 1. \tag{2.10}$$

So, F_n is convex of order $\sum_{i=1}^{n} \alpha_i (\beta_i - 1) + 1$. □

Corollary 2.2. *Let α_i, $i \in \{1, \dots, n\}$ be real positive numbers and $f_i \in S_p(\beta)$ for $i \in \{1, \dots, n\}$. If*

$$0 < \sum_{i=1}^{n} \alpha_i \leq \frac{1}{1 - \beta}, \tag{2.11}$$

then the function F_n is convex of order $(\beta - 1) \sum_{i=1}^{n} \alpha_i + 1$.

Proof. In Theorem 2.1, we consider $\beta_1 = \beta_2 = \cdots = \beta_n = \beta$. □

Remark 2.3. If $\beta = 0$ and $\sum_{i=1}^{n} \alpha_i = 1$, then

$$\mathbf{Re}\left(\frac{zF_n''(z)}{F_n'(z)} + 1 \right) > 0, \tag{2.12}$$

so F_n is a convex function.

Corollary 2.4. *Let γ be a real number, $\gamma > 0$. Suppose that the functions $f \in S_p(\beta)$ and $0 < \gamma \leq 1/(1 - \beta)$. In these conditions, the function $F_1(z) = \int_0^z (f(t)/t)^\gamma dt$ is convex of order $(\beta - 1)\gamma + 1$.*

Proof. In Corollary 2.2, we consider $n = 1$. □

Corollary 2.5. *Let $f \in S_p(\beta)$ and consider the integral operator of Alexander, $F(z) = \int_0^z (f(t)/t) dt$. In this condition, F is convex by the order β.*

Proof. We have

$$\frac{zF''(z)}{F'(z)} = \frac{zf'(z)}{f(z)} - 1. \tag{2.13}$$

From (2.13), we have

$$\mathbf{Re}\left(\frac{zF''(z)}{F'(z)}+1\right)=\mathbf{Re}\left(\frac{zf'(z)}{f(z)}-\beta\right)+\beta>\left|\frac{zf'(z)}{f(z)}-1\right|+\beta>\beta. \tag{2.14}$$

So, the relation (2.14) implies that the Alexander operator is convex. □

References

[1] F. Ronning, "Uniformly convex functions and a corresponding class of starlike functions," *Proceedings of the American Mathematical Society*, vol. 118, no. 1, pp. 189–196, 1993.

[2] D. Breaz and N. Breaz, "Two integral operators," *Studia Universitatis Babeş-Bolyai, Mathematica*, vol. 47, no. 3, pp. 13–19, 2002.